

Pre-service Teachers' Perceptions of the Use of Representations and Suggestions for Students' Incorrect Use

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ABSTRACT

In this study, we investigated how elementary pre-service teachers (PSTs) perceive using representations in teaching mathematics and what fractional representations (e.g., manipulatives or models) they suggest to guide students' incorrect use of representations in learning fractions. A written questionnaire was administered to 151 PSTs at a large Southwestern university in the US. An inductive content analysis approach including both qualitative and quantitative analyses was used to analyze the data. Findings suggested that fraction-related topics were the PSTs' main choices for using representations, and they valued understanding concepts and making connections between representations and concepts. Also, the findings showed the PSTs' tendency to use models procedurally and their predominant dependency on a few types of models (e.g., wedged circular models) in guiding students who use representations incorrectly. Implications for designing mathematics methods courses that support effective use of representations are discussed.

Keywords: representations, fraction models, manipulatives, preservice teachers, teacher preparation

INTRODUCTION

The ability to execute a wide range of instructional practices that are deeply rooted in profound mathematical knowledge for teaching is integral to effective mathematics teaching (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008). In this paper, we call attention to the issue of pre-service elementary teachers' combined knowledge of teaching and mathematics. In particular, we examine their ability to analyze students' work, to choose and use appropriate mathematical representations in explanations, and to evaluate the instructional advantages and disadvantages of representations used to teach a specific mathematical idea.

The term *representation* here refers broadly to the external and internal portrayals of mathematical phenomena that enable teachers to manage, document, communicate, and interpret mathematical ideas. These may be expressed in different modes, including experience-based scripts (i.e., real-life contexts), manipulable models (i.e., manipulatives), pictures or diagrams, verbal exchanges, and written symbols (Lesh, Post, & Behr, 1987; van den Heuvel-Panhuizen, 2003).

The use of mathematical representations to develop conceptual understanding of key mathematics concepts has been widely discussed, with a general consensus that, due to the abstract nature of mathematics, representations are essential as placeholders of thought and windows into students' understanding (e.g., Duval, 2006; Greeno & Hall, 1997; McKendree, Small, & Stenning, 2002; Woleck, 2001; Zhang, 1997). Professional organizations also support their use in mathematics instruction (National Council of Teachers of Mathematics [NCTM], 2014).

However, research urges caution rather than "magical hope" (Ball, 1992) in the use of various representations, which may have both affordances and constraints for supporting student learning (e.g., Kamii, Lewis, & Kirkland, 2001). Moreover, using representations in mathematics instruction is a complex teaching practice that includes choosing and interpreting appropriate representations for specific mathematics concepts, using them to interact

Contribution of this paper to the literature

- This paper contributes to extending the previous research by providing insights into their perceptions of what constitutes effective use of representations in the teaching and learning of mathematics and giving implications for teacher preparation, which in turn will have an impact on pre-service teachers' future practices.

with students, and identifying their effectiveness and limitations. In addition, as teachers' own knowledge of mathematical content is necessarily intertwined with their teaching practices (Berenson et al, 1997), teachers with weak content knowledge will have more difficulty meaningfully employing various representations.

The topic of fractions is considered a high-leverage content, crucial for students' mathematical development (Lamon, 2007; McNamara & Shaughnessy, 2010; TeachingWorks, n.d.), and one of the most challenging topics to teach and learn. Not surprisingly, therefore, numerous fraction-related concrete representations, known as manipulatives, are commercially available. Many research studies have reported the usefulness of manipulatives as well as other curriculum materials and models in teaching fractions (Clarke & Roche, 2009; Cramer & Henry, 2002; Hackenberg & Lee, 2015; Izsak, 2008; Lamon, 2007; Lee, 2017; Lee & Hackenberg, 2014; Watanabe, 2002).

However, many studies also have reported teachers' lack of competence in using representations to support student learning (Borko et al., 1992; Izsak, 2008; Lee, Brown, & Orrill, 2011). In particular, studies have shown that pre-service teachers who demonstrate good knowledge of fraction operations and computation skills often show low knowledge and skills in the use of representations (e.g., Lo & Lou, 2012; Rosli, Han, Capraro, & Capraro, 2013). Similarly, as will be discussed below, the participants of this study themselves suggested in an initial survey that fractions might be a good example of mathematics content that can benefit from using various manipulatives and models. However, selecting specific modes of representations, such as manipulatives or models, is only part of the story, the rest being teachers' knowledge of how to use them effectively. Thus, bridging this gap between pre-service teachers' computational and pedagogical competencies is a critical issue to the mathematics education community.

Moreover, when considering that teachers need to fully understand various models and their uses in teaching and learning fractions as the usefulness of pedagogical representations is context-dependent, investigating pre-service teachers' current knowledge of representations and drawing corresponding suggestions to effectively design mathematics pedagogy courses are important to help pre-service teachers learn more about the effects of the various models on teaching and learning fractions and be ready for successful mathematics teaching.

Therefore, in this study, we explored pre-service teachers' perceptions of the use of various representations, ability to analyze and interpret students' mathematical work involving the use of representations, and competence in utilizing representations to provide effective instructional guidance. To do so, we first examined pre-service teachers' conceptions of using representations in general and in the context of teaching fractions specifically. In the hope that this study has implications for teacher preparation, which in turn will have an impact on pre-service teachers' future practices, we investigated the following research questions:

(1) How do pre-service teachers perceive using representations (e.g., manipulatives or models) in teaching mathematics?

(2) What fractional representations (e.g., manipulatives or models) do pre-service teachers suggest to guide students toward correct use of representations in learning fractions?

THEORETICAL BACKGROUND

Knowledge of Representation

As noted, the use of various representations is commonly considered necessary in mathematics instruction because of the abstract nature of mathematics and mathematical discourse (e.g., Duval, 2006; McKendree, Small, & Stenning, 2002; Zhang, 1997). Accordingly, teachers' and students' competence in using representations has long been emphasized in mathematics education (e.g., Collins, 2011; Huinker, 2015; Lesh, English, Sevis, & Riggs, 2013). Educators refer to this competence using varied terms such as *representational flexibility*, *representational fluency*, *representational thinking*, and *representational competence* (Collins, 2011; Greer, 2009; Huinker, 2015; Nathan, Alibali, Masarik, Stephens, & Koedinger, 2010; Pape & Tchoshanov, 2001). Despite different terms, at the core of this concept are flexibility and variability in the meaningful use of representation, which are considered to be evidence of mathematical understanding and critical characteristics of a good problem solver (e.g., NCTM, 2000, 2014; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010).

While using representations is recognized as important for supporting students' sense-making processes, the literature also documents that limited uses or misuses of representations, often due to teachers' limited content knowledge, can negatively affect teaching and learning (Bosse, Lynch-Davis, Adu-Gyamfi, & Chandler, 2016; Izsák, 2008; Lee, Brown, & Orrill, 2011; Zazkis & Gadowsky, 2001). However, while adequacy of content knowledge is a necessary condition for teachers' effective use of representation, it is not sufficient. Teachers need to be fluent in using not only their own solution representations but also pedagogical representations. Cai (2005) explains *solution representations* as "the visible record generated by a solver to communicate thinking of the solution processes" (p.137) and *pedagogical representations* as "representations that teachers and students use as expressions of mathematical knowledge that help explain concepts, relationships, connections or problem-solving processes" and argues that "the use of pedagogically sound representations is an important decision to make as one selects instructional strategies for the mathematical classroom" (p.139). All in all, competencies in using representations in mathematics instruction include teachers' ability to present their own solution representations, to identify and analyze students' solution representations, and to select desirable pedagogical representations. Teachers' mastery of these competencies depends largely on their beliefs, conceptions, and content knowledge (Cai, 2005).

In this study, we analyzed pre-service teachers' representations proposed in the context of comparing two fractions in terms of whether they provided solution representations or pedagogical representations.

Fraction Concepts

A fraction itself does not tell anything about the actual size of the whole or the actual size of the parts, but only represents the relationship between the part and the whole (van de Walle, Karp, & Bay-Williams, 2013). In this regard, attention to the relationship between the part and the whole is important in defining the concept of fractions. As a way to pay attention to this relationship, some researchers (Steffe & Olive, 2010; von Glasersfeld, 1995) have demonstrated that to develop the concept of fractions, students need to construct important mental actions, called *operations*, such as *partitioning* (i.e., dividing the given quantity into equal parts), *iterating* (i.e., repeating a fractional part or unit to make a larger fraction), and *disembedding* (i.e., taking a part from a whole without mentally destroying the whole). For example, when asked to identify $\frac{3}{4}$ of a bar, learners who have fully developed the concept of fractions would partition the given bar into equal parts, remove a part from the whole while mentally or materially maintaining the whole to which the parts can be compared, and iterate this action three times (Steffe & Olive, 2010) although the process could be shown in a simple fashion such as dividing the given bar into four equal parts and shading three parts. However, students who are on the way of developing the concept of fractions may also divide the given bar into four parts (not necessarily equal) and identify three of those parts by shading, but think of three-fourths as three parts *embedded* within what they see as the whole rather than considering it as a measurable quantity or a fractional number (Hackenberg, 2013; Steffe & Olive, 2010).

To compare the relationships between the part and the whole of two different fractions, both fractions must have the same sized whole, so students should be able to construct the concept of "equal-sized whole for both fractions" as well as "equal partitioning within a fraction," including the operations of partitioning, disembedding, and iterating. Thus, equal partitioning and equal-sized whole are the key mathematical ideas that teachers need to know in order to create models for comparison of two fractions (van de Walle et al., 2013). For this reason, in this paper, we used the two factors of equal partitioning and equal whole as criteria for evaluating the correctness of pre-service teachers' models for comparing fractions.

Fraction Models

In the domain of fractions, students' solution representations often serve as windows for teachers to identify students' understandings or misconceptions (Woleck, 2001) regarding such concepts as equal partitioning, referent units, or invariance of the whole. Also, using effective representations can deepen students' understanding of fractions; using two different models and asking students to make connections between them can broaden their concepts of fractions (van de Walle et al., 2013). Prior research has reported that using models effectively in fraction tasks is critical (Cramer & Henry, 2002; Petit, Laird, & Marsden, 2010; Siebert & Gaskin, 2006). Thus, when teachers plan their pedagogical representations of fractions, they must examine and select representations which are most applicable and useful. Given the importance of representations in teaching and learning mathematics, therefore, we further examine three major types of pedagogical representations in fractions, area, length, and set models, with particular attention to their affordances and constraints as documented in the literature.

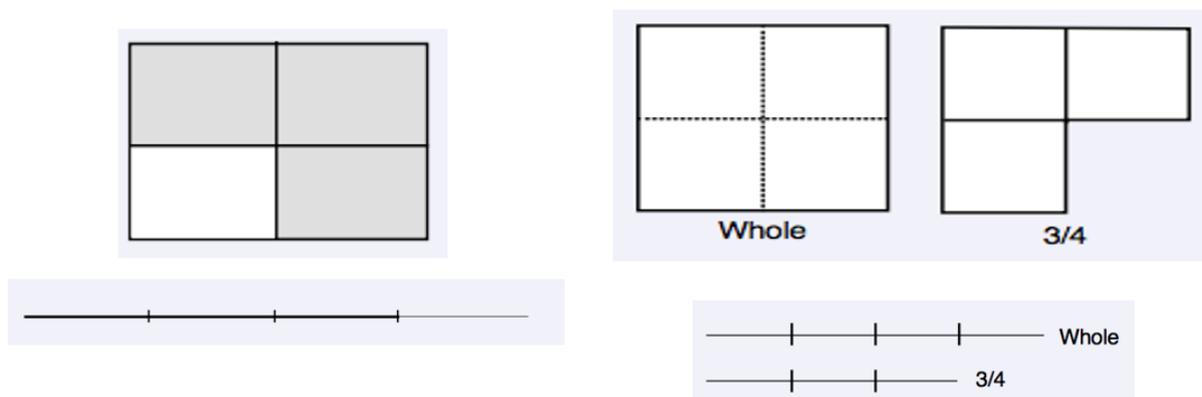
The *area models* (e.g., circular pie pieces, rectangular arrays, geoboards, drawings on grids or dot paper, pattern blocks, etc.) subdivide a whole area into equal parts. These models are most commonly presented to students as various geometric shapes and are dominant in curriculum materials (Hodges, Cady, & Collins, 2008) and teacher education textbooks (Zhang, Clements, & Ellerton, 2015). The area model has been supported for its effectiveness in the teaching and learning of fractions, in particular in the early-grade mathematics curricula (e.g., see Rational

Number Project), because it elucidates the part-whole concept of fractions and the meaning of the relative size of a part to that of the whole (Cramer, Wyberg, & Leavitt, 2008; van de Walle et al., 2013). But some constraints have also been noted. For example, fractioned circles are among the most popular manipulatives in teaching and learning fractions (Bray & Abreu-Sanchez, 2010; Cramer & Henry 2002; Cramer, Post, & del Mas, 2002). However, they are not useful for dealing with large denominators or denominators that cause difficulty in showing equal partitioning. This limitation is referred to as “the model breaking point,” at which the utility of a model for illustrating mathematics concepts diminishes and even ends (Vig, Murray, & Star, 2014). Also, students who are learning fractions for the first time may have difficulty equally partitioning the circle models because they have to consider the center angles of a circle (van de Walle, et al., 2013). Also, Martin and Schwartz (2005) reported that students who had learned fractions with pie wedge models showed lower ability to transfer skills to other manipulatives because the structure of pie wedges gives a part-of-whole interpretation “for free,” and thus does not give students opportunities to interpret groupings and whole structures by themselves. These researchers point out that the heavy emphasis on area-model approaches does not support students’ construction of strong and flexible fraction concepts, especially with tasks in non-area situations. In brief, such emphasis possibly restricts students’ thinking rather than expanding it.

Another major fraction model is the *length model* (e.g., fraction strips, Cuisenaire rods, number lines, etc.), in which measurements of lengths rather than of areas are compared. Linear models, which highlight the length of the given representations or are shown as number lines, allow students to integrate their knowledge of fractions and whole numbers in a continuous model (Lamon, 2005; Saxe, Diakow, & Gearhart, 2012; Siegler, Thompson, & Schneider, 2011; Wu, 2011). Some research has suggested that with a linear model it is difficult for students to develop an accurate visualization of the value of a given fraction (Behr & Post, 1992). However, another strand in the literature emphasizes the utility of linear models in that their continuity helps students understand fractions that are greater than one more easily than the discrete wholes used in area models (Lee, 2017; Siegler et al., 2011). In particular, several researchers (Clarke, Roche, & Michell, 2008; Usiskin, 2007; Watanabe, 2006) have supported number line models for three reasons: (1) the number line can help students understand a fraction as a single number rather than one number over another number (Cramer & Whitney, 2010; Petit et al., 2010; Siegler et al., 2011); (2) the number line allows comparisons of its relative size with that of other numbers, which is not clear when using area models (Usiskin, 2007; Watanabe, 2006); and (3) the number line helps students expand the number system beyond whole numbers by reinforcing that there is always one more fraction to be found between two fractions (Siegler et al., 2010). While some Asian countries’ school curricula (e.g., Grow-Maienza & Beal, 2005; Watanabe, 2007) and recent Common Core State Standards for Mathematics (NGA & CCSSO, 2010) emphasize the linear representations of fractions, it is notable that some countries including the US historically have not emphasized linear representations (Petit et al., 2010; Siegler et al., 2011).

Set models (e.g., two-color counters, objects, tallies, etc.) utilize discrete collections within sets of countable objects. For example, six objects are one-third of a set of 18 objects because the set of 18 represents the whole and there are three sets of six objects in the whole. That is, set models require students to perceive a collection of discrete objects as a single unit, which may pose some challenges (Behr, Wachsmuth, & Post, 1988) because students tend to attend to the size of the set (i.e., 18 objects in the example) but not to the number of *equal sets* in the whole (i.e., three sets of six objects). The literature offers mixed results and suggestions regarding the effectiveness of the set model. While many researchers recognize the importance of set models due to the applicability of students’ prior knowledge of whole number strategies and the connections with many real-world uses of fractions (Behr & Post, 1992; Charles & Nason, 2000; Hunting, 1986; Mack, 1995; van de Walle et al., 2013), others recommend introducing set models later than other models (Bezuk & Cramer, 1989; Cramer et al., 2008). Some educators (e.g., van de Walle et al., 2013) do not recommend using a set model to teach the basic concept of fractions or comparing fractions because it may require some understanding of ratio or the concept of fraction multiplication.

While the forms of models differ, the methods of representing a whole and its parts can be varied. Watanabe (2002) explains two distinct methods for representing fractions, depending on how the relationships between the whole and fractional parts are shown: (1) the part-whole method and (2) the comparison method. In the part-whole method, the fractional part is embedded in the whole, whereas in the comparison method the whole and part are constructed separately (see [Figure 1](#)).



Representation of $\frac{3}{4}$ using the part-whole method

Representation of $\frac{3}{4}$ using the comparison method

Figure 1. Two methods for representing a whole and its parts [Note: the pictures are adapted from "Representations in Teaching and Learning Fractions" by Watanabe (2002, pp. 458-459)]

Watanabe (2002) asserts that children may benefit from a more explicit treatment of the different ways that fractions can be represented, highlighting the importance of teachers' conscious consideration of the appropriateness of these methods from a developmental perspective. For example, in the part-whole method, the fractional part is both an entity in itself and a quantity in a larger entity. This nested thinking might be difficult for young students.

Since effectively representing fractions is a complex teaching practice that involves knowledge of content, curriculum, and students, it would not be enough to simply be aware of the importance of using representations. Pre-service teachers are familiar with specific representations of fractions through their past experiences and examination of common curriculum materials. However, there should be more conscious and intentional reflections upon the purposes for and methods of using various representations for teaching and learning fractions. In this study, we analyzed pre-service teachers' fractional models in terms of three types of models and further identified whether they used the part-whole method or comparison method in proposing each type of fractional model.

METHOD

Participants and Context

Participants in this study were 151 pre-service elementary teachers (PSTs) at a large Southwestern university in the U.S. All were enrolled in an elementary mathematics content course that covered concepts of variables, unknowns, patterns, functions, rates of change, co-variational reasoning, and algebraic reasoning. Prior to this course, they had taken a course that dealt with number and operations, in which they learned about fractions and fraction operations as well as various fraction representations to help their own understanding of fractions. After the target course, the PSTs were scheduled to take a mathematics methods course, which was designed to help them understand the teaching approaches, strategies, and issues in teaching mathematics. Thus, this study aimed to probe what views and understandings PSTs bring to a methods course and to gain insights into this process that would help in the design of both a content course and a methods course to support the development of their knowledge for teaching elementary mathematics.

Data Source

The participating PSTs completed a written questionnaire with two parts: Part 1 consisted of two open-ended questions concerning the use of representations (e.g., manipulatives or models) in teaching mathematics, and Part 2 consisted of two tasks that asked PSTs to interpret and evaluate students' representations as well as to provide another representation to the students (see [Figure 2](#)). The two questions in Part 1 and the second task in Part 2 (see below) were created by the authors and the first task in Part 2 was taken from prior research (Cross, Lee, Zeybek, & Adefope, 2015).

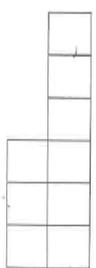
Part 1

(1) What topics would be the most effective for elementary students to learn mathematics using representations (e.g., manipulatives or models)? Describe the topics and explain how to teach the topics using representations.

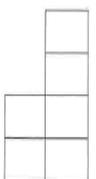
(2) What do you think is the most important aspect of teaching a math lesson to elementary students using representations (e.g., manipulatives or models)?

Part 2

Task 1: Susan, a 3rd grade student, is given the following question: "Which is larger $\frac{2}{4}$ or $\frac{3}{6}$? How do you know?" Susan created the following models with the multi-link cubes to represent the fractions and help her with comparing the fractions. (Note: The models with actual multi-link cubes were presented in the way shown below.)



$\frac{3}{6}$

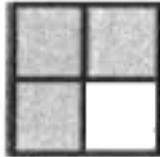


$\frac{2}{4}$

(1) Do you think Susan's models will help her to compare the fractions?

(2) If you don't think Susan's models will help her, please create a model that you think would be useful for comparing $\frac{2}{4}$ and $\frac{3}{6}$.

Task 2: You asked a 3rd grade student, Lisa, to make a drawing to compare $\frac{3}{4}$ and $\frac{5}{6}$. She draws the following picture and claims that " $\frac{3}{4}$ and $\frac{5}{6}$ are the same amount."





(1) Do you think Lisa's models are correct?

(2) If so, explain. If not, create a model that you think would be useful for comparing $\frac{3}{4}$ and $\frac{5}{6}$ and explain why your model would be helpful.

Figure 2. Problems used for this study

We purposefully selected the two models in Tasks 1 and 2 for two reasons. First, in teaching fractions, the comparison method has been less used than the part-whole method (Watanabe, 2002). Thus, we wondered how PSTs would interpret an example of the comparison method (Task 1) as well as an example of the part-whole method (Task 2). Second, we tried to be balanced in selecting types of models. Task 1 is related to the length model for comparing equivalent fractions because the lengths of the linked cubes were highlighted, whereas Task 2 is an area model for comparing two fractions using an alternative strategy such as one part away from being a whole (see Clarke & Roche, 2009). Here, we did not use a set model because the focus of our study was specifically on comparing the sizes of two fractions by using fractional models. As noted, according to the literature (van de Walle et al., 2013), using a set model is not recommended for teaching the basic concept of fractions or comparing fractions in that it could require an understanding of ratio or the concept of fraction multiplication.

Data Collection and Analysis

The written questionnaire including the problems in Figure 2 was administered to all PSTs in six sections of a mathematics content course at the beginning of spring 2016, fall 2016, and spring 2017. All PSTs completed the questionnaire within the given time of one hour.

To analyze the data, we used an inductive content analysis approach that included both qualitative and quantitative analyses (Grbich, 2007). First, to analyze the open-ended question, we read all of the PSTs' responses and identified common themes among them. Then we developed a coding scheme and used it to analyze the responses. When there were discrepancies between the two researchers' codes, we discussed them until we reached 100% agreement on the coding. Finally, to find patterns, we calculated the frequencies of codes in the PSTs' responses.

To analyze the PSTs' responses to students' fractional models, we first evaluated the correctness of the responses and categorized the models that PSTs provided in terms of two aspects: (1) whether the same sized whole and equal partitioning were used for both fractions; and (2) the types of models provided, such as length model (e.g., rectangular strip, number lines), area model (e.g., pie model, rectangular array), set model (e.g., discrete models), and a drawing based on an algorithm to find the common denominator. Particularly in determining the types of

Table 1. Number of Responses about Topics that PSTs Suggested for Using Representations

Topics	Number of Responses
Counting	15 (4%)
Whole Number Operations	77 (25%)
Place Value (or base system)	20 (7%)
Fractions (e.g., basic concept, equivalent fractions, fraction comparison)	108 (35%)
Fractions (operations)	11 (3%)
Other fractional numbers (e.g., decimals, percent)	6 (2%)
Measurement (including measuring perimeter, area, volume)	17 (5%)
Geometry (shapes)	23 (8%)
Word Problems	5 (1%)
Other topics (e.g., money, algebra, polynomials, equation, function, negative number, pattern, Pythagorean theorem)	19 (6%)
N/A	4 (1%)
Total	305

Note. The total number of responses is higher than the number of participants because some PSTs provided multiple responses

Addition - use M & M's (or any food) use algebra tiles & show how 2 groups of #'s make 1 group
 Fractions - you can use circles (pizza) or linear models to show how fractions equal each other & a whole
 Subtraction - same as addition but w/ taking away 1 group from the large group.
 positive / negative #'s - you can use the red & yellow double sided tiles to show positive & negative #'s & how to add & subtract them

Figure 3. An example from a pre-service teacher

models used by PSTs, when their drawings were not clear, we referred to their explanation on what attribute (length, area, or count) they were focusing on in order to supplement our analysis of their drawings. Then we developed a coding scheme for PSTs' use of pedagogical representations along with examples from their work and used it to code all data (see [Tables 6](#) and [7](#) in the findings and discussion section). Finally, we counted frequencies of types of models suggested by PSTs and interpreted the data quantitatively as well as qualitatively.

FINDINGS AND DISCUSSION

PSTs' Perceptions of Using Representations in Teaching Mathematics

Topics that PSTs suggested for using representations

Table 1 shows PSTs' suggested topics in which representations (e.g., manipulatives or models) can be used effectively. If we consider all fraction operations (fraction addition, subtraction, multiplication, division) and other fractional numbers (decimals and percent), the topics related to fractions account for 40% of the responses. However, with regard to the topic of fractions, most PSTs pointed out the basic part-whole concept of fractions and very few (3 out of 108 responses) mentioned that manipulatives could be used to compare fractions, which is related to the focus of our study. This result shows that these PSTs clearly recognized the importance of manipulatives in teaching part-whole fraction concepts, but they needed to be exposed to a variety of ways to use manipulatives or models in teaching fraction comparison or fraction operation.

Although PSTs were given enough time to complete the questionnaire, in their responses, the majority simply listed topics and the names of manipulatives or models without describing how to use them, which may show that they did not have a clear image of using manipulatives or models. Only seven PSTs described how to use manipulatives in specific ways depending on the topics (see [Figure 3](#)).

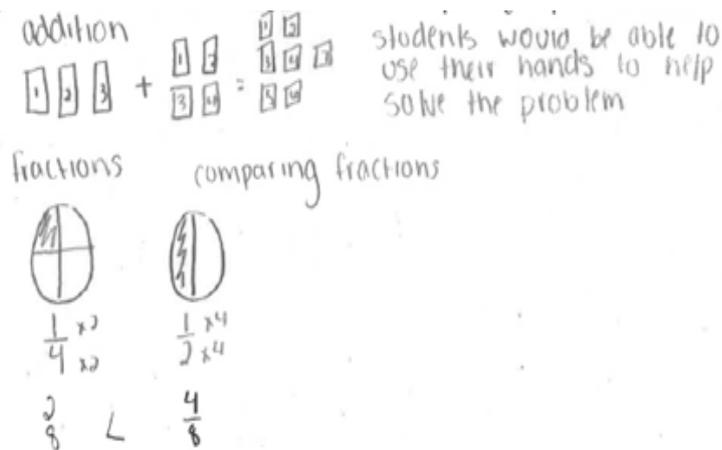


Figure 4. An example from a pre-service teacher

Lastly, three PSTs gave examples of how to use manipulatives, but their examples were not related to conceptual reasoning. For example, as shown in **Figure 4**, one PST used fractional circles to compare the sizes of fractions but employed a computational approach to finding common denominators, which was not consistent with the presented model. This example suggests the PST's lack of pedagogical content knowledge for connecting conceptual understanding with mathematical notation or algorithms through the use of manipulatives or models.

Most PSTs simply provided topics. However, even if they provided detailed explanations for using manipulatives or models to teach fraction comparison, they did not sufficiently address conceptual understanding. This may be an indication of the PSTs' lack of knowledge of how to use manipulatives or models effectively in teaching fractions, possibly because they had insufficient experience with manipulatives or models in their K-12 school mathematics classes or in their teacher preparation program. Pajares (1992) found that teachers' own learning experiences influence their perceptions of teaching, which in turn affect their actions. Mathematics has been taught in various ways nationally and internationally (Kawanaka, Stigler & Hiebert, 1999; Noyes, 2006); thus, students have different experiences in learning mathematics in their schools, depending on the school's location and teachers. This finding is consistent with Soto-Andrade's (2007) finding that participants' different perceptions of a mathematics subject as expressed in metaphors reflected differences in their previous experiences with that subject. In a similar vein, the variations in PSTs' perceptions of the use of manipulatives or models might be explained by their previous educational backgrounds.

The most important aspects of teaching mathematics with representations

Table 2 shows PSTs' responses regarding what they considered important in teaching mathematics with manipulatives (e.g., manipulatives or models). A fourth of their responses indicated *understanding and reasoning*. For example, PSTs addressed emphasizing mathematical concepts over procedures, understanding the reasoning behind students' use of models or manipulatives, and making connections between models and mathematical concepts (e.g., Helping students make connections from visuals or manipulatives to written-out problems). The next largest group of responses (15%) indicated the importance of *controlled and focused management* by helping students understand how to use the manipulatives and models accurately, demonstrating their use, and establishing consistent rules for using them purposefully rather than as toys. Almost as many responses (11%) emphasized *openness to students' exploration of manipulatives* such as giving children time to handle and experiment with the manipulatives on their own and discover answers for themselves rather than being guided. The importance of *selecting proper manipulatives/models* at different levels of progression, from concrete, pictorial, to abstract representation, was emphasized in 8% of the responses. Some examples were related to using scaling models appropriately, helping students progress through the different levels of representations, and helping students graduate from using manipulatives to solving problems without them.

Table 2. Frequencies of Aspects in Teaching Mathematics with Manipulatives or Models Considered the Most Important by PSTs

Categories	Number of Responses
A. Understanding & Reasoning	55 (25%)
B. Hands-on experiences or visuals	36 (16%)
C. Controlled and focused management	35 (15%)
• Giving clear directions and showing how to use manipulatives	26
• Keeping students focused on purposeful use of manipulatives	9
D. Openness to students' exploration of manipulatives	25 (11%)
• Openness to various uses of manipulative/models or of modes of representations (including symbolic, visual)	6
• Student exploration/experiment	19
E. Motivation: Making learning enjoyable, fun, engaging	9 (4%)
F. Real-life connection: Relating to real-life; Real-life application; Provide context	9 (4%)
G. Equity	10 (5%)
• Availability: Making manipulatives available to students	2
• Access to lesson: Ensuring all students' participation/Accommodating all students with different learning styles	8
H. Selection and Organization	17 (8%)
• Selection of proper and effective models and manipulatives	5
• Consideration of progression: matching types of representation to students' levels of progression	12
I. Misc.	10 (5%)
• Promoting problem solving, communication, critical thinking	
• Scaffolding from previous experience	
• Providing activities/games	
• Fostering cooperation with others/Helping each other	
J. N/A	15 (7%)
• No specifics provided (other than saying its importance)	
• Listed math topics only	
Total	221

Note. The total number of responses is higher than the number of participants because some pre-service teachers provided multiple responses

Table 3. PSTs' Reasons for Considering Susan's Models Helpful

Categories for the Reason	Number of Responses
A. The same proportional relationship between numerator and denominator (Representing a half)	29 (64%)
B. Visual representation used (without further explanation)	13 (30%)
C. No reason or unclear reason	3 (6%)
Total	45

Identifying Correctness of Using Representations for Comparing Two Fractions

To investigate the PSTs' awareness of correct use of representations for comparison of two fractions, we asked two questions in connection with Susan's model (Task 1) and Lisa's model (Task 2). Susan's model was drawn as multi-link cubes to show the fractions of $\frac{3}{6}$ and $\frac{2}{4}$, respectively. Lisa used an area model in which three parts were shaded in the 2 by 2 square and five parts are shaded in the 2 by 3 rectangle.

Task 1: Susan's model

When asked whether Susan's models would help her to compare the two fractions of $\frac{3}{6}$ and $\frac{2}{4}$, 100 PSTs (66%) answered that the models were not helpful, 45 PSTs (30%) said that they were helpful, and 6 PSTs (4%) pointed out both positive and negative aspects, such as "It can help but at the same time that might confuse her," "Yes but it may be a little confusing because they may just count the blocks," and "I think Susan's models will help her compare the fractions, but I don't think it is the best method."

Among the PSTs who considered Susan's models helpful, more than half justified their answers by specifically referring to the same proportional relationship between numerator and denominator in both fractions (see Table 3). For example, in Figure 5, the PST in the upper section focused on both fractions having the same proportional relationship between the part and the whole, while the other PST below indicated that both fractions could be simplified to $\frac{1}{2}$ and were therefore equivalent because in both models, the left cubes were half way up the right cubes.

yes, the will show her that $\frac{3}{6} + \frac{2}{4}$ are the same # of pieces to their whole so they are the same fraction for different wholes

yes b/c you can visually see that there is exactly half the # shown in the denominator stand than in the numerator stand = $\frac{3}{6} = \frac{1}{2}$ $\frac{2}{4} = \frac{1}{2}$

Figure 5. Examples of PSTs' reasons based on the same proportional relationship between numerator and denominator

Table 4. PSTs' Reasons for Not Considering Susan's Models Helpful

Categories for the Reason	Number of Responses
1. Models do not <i>visually</i> reflect final result ($3/6=2/4$): In the model, $3/6$ looks larger than $2/4$ although both fractions simplify to $1/2$.	37 (37%)
2. Models are not based on the same sized whole.	31 (31%)
3. Models do not represent the part-whole concept of fractions (e.g., using an incorrect number of blocks to represent the given fractions; not representing "parts out of whole; numerator is not shaded to represent a fraction.)	20 (20%)
4. General reasons are given without details (e.g., Not showing comparison, Misleading and causing confusion)	9 (9%)
5. Models do not show the same additive relationship between numerator and denominator in the two fractions (e.g., $4-2=2$ vs. $6-3=3$)	3 (3%)
Total	100

Thirteen PSTs simply explained that the use of visual representation was helpful without explaining how. One wrote, "By drawing it out, it helps her visualize it to see which is bigger," and another, "Visuals are helpful because the student is able to see the fraction to the whole and make a comparison."

Among the PSTs who answered that the models were not helpful, 37 pointed out that they did not *visually* reflect the final result of comparing two fractions (e.g., $3/6=2/4$) (see Table 4). That is, the representations did not visually depict the two fractions as equal. For example, a PST responded, "No, because the model makes $3/6$ look larger, even though both fractions can be simplified to $1/2$."

Less than a third ($n=31$) of the PSTs pointed out that the models showed differently sized wholes, which was the salient point. For example, one commented, "No, because her whole in each model is not equal, so the comparison will be off." Twenty PSTs indicated that the models did not represent the part-whole concept of fractions. Some PSTs pointed out that the total number of blocks for each fraction should be the same as their denominator. That is, to represent $3/6$ (or $2/4$), the total number of blocks needed should be six (or four), not nine (or six) as in the model (see Figure 2). For example, one said, "No, because the models are showing $3/9$ and $2/6$ which is not what Susan is trying to compare." and another, "No, there should not be a total of 9 squares and 6 squares in each model. There needs to be 6 squares and 4 squares to properly represent each fraction." Other PSTs argued that the models did not clearly illustrate parts of a whole (e.g., " $3/6$ does not refer to 3 blocks next to 6 but it means 3 out of 6 equal sized parts are used up and the same goes for $2/4$. But Susan didn't shade in the part that is supposed to be shaded"). Here, although the key misconception of the model was the use of differently sized wholes in comparing two fractions, PSTs were focusing on the total number of blocks used to represent each fraction based on the part-whole concept of fractions. These responses show that a fifth of the PSTs may not have had experience using the *comparison method* to represent fractions (Watanabe, 2002) as they seemed to consider only a *part-whole method* as a correct representation.

Nine PSTs provided vague or incomplete reasons. Two pointed out that the models did not show comparison (e.g., "No, I do not think it will help because it's not showing the comparison between the two fractions"), and seven PSTs commented that the models would cause confusion without giving reasons (e.g., "I like her train of thought with these models, however I think they are very misleading and would cause much confusion.").

In addition, three PSTs indicated that the models demonstrated different relationships between numerator and denominator in the two fractions $2/4$ and $3/6$. That is, the PSTs *additively* compared the number of blocks

No, I don't think her models will help her compare the fractions because the representation of $\frac{3}{6}$ is off by 3, the $\frac{2}{4}$ representation is over by 2 blocks as well, if she is counting, 2 blocks as 1 unit.

Figure 6. An example of PSTs' responses claiming different additive relationships between numerator and denominator in the two fractions, $\frac{2}{4}$ and $\frac{3}{6}$

Table 5. Pre-service Teachers' Reasons for Considering Lisa's Model Correct

Lisa's model is correct because...	Number of Responses
A. Each fraction ($\frac{3}{4}$, $\frac{5}{6}$) is shown correctly based on the part-whole concept of fraction.	25 (77%)
B. Models show a correct result such as $\frac{5}{6}$ is larger than $\frac{3}{4}$.	3 (9%)
C. It is simply mentioned that the models are correct without providing either justification or other models.	5 (15%)
Sub total	33
Lisa's model is correct, but...(Addressing both positive and negative aspects)	Number of Responses
D. Correct, but need to help her to see the difference between $\frac{3}{4}$ and $\frac{5}{6}$ using a pie model (two circles of the same size but segmented differently)	6 (29%)
E. Correct, but need to help her to see the difference between $\frac{3}{4}$ and $\frac{5}{6}$ using fraction bars (two bars of the same size but segmented differently)	11 (52%)
F. Answers/drawings are correct, but the interpretation is not correct	4 (19%)
Sub total	21

representing the numerator and denominator in each fraction, i.e., subtracted the number representing the numerator of each fraction from the number representing the whole and got different answers (Figure 6). The PSTs seemed to be unaware that the comparison of two fractions should be based on the *proportional* relationship between numerator and denominator in two fractions (i.e., [the number of blocks representing a part in each fraction] \div [the number of blocks representing a whole in each fraction] = [ratio]; $2 \div 4 = \frac{1}{2}$ vs. $3 \div 6 = \frac{1}{2}$).

Task 2: Lisa's model

When asked whether Lisa's model was correct, 95 PSTs (63%) answered, "incorrect" and 33 PSTs (22%) answered, "correct." Also, 21 PSTs (14%) responded that the model was correct but could be refined for effectiveness, while two (1%) did not provide any answers.

Among those who considered the model correct (see Table 5), 25 out of 33 (77%) explained that the fractions of $\frac{3}{4}$ and $\frac{5}{6}$ were expressed correctly using the area model by coloring three parts out of four and five parts out of six. These responses focused on whether the fractional part is correctly shaded in the whole based on the part-whole method (cf. Watanabe, 2002) rather than the important idea that in comparing two fractions, the models or manipulatives should have the same sized whole. Also, three PSTs said, "the model showed that $\frac{5}{6}$ is larger than $\frac{3}{4}$ because $\frac{5}{6}$ has more shaded unit squares." These responses imply two possibilities: First, the PSTs might have been misapplying whole number comparison to fraction comparison. That is, they might have focused on comparing only numerators by considering the numerator and denominator as separate values rather than as elements of a single value. Second, although the PSTs recognized fractions as a single value, they might have lacked knowledge of pedagogical representations, which led them to use a representation to illustrate the final result (e.g., $\frac{5}{6} > \frac{3}{4}$) rather than to explore the process of comparing two fractions by considering the different denominators (Lee, 2017; Rosli et al., 2013). Five PSTs simply responded that the model was correct without providing any justification.

Also, among the 21 PSTs who responded that the model was correct but could be improved, 17 emphasized the use of models with the same sized wholes. Of these, six specifically suggested using a pie model and eleven suggested using fraction bars. Finally, four PSTs responded that the drawings or answers were correct but the interpretation of the models was not correct without providing details.

Table 6. PSTs' Fractional Knowledge in Comparing the Size of Two Fractions in Task 1

Purpose of Representation	Correctness of the Models	Fraction Concepts	Number of Responses
Pedagogical representation	Correct models	Both equal partitioning & same sized whole	95
		Only equal partitioning	30
	Incorrect models	Only same sized whole	2
		Neither equal partitioning nor same sized whole	3
Solution representation	Correct models	Both equal partitioning & same sized whole	6
Total			136

Note. The total number of responses is higher than the number of participants because some PSTs provided multiple responses

Types of Fractional Models Suggested by PSTs to Guide Students toward Correct Use of Representations in Comparing Fractions

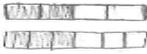
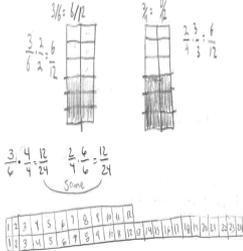
Task 1: Susan's model

In Susan's response to the problem, she represented the two fractions $2/4$ and $3/6$ with length/linear models and used the comparison method, in which the whole and parts are constructed separately instead of having parts embedded in the whole (Watanabe, 2002). Her models represented the equal-partitioning concept but not the principle of invariance of the whole. When asked to suggest models that would be useful for comparing $2/4$ and $3/6$, PSTs proposed various forms of models. Excluding PSTs who did not propose models because they believed that Susan's work was correct ($n=16$) or simply did not respond ($n=8$), a total of 136 responses from 127 PSTs were analyzed in terms of two criteria, concepts required for fraction comparison and type of fraction models.

As the invariance of the whole (equal-whole: EW) and equal-partitioning (EP) are two critical elements in comparing fractions, we noted whether these concepts were successfully represented in the PSTs' models and found that 74% of PSTs demonstrated both concepts and 26% of PSTs showed one or the other but not both in their models (see Table 6).

In terms of types of fractional models, Table 7 shows the frequencies of PSTs' correct models categorized as either area models, length models, or set models, along with a fourth category indicating models presented to illustrate results based on algorithms rather than the process of problem solving. The most common models were area and length/linear models. For area models, circular drawings with pie-like wedges were dominant. For linear models, placing rectangular strips either vertically side-by-side or top-down was the preferred method. In addition, only one PST suggested using a number line verbally without illustrating an actual model.

Table 7. Proposed Correct Models by PSTs in Task 1: Comparing $\frac{2}{4}$ and $\frac{3}{6}$

Types of models	Features of Model	Examples	Number of Responses
Area Model (32)	Pie wedges		30
	Rectangular arrays		2
Length/ Linear Model (63)	Vertical strips		25
	Stacked strips		7
	Horizontal strips		8
	Top-down strips		23
Algorithms (6)	Standard algorithms		6

The majority of PSTs ($n=130$) proposed *pedagogical* representations (Cai, 2005). However, six PSTs presented *solution* representations accompanied by their work of finding common denominators using the standard algorithm. That is, these PSTs thought about common multiples of 4 and 6 (e.g., 12 or 24) symbolically without using a drawing and then constructed their models based on their computations. In addition, although the given example used the comparison method (e.g., showing parts and the whole separately), almost all of the PSTs (98%) employed the *part-whole method* by embedding parts into the whole.

To analyze PSTs' incorrect models, we subdivided them according to type (see Table 8) and found that the source of the majority of incorrect representations was a failure to apply the principle of invariance of the whole (equal-whole). Also, when we calculated the error rate for the two most common types of models, we found that PSTs produced fewer incorrect representations (about 14%) when using the pie model than when using fractional bar model (29%). One explanation for this result may be that it is easier to represent the invariance of the whole with the pie model than with the bar model. Also, number line and set models using chips were rarely used and had a high error rate, which may suggest that the PSTs had not had enough experience to work with these models, and if they tried, they had difficulty using them.

Table 8. Proposed Incorrect Models by PSTs in Task 1: Comparing 2/4 and 3/6

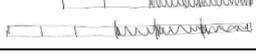
Model used	Features of Model	Reason of incorrectness	Examples	Number of Responses	Error Rate
Area Model (7)	Pie wedges	Fail to represent EW		3	5 / 35 (14%)
		Fail to represent EP		2	
	Rectangular arrays	Fail to represent EW		2	2 / 4 (50%)
	Vertical strips	Fail to represent EW		5	
Length/ Linear Model (26)	Stacked strips	Fail to represent EW		3	26 / 89 (29%)
	Horizontal strips	Fail to represent EW		3	
	Top-down strips	Fail to represent EW		14	
	Number line	Using a number line was suggested without the actual representation	N/A	1	
Set Model (2)	Chips	Fail to represent EW and EP		2	2 / 2 (100%)

Table 9. PSTs' Fractional Knowledge in Comparing the Size of Two Fractions in Task 2

Purpose of Representation	Correctness of the Models	Fraction Concepts	Number of Responses
Pedagogical representation	Correct models	Both equal partitioning & same sized whole	65
		Only equal partitioning	9
	Incorrect models	Only same sized whole	3
		Neither equal partitioning or same sized whole	4
Solution representation	Correct models	Both equal partitioning & same sized whole	12
Total			93

Task 2: Lisa's model

In comparing 3/4 and 5/6, Lisa represented the two fractions using part-whole area models in which areas of rectangular arrays are used and parts are embedded in the whole (Watanabe, 2002). As with Task 1, Lisa's model represents the equal-partitioning concept but not the invariance of the whole. When asked to suggest models that would be useful for Lisa to compare 3/4 and 5/6, of the 95 PSTs who considered the models as not useful, 93 proposed alternative models, which we analyzed in terms of fraction concepts and types of fraction model as we did for Task 1.

Regarding the fraction concepts shown in the PSTs' fractional models, 83% of PSTs demonstrated both concepts of equal partitioning and the same sized whole and 17% of PSTs showed one or the other concept but not both in their models (see Table 9). Among all PSTs, 81 proposed *pedagogical* representations, and 12 proposed *solution* representations by finding common denominators using the standard algorithm or converting fractions into percentages. More specifically, 11 PSTs figured out common multiples of 4 and 6 and drew their models to show their calculation result, and the one PST who suggested the number line model had first converted the given fractions to percentages by calculation and then constructed the number line model. Moreover, almost all of the PSTs (99%) used the *part-whole method* in which parts are embedded into the whole.

Table 10. Proposed Correct Models by PSTs in Task 2: Comparing $\frac{3}{4}$ and $\frac{5}{6}$

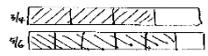
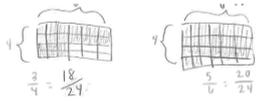
Model used	Features of Model	Examples	Number of Responses
Area Model (31)	Pie wedges		20
	Rectangular arrays		11
Length/ Linear Model (35)	Vertical strips		4
	Horizontal strips		3
	Top-down strips		27
	Number line/ Algorithm		1
Algorithms (11)	Signs and symbols only		11

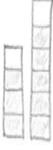
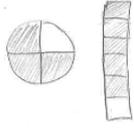
Table 10 classifies the different types of models suggested as area models, length models, set models, and models illustrating solutions arrived at algorithmically.

A sizable majority (83%) of responses that provided some types of representation as guidance for students correctly illustrated both invariance of the whole and equal partitioning. As displayed in Task 1, the most common models were area and length/linear models. For area models, circular drawings with pie-like wedges were dominant followed by rectangles. For linear models, placing rectangular strips top-down was the most popular chosen method.

The PSTs provided fewer types of models in Task 2 than in Task 1. For example, the stacked-strip model that showed parts and wholes separately was not used at all in Task 2, perhaps because of the influence of Lisa’s model, or perhaps reflecting individual PSTs’ levels of familiarity and comfort. In addition, the number of models that depended on algorithms was almost doubled in Task 2.

When analyzing PSTs’ incorrect models (see **Table 11**), we found that length/linear models were the most frequent, followed by area models. Also, regardless of the types of models, about 60% failed to represent the same sized whole. When calculating the error rate of the most common two types of models, we realized that it was slightly higher for length/linear models than for area models, which is similar to the analysis of Task 1. This result may indicate that it is especially important to be careful to observe the principle of an equal sized whole in drawing length/linear models. Also, set models using chips were identified only once. Interestingly, in the miscellaneous category, one PST used different types of models to represent the two fractions, a pie model for $\frac{3}{4}$ and a vertical strip for $\frac{5}{6}$, and then the given fractions were compared. This was the only case of changing forms in either task.

Table 11. Proposed Incorrect Models by PSTs in Task 2: Comparing 3/4 and 5/6

Model used	Features of Model	Reason of incorrectness	Examples	Frequency	Error Rate
Area Model (3)	Pie wedges	Fail to represent EP		3	3 / 23 (13%)
Length/ Linear Model (7)	Vertical strips	Fail to represent EW		2	7 / 42 (17%)
	Horizontal strips	Fail to represent EW		1	
	Top-down strips	Fail to represent EW		4	
Set Model (1)	Chips	Fail to represent EW and EP		1	1 / 1 (100%)
Explanation only (3)	Written explanation	Fail to represent EW and EP by presenting common denominators without specific representations	N/A	3	3 / 3 (100%)
Miscellaneous responses (2)		Fail to represent EW <ul style="list-style-type: none"> Using different fractions than the given ones (comparing 1/3 and 1/6) Two different models (pie & strip) were used to compare 		2	2 / 2 (100%)

SUMMARY

At the outset of the study, the goal was to investigate PSTs' perceptions of the use of representations in teaching mathematics, their knowledge of pedagogical representations, and their suggestions of types of fractional models to guide a student who is using manipulatives/ models incorrectly. In the first part of the questionnaire, PSTs were asked to suggest topics in which manipulatives or models can be used effectively and to identify the most important aspects of teaching mathematics with manipulatives or models. The results show that fraction-related topics were the PSTs' main choices for using manipulatives/models, and the PSTs valued understanding concepts and making connections between the mathematical concepts and models/manipulatives.

While PSTs' recognition of the importance of meaningfully teaching fractions as shown in this snapshot is promising, the results warrant a closer look. While fraction-related topics were the PSTs' main choices, it is notable that these topics were predominantly limited to such basic introductory concept of fractions as a "part out of whole." It was rare to see PSTs' suggestions for the use of manipulatives for more advanced topics such as comparison or operations involving fractions. Furthermore, although conceptual understanding was highlighted as the most important aspect of using manipulatives or models, the PSTs rarely provided detailed explanations of how to use manipulatives to make sense of the math concepts. These findings may suggest two things: (1) the PSTs might have believed that manipulatives or models would be more useful for concepts taught in the lower grades than for concepts taught in the upper grades; (2) the PSTs might have had limited content and/or pedagogical knowledge that they needed to teach advanced fraction knowledge using representations. Either possibility may corroborate the prior findings that teachers tend to have limited representational competence (Collins, 2015; Huinker, 2015; Izsák, 2008; Lee, Brown, & Orrill, 2011; Rosli et al., 2013).

PSTs' responses to Tasks 1 and 2 were examined further to probe their pedagogical content knowledge, in particular their ability to anticipate students' misconceptions and address them by proposing appropriate manipulatives or models. The linear model provided in Task 1 used the comparison method (i.e., showing the whole and part separately) whereas the area model in Task 2 used the part-whole method (i.e., embedding the fractional part in the whole). Both were incorrect representations for comparing two fractions because they depicted unequal wholes. Approximately two-thirds of the PSTs recognized the incorrect use of models in both tasks (66% and 63% respectively), but not all for valid reasons. When asked to propose models to guide students, the majority

of PSTs illustrated both the invariance of the whole and equal partitioning principles. However, 26% of the responses to Task 1 and 17% to Task 2 failed to show one or both principles in their representations.

It is noteworthy that the most common models were area (especially circular drawings with wedges) and length/linear models (using various arrangements of rectangular strips rather than a number line). To identify whether PSTs used the same representation in both task contexts, we compared their proposed models for Tasks 1 and 2. Excluding the 65 PSTs who did not provide specific models in one or both tasks because they thought the original work was correct, 86 PSTs' representations for the two tasks were compared. About 47% of the PSTs used the same model in both tasks. As shown in **Tables 7** and **10**, the PSTs predominantly used wedged circles and rectangular strips, whereas a small minority used other models such as the number line in a very limited way. This finding may confirm prior research that U.S. school mathematics textbooks tend to provide a narrow range of types of fractional models such as area models (Hodges et al., 2008; Zhang et al., 2015). Thus these PSTs might not have had opportunities either to explore fractions with various fractional models or to connect the models to related concepts. Also, when considering the significant emphasis on the number line model in recent recommendations for schools mathematics in the US (e.g., Common Core State Standards), the PSTs' responses to these tasks suggest that their experiences and thinking were not quite aligned with current standards for school mathematics.

IMPLICATIONS FOR MATHEMATICS TEACHER PREPARATION

Considering that the PSTs' work in this study reflects their understanding and perceptions just prior to taking a methods course, the results provide some insights into how to design a methods course that helps PSTs refine their conceptual understanding of mathematics and acquire adequate pedagogical content knowledge. Thus, based on this study, we provide four recommendations for mathematics educators to consider in helping prospective teachers effectively use manipulatives or models for comparing two fractions. First, mathematics educators need to explicitly cover the *comparison method* as well as the *part-whole method*. In this study, it was noted that even though the nature of the representations provided in the two tasks changed (i.e., Task 1's comparison method vs. Task 2's part-whole method), in suggesting guidance for students, the PSTs tended to use methods with which they were familiar but which were not necessarily most suitable for children such as the part-whole method. According to Watanabe (2002), children may have more difficulty with the part-whole method because the fractional part in the model is depicted as both an entity in itself as well as a quantity in a larger entity. However, many PSTs in this study demonstrated an unfamiliarity with the comparison method when they indicated that its use, rather than a failure to adhere to the same-size whole principle, was a reason for the incorrectness of some length and area models. Also, the majority of PSTs predominantly used a part-whole method in proposing appropriate representations across the two tasks.

Second, mathematics educators need to address *number line models* in more depth. Several studies (e.g., Grow-Maienza & Beal, 2005; Siegler et al., 2011; Tunc-Pekkan, 2015; van de Walle et al., 2013; Watanabe, 2007) and mathematics standards (NGA & CCSSO, 2010) have emphasized that number line models can be more useful to teach fractions because they are based on continuity in contrast to the discrete whole of the area model. However, the PSTs in our study made very limited use of number line models in their pedagogical suggestions for addressing students' misconceptions concerning the use of models/manipulatives for comparison of fractions.

Third, mathematics educators need to help PSTs implement effective *pedagogical representations* as distinguished from solution representations. Cai (2005) emphasized that teachers should be able not only to analyze students' solution representations but also to provide appropriate pedagogical representations that help explain conceptions and relationships in the process of solving problems. In this study, some PSTs provided only solution representations to illustrate the results of applying the proper algorithm to find common denominators in Tasks 1 and 2. In particular, the number of models that depended on algorithms was greater in Task 2 although the task could be solved by using an alternative strategy such as one part away from being a whole (cf. Clarke & Roche, 2009).

Finally, mathematics educators need to engage PSTs in *drawing models* by focusing on the principles of invariant wholes and equal partitioning in comparing fractions. Although pre-made fractional models can help students easily compare two fractions at the introductory level by lining up pieces representing the fractions, this method runs the risk that students do not pay attention to the premise that the two fractional models should comply with the invariant size of whole (van de Walle et al., 2013). In this study, many PSTs appeared unaware of the importance of the invariant whole in comparing two fractions, both in identifying students' misconceptions concerning the use of models/manipulatives and in providing effective pedagogical representations to address the misconceptions.

All in all, the PSTs' performance in this study suggests several areas for further study. As noted, the PSTs' suggestions for the use of representations were limited to teaching and learning basic part-whole concepts of fractions, and they suggested more algorithm-based guidance in Task 2 than in Task 1. As understanding fractions is a foundational topic that requires adequate attention for the benefit of students and teachers at all levels, this

result raises concern and warrants further examination. It may simply indicate the PSTs' weak knowledge of content; however, it could be an indication of the PSTs' perceptions of the effectiveness of manipulatives or models. As Vig et al. (2014) assert, it is important to identify the *breaking points* of a model to illuminate both its affordances and its constraints. Accordingly, the results of this study have led to a follow-up investigation on when and how PSTs perceive the usefulness of models in teaching and learning fractions, which is currently underway. Implications for mathematics teacher preparation include providing PSTs with opportunities to reflect on the affordances and constraints of models by examining the relationship between models and the underlying mathematics concepts which they are intended to reflect.

We also noted PSTs' predominant dependency on a few types of models (e.g., wedged circular models) in guiding students regardless of the format and structures used in the problems they analyzed. PSTs' tendency to use models procedurally and to overuse certain models is a matter of concern. While we recognize that there have been inconclusive results in research on the effectiveness of different fraction models, what we are concerned with is PSTs' lack of representational fluency in interpreting student work and providing appropriate guidance. In this regard, there should be an effort to enhance PSTs' representational fluency in mathematics teacher preparation programs. This effort may include a close examination of the fraction models typically used in textbooks for mathematics teacher preparation and their benefits and limitations in supporting PSTs' ability to reason flexibly about various fraction-related concepts, in particular fraction magnitude. As effective mathematics teaching requires the teacher to have sufficient content knowledge as well as an ability to adequately represent mathematical concepts and relationships to support student understanding, this study provides grounds for renewed attention to PSTs' content and pedagogical knowledge about teaching and learning fractions.

We hope that the findings of this study provide useful insights into PSTs' conceptions of representations in teaching mathematics and understanding of their effective use. By design, this study did not provide specific intervention to look at the changes in PSTs' perspectives over time. Future studies may develop more explicit interventions for PSTs and identify the effectiveness of such interventions. In particular, it would be interesting to track how PSTs' respond to similar tasks before, during, and after taking content courses, before and after taking the methods course, and so on.

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