Eurasia Journal of Mathematics, Science & Technology Education, 2015, **11**(2), 227-238

Designing Learning Strategy to Improve Undergraduate Students' Problem Solving in Derivatives and Integrals: A Conceptual Framework

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Received 19 September 2013; accepted 11 April 2014; published 25 March 2015

Derivatives and integrals are two important concepts of calculus which are precondition topics for most of mathematics courses and other courses in different fields of studies. A majority of students at the undergraduate level have to master derivatives and integrals if they want to be successful in their studies However, students encounter difficulties in the learning of derivatives and integrals. Most of these difficulties arise from the students' weakness in problem solving. This paper presents a learning strategy which has been designed to overcome these difficulties based on mathematical thinking and generalization strategies with prompts and questions.

Keywords: learning difficulties, problem solving, learning strategy, mathematical thinking, generalization, derivatives and integrals

INTRODUCTION

Calculus is an important subject since it exists in most of university courses such as; economy, engineering, statistics, science, and all mathematical courses like; analysis, numerical analysis, statistic, differential equation and operation research (Tall, 1992, 1993, 1997, 2010a; Tall and Yudariah, 1995; Karamzadeh, 2000; Tarmizi, 2010). According to Tall (2012), imagining university courses without calculus is unfeasible. Therefore, derivatives and integrals are two important topics in university mathematics which are prerequisites in order to learn other concepts in the different fields of studies (Tall, 1993, 1997, 2004a, 2011).

Many university courses depend on the knowledge on derivatives and integrals and also their applications (Tall, 2004a, 2010b, 2011; Metaxas, 2007; Pepper et al., 2012). In fact, traces of derivatives and integrals are visible in the advanced mathematics and even other subjects. Therefore, learning derivatives and the integrals can be helpful and useful for students in order for them to learn other mathematical courses at university level (Tall, 2010a, 2011; Tarmizi, 2010). However, there are some obstacles in the learning of calculus and its concepts especially derivatives and integrals (Tarmizi, 2010; Tall, 2010a, 2012; Pepper et al., 2012).

Derivatives and integrals are two difficult concepts of mathematics for many undergraduate students. The difficulties in learning derivatives and integrals among undergraduate students are due to their weakness in solving problems involving these concepts (Tall, 1993, 1997, 2011; Willcox and Bounova, 2004; Yazdanfar, 2006; Metaxas, 2007; Roknabadi, 2007; Tarmizi, 2010; Rubio and Chacon, 2011; Pepper et al, 2012; Azarang, 2012).

Many researchers (Tall, 1992, 1997, 2012; Stacey, 2006; Metaxas, 2007) have noted that students possess

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doi: 10.12973/eurasia.2015.1318a

ISSN: 1305-8223

State of the literature

- Students' difficulties in the learning of derivatives and integrals have been appearing through problem solving.
- Lack of making connection between graphical and symbolical aspects, more focus of symbolical aspect, weakness of recalling previous knowledge and the lack of suitable framework are remarkable reasons of students' difficulties in problem solving.
- Different learning strategies can be designed to help students in learning of derivatives and integrals based on mathematical thinking and generalization strategies

Contribution of this paper to the literature

- The designed strategy (MGSDI) can help students to rectify their difficulties by modifying generalization strategies and combining them with three worlds of mathematics in the learning of these topics.
- In this study, the postures of using prompts and questions have been shown based on MGSDI. Appropriate prompts and questions have been introduced for derivatives and integrals in embodied and symbolic worlds of mathematical thinking.
- Also, the activities of prompts and questions tried to highlight specialization and generalization and their relationships through mathematical thinking worlds.

difficulties in the learning of derivatives and integrals concepts, because the teachers and students focus on symbolic aspect rather than graphical. Moreover, the inability to make connection and relation between graphical aspect and symbolical aspect is another reason for the weakness (Tall, 1997, 2012; Metaxes, 2007; Shahshahani, 2012). The need for a heuristic and appropriate framework or plan to solve problems and weakness in using previous knowledge and information in new areas are other reasons for the difficulties. (Polya, 1988; Tall and Yudariah, 1995; Tall, 2001, 2004a, 2007; Kirkley, 2003; Villers and Garner, 2008; Mason, 2010; Tarmizi, 2010).

Some methods are being introduced to support students to overcome their difficulties in the learning of derivatives and integrals. There is quite an extensive study on promoting mathematical thinking to help students' understanding of calculus, especially derivatives and integrals (Dubinsky, 1991; Schoenfeld, 1992; Tall, 1986, 1995, 2002a, 2002b; Watson and Mason, 1998; Yudariah and Tall, 1999; Gray and Tall, 2001; Mason, 2002; Roselainy, 2008; Mason et *al*, 2010; Kashefi et *al*, 2013). Although some new methods such as using mathematical thinking (Tall, 2004; Mason et *al*, 2010) and generic skills (Kashefi et *al*, 2013) have been invented to support students' learning in derivatives and integrals, the difficulties still exist at undergraduate level according to Orton (1983), Yudariah (1995), Yazdanfar (2006), Aghaee (2007), Parhizgar (2008), Roselainy (2008), Tall (2008, 2012), Javadi (2008), Ghanbari (2012), Tarmizi (2012) and Azarang (2012).

Mathematical thinking is an active process involving highly complex activities, such as specializing, conjecturing and generalizing which improves students' understanding (Tall, 2002a, 2002b; Yudariah and Roselainy, 2004; Stacey, 2006; Mason, Burton, and Stacey, 2010; Kashefi et al, 2013). According to Tall (2004b, 2008), mathematical thinking process occurs in three worlds of mathematics namely embodied world, symbolic world and formal world which is called the theory of three mathematical thinking worlds. Based on the theory of three worlds of mathematics, there are two approaches of derivatives and integrals; graphical and symbolic (Lithold, 1968; Silverman, 1998; Thomas, 2009; Tall, 2011). Graphical view appears as an embodied notion such as; curve, diagram and graph (Tall, 2002a, 2004b, 2010a, 2012; Stewart, 2008; Tarmizi, 2010). The second approach is symbolic which deals with algebraic forms of functions such as limit, derivatives, integrals and multi integration (Yudariah, 1997; Watson, 2000; Tall, 2004a; Stewart, 2008).

Mathematical thinking is related to improving generalization in the learning of mathematics (Watson and Mason, 2006; Mason et al, 2010; Tall, 2012). Tall (2002a) asserts that generalization strategies in mathematical thinking worlds are expansive, reconstructive and disjunctive generalization. In fact, generalization is an important element of the mathematical thinking process and problem solving methods. It can be used to support students to overcome their difficulties in the learning of calculus especially derivatives and integrals (Polya, 1988; Cruz and Martinon1998; Larsen, 1999; Karamzadeh, 2000; Tall, 2002b, 2004b; Sriraman, 2004; Mason et al, 2010; Kabael, 2011).

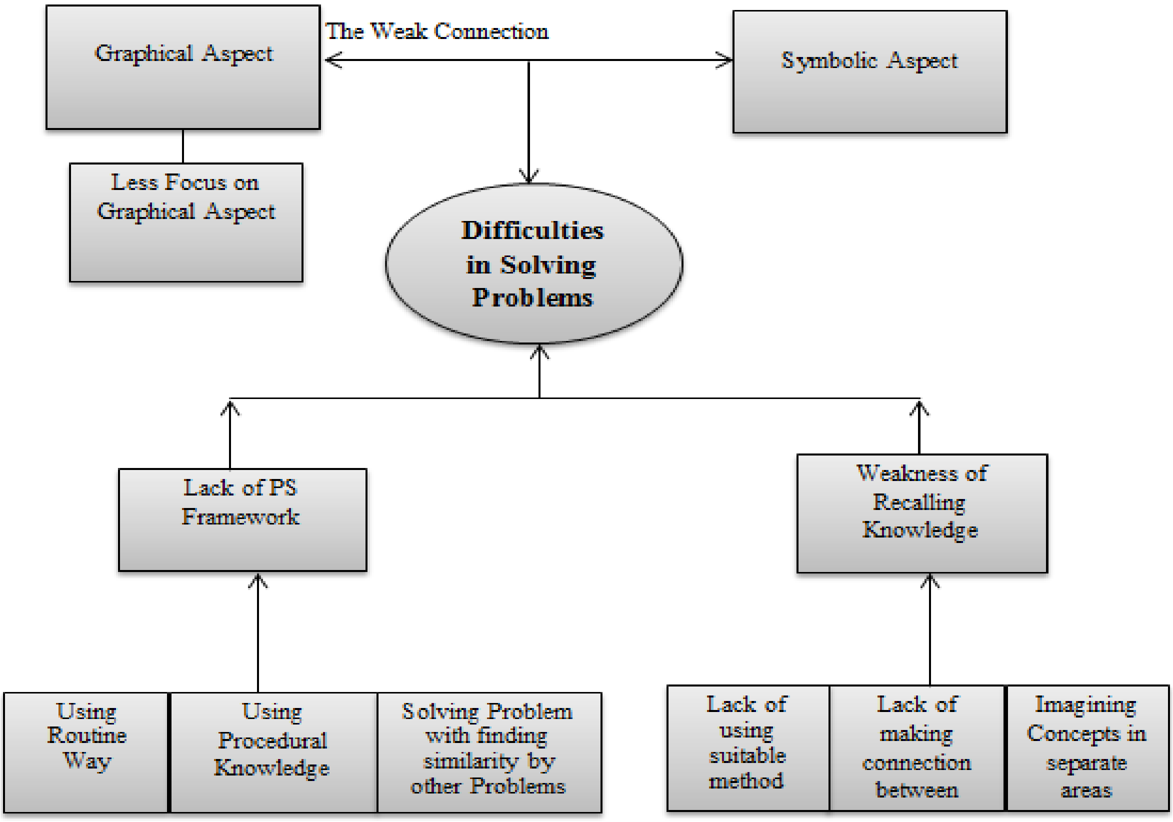


Figure 1. Difficulties in solving problems of derivatives and integrals

Considering the mathematical thinking activities, Watson and Mason (1998) have developed various general questions and prompts which can be used to motivate the development of Mathematical sense among students. The prompts and questions offered examples of some generated questions which are used as a way to know how suitable questions and structure should be chosen (Roselainy, 2008). The main delivery methods for applying prompts and questions through mathematical thinking are expounding and explaining. In addition, effective learning can be achieved through other interaction styles and six important interaction modes (Mason, 1999, 2002; Roselainy, 2008) and thus they should be used for effective teaching based on prompts and questions.

This study is designed based on specialized and modified forms of generalization strategies and mathematical thinking to support undergraduates in improving their problem solving in the learning of derivative and integral. Prompts and question are suitable methods to improve the learning strategy.

Students' Difficulties in the Learning of Derivative and Integral

The analysis of information on students' difficulties in learning Derivatives and Integrals has shown that these difficulties are due to their weakness in problem solving (Metaxas, 2007; Rubio and Chacon, 2011; Pepper et *al*, 2012; Tall, 1993, 1997, 2011; Willcox and Bounova, 2004; Javadi, 2008; Tarmizi, 2010; Ghanbari, 2012). Many researchers (Tall, 1992, 1997, 2012; Stacey, 2006; Yazdanfar, 2006; Metaxas, 2007; Roknabadi, 2007) have highlighted that students' problem solving skill in the learning of Derivatives and Integrals is insufficient because the teachers and students deal with the algebraic aspect rather than graphical (see Figure 1). Moreover, the lack of relationship between graphical and algebraic aspects is another reason for students' difficulties (Tall, 1997, 2012; Metaxes, 2007; Shahshahani, 2012).

The lack of methods to make strong connections between graphical aspect and symbolic aspect is seen among undergraduates yet. Therefore, there is a necessity to use the properties of graphical aspect in teaching and learning of derivatives and integrals among undergraduate students. In addition to these reasons for problem solving, the absence of problem solving plan is another important difficulty among students in the learning of derivatives and integrals (Polya, 1988; Yudariah and Tall, 1995; Tall, 2001, 2004a, 2007; Kirkley, 2003; Villers and Garner, 2008; Parhizgar, 2008; Mason, 2010; Tarmizi, 2010; Azarang, 2012; Ghanbari, 2012).

Table 1. Students' difficulties in learning derivatives and integrals, the reasons and proposed solutions

Students' Difficulties in Learning Derivatives and Integrals	Reasons	Proposed Methods
Problem Solving	*Focusing on Symbolic Aspect*	Using Expansive Generalization in Embodied World
	Lack of Connection between Graphical and Symbolical Aspects	Using reconstructive generalization
	Weakness of recalling previous knowledge	Generalization Strategies
	Lack of Problem Solving Framework	Mathematical thinking Process Specialization and Generalization

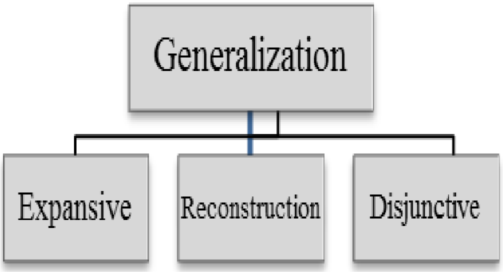


Figure 2. Generalization's components from Tall's viewpoint

Lack of problem solving framework or pattern and the inability to recall previous knowledge in new situations are other causes of the difficulties faced by students. Hence, students need to be aware of problem solving framework in order to follow the process of solving problems of derivatives and integrals. Most undergraduate students fail to get the answers because they are clueless on what should be done. Therefore, there is a strong requirement to introduce and use a problem solving plan in the learning and teaching of derivatives and integrals. The above weaknesses are highlighted in Figure 1. Recalling previous knowledge is difficult for students due to some factors such as such as they are imagining concepts in separate areas, they are unable to make connection between earlier and later information, the inability to use suitable method and strategy to recall information and so on. Figure 1 shows more information about the difficulties in problem solving.

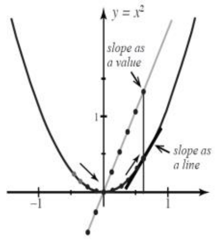
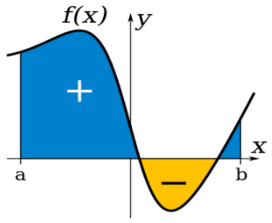
Appropriate Methods to Overcome Students' Difficulties

Based on the points in the previous heading, there is a strong necessity to find suitable strategy to improve students' problem solving abilities within the learning of derivatives and integrals. Mathematical thinking (Tall, 2004a; Mason et *al*, 2010) and generalization (Tall, 2002a) have been proposed to rectify these difficulties in learning of derivatives and integrals. Generalization strategies such as expansive and reconstructive allow connection to be made between graphical aspect and symbolic aspect. Generalization has the potential to increase focus on graphical aspect, because the expansive as a strategy of generalization can be used in graphical aspect to work more on that aspect (Tall, 2008; Villiers and Garner, 2008).

Mathematical thinking is a suitable method to help people to use generalization strategies (Tall, 2004a; Mason et *al*, 2010), because using generalization can conduct students to avoid using disjunctive strategy to use expansive and reconstructive strategies in graphical and symbolical worlds of mathematical thinking (Tall, 2002a). Generalization can be used to recall previous knowledge and making connections between concepts (Polya, 1982; Karamzadeh, 2000; Stacey, 2006), because generalization involves making connection between previous concepts and news.

Mathematical thinking is a suitable method to help people to use generalization (Tall, 2004a; Mason et *al*, 2010). Three worlds of mathematics (Tall, 2008) and mathematical thinking process (Mason et *al*, 2010) can be helpful and beneficial to overcome students' difficulties in problem solving. Mathematical thinking worlds cover both graphical and symbolic aspects of derivatives and integrals through embodied and symbolic worlds (Tall, 2012). Students can use the properties of these worlds to make connection between them to rectify the mentioned difficulties. Besides, mathematical thinking process (Watson 2002; Mason et *al*, 2010) such as specializing and generalization can be used as a problem solving framework. More details can be found in Table below.

Table 2. Posture of derivatives and integrals in mathematical thinking worlds

Concept	Embodied world	Symbolic world	Formal world
Derivatives		Slope from x to $x+h$ $=\dfrac{f(x+h)-f(x)}{h}$ $=\dfrac{(x^2+2xh+x^2)-x^2}{h}$ $=2x+h$ For small h, the slope stabilises to $\mathbf{2x}$.	Using ε and δ in analysis
Integrals		$A(a,b)=\int_a^b f(x)\,dx$	Using ε and δ in analysis

Designing Learning Strategy

Mathematical thinking can be a useful method to reduce students' difficulties in the learning of derivatives and integrals, because it can cover both graphical and symbolic aspects. Moreover, it can support generalization to help students in recalling previous knowledge in the problem solving process. Therefore, mathematical thinking worlds and generalization strategies need to be introduced more when designing a learning strategy.

Mathematical Thinking Worlds

Tall and Watson compare the embodied and symbol at university level with formal view or formal approach and introduce it as an axiom (Watson, 2000; Tall, 2004a, 2004b, 2012; Stewart, 2008). They emphasize that those three points of view (geometric, symbolic and axiomatic) cannot be considered as mathematical concepts. However, they are three different cognitive developments which happen in mathematical thinking in three separate worlds.

In the development of mathematical thinking theory, Tall has studied and carefully used a majority of mathematical topics (Tall, 2004b). Tall (2004b) establishes the theory of mathematical thinking worlds namely embodied world, symbolic world and formal world; he also invents the word of procept which is a combination of the word process and concept in symbolic world and this word is important for conceptual understanding. The word of world has been chosen to emphasize distinct ways of thinking about thinkable concepts in mathematics (Tall, 2005; Mejia-Romas, 2006).

Table 2 shows the postures of derivative and integral through three worlds of mathematical thinking. Derivative and integral and their properties are shown with graphs and histograms in the embodied world of mathematical thinking (Stewart, 2008; Tall, 2008, 2012). In the symbolic worlds, the postures of derivatives and integrals are the symbols of them. These symbols can be their general forms and properties which are shown with numbers and algebraic letters. Although, mathematical thinking involves the formal world, the main emphasis in the learning calculus is on embodied world and symbolic world and its concepts such as derivatives and integrals (Tall, 2004a, 2008, 2012).

Using generalization strategies is another useful method to improve students' problem solving. Generalization can make a connection between graphical and symbolic aspects (Tall, 2008; Kashefi et al, 2013). It can be a useful method for recalling previous knowledge in problem solving activities (Polya, 1988; Karamzadeh, 2000).

Generalization and Its Strategies

Tall (2002a, 2004b, 2012) asserts that mathematical thinking can support generalization in mathematics. Generalization strategies are used in mathematics to show processes at broader contexts. It can help problem solvers to know about the products of those processes (Tall, 2002a). According to Tall the components of generalization can appear in three types; expansive, reconstruction and disjunctive.

First of all, expansive generalization involves extending the existing information of learners without any change of their previous ideas. On the other hand, in expansive generalization; the new information should be similar to the current information in the same area. For example, in learning vector space firstly students are taught on R and R^2 then educators will add another component to plane (R^2) and introduce apace (R^3).

Furthermore, when a person extends a concept by changing his or her previous ideas or knowledge, (s)he reconstructs his or her cognitive area of knowledge. This change is named reconstruction generalization. To illustrate, in transitions from figure to symbol (or vise versa), this kind of generalization is happened (Harel and Tall, 1991; Tall, 2008). According to Tall (2002a, 2008), this generalization can be used as an ideal generalization to make a connection between embodied and symbolic aspects to overcome students' difficulties in recalling previous knowledge in new situations (Polya, 1982; Karamzadeh, 2000).

Next, disjunctive generating problems can be solved and analyzed in advanced knowledge, but this generalization has less effects on students learning in comparison to expansive and reconstructive (Tall, 2002a). In order to use this kind of generalization, students solve new problems by adding numbers of disconnected pieces of information illogically. Mason and colleagues (2010) have tried to correct this generalization in the third stage (review) of their framework. Using disjunctive generalization leads students to solve problems routinely instead of applying suitable problem solving framework (Tarmizi, 2010).

To sum up, generalization from Tall's viewpoint has three components such as; expansive, reconstruction and disjunctive. The relationship between the components of generalization according to Tall's perspective on generalization is shown in Figure 2 (Harel and Tall, 1991; Tall, 2002).

Consideration on generalization (Tall, 2002a) is important in the teaching of mathematical concepts such as derivative and integral. Besides, generalization in the problem solving process such as specialization, conjecturing and generalization (Mason, Burton and Stacey, 2010) are useful in overcoming students' difficulties in the learning of derivative and integral (Roselainy, 2008; Kashefi et *al*, 2013). Hence, specialization and conjecturing are foundations to achieve generalization.

Specialization in the teaching of mathematics can be used in various cases or examples (Roselainy, 2008; Mason et al, 2010; Mason, 2012; Kashefi et *al*, 2013). In other words, specialization means referring to more examples in order to learn the concepts or solve the problems. These examples are specific and particular instances of more general situations in a concept (Mason et *al*, 2010). According to Mason, Burton and Stacey (2010) if specialization happens successfully via a useful conjecturing, it can be helpful in making

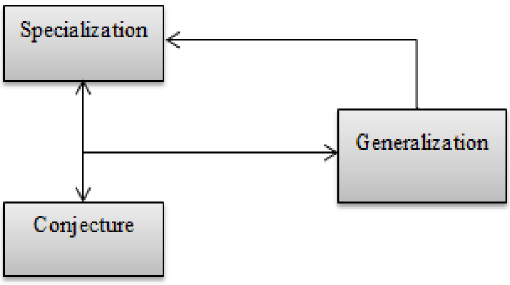


Figure 3. Generalization's process

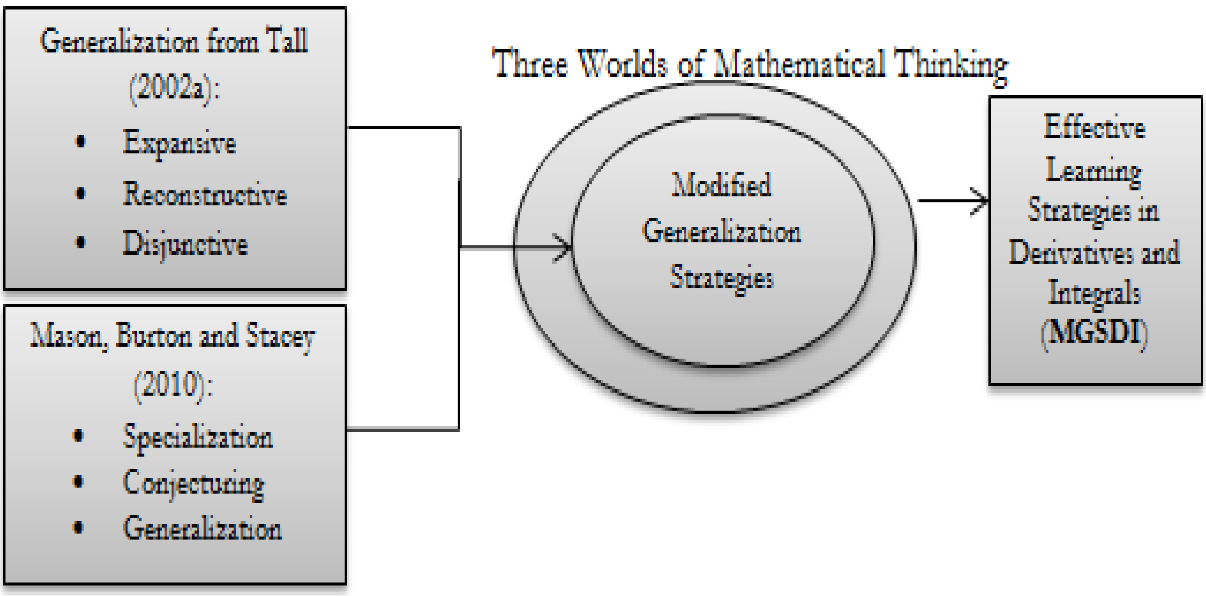


Figure 4. Design strategy for improving problem solving

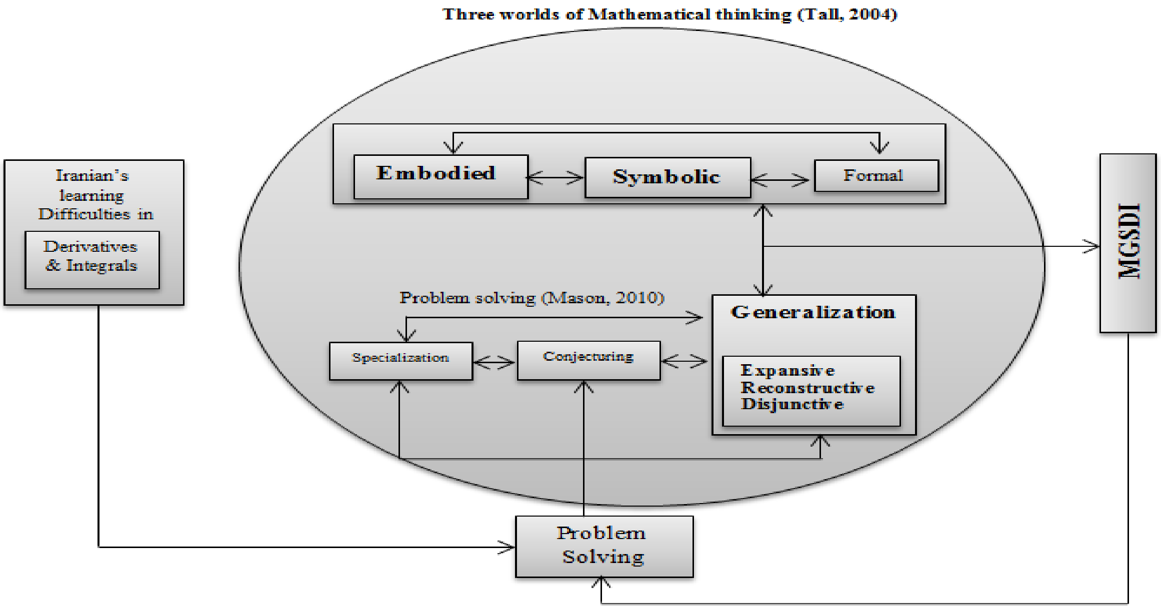


Figure 5. The detail strategy of MGSDI

generalizations. Conjecturing is forming an opinion or supposition about similarities and differences between given examples in specialization to find the suitable way for solving problems. When the solutions of problems are written in general form, it can be the starting point of using generalization.

Generalization is the main process after specialization. When students face new concepts or new problems after specializing and conjecturing, formulation is formed in their minds. It is the beginning point in utilizing generalization (Mason and colleagues, 2010). Generalization activity can be extended to previous knowledge or to new related concepts (see Figure 3).

Embedding Mathematical Thinking Worlds and Generalization Strategies

Based on the results of Table 1, mathematical thinking and generalization strategies can be useful methods to overcome students' difficulties in the learning of derivatives and integrals. It is important to possess the quality of combining and blending these methods together to design a suitable learning strategy for derivatives and integrals.

The generalization strategies (Tall, 2002a) have been integrated with generalization process (Mason et *al*, 2010) to establish modified generalization strategies. The modified generalization is a combination of generalizations. The proposed framework of learning strategy in derivatives and integrals contains the theory of mathematical thinking (Tall, 2004b), the perspective of Tall about generalization (Tall, 2002a), and the framework of Mason, Burton and Stacey (2010) in using generalization. Figure 4 illustrates the design strategy.

This study has designed learning strategies called Modified Generalization Strategies in derivatives and integrals (MGSDI) specifically to support undergraduates in improving their problem solving skills in calculus topics. Modified generalization strategies which consist of a series of complex learning activities are first formulated based on Tall's three components of generalization (expansive, reconstructive and disjunctive) and Mason's generalization process namely specialization, conjecturing and generalization.

Mason generalization process with two preliminary processes (specialization and conjectured) has been modified by embedding Tall's three generalization strategies. Generalization can be achieved in Mason's process by using at least one strategy of Tall's. In other words, the problem solving framework of Mason is being improved when the generalization process is modified. Although specialization and conjecturing belong to the design strategy, the main focus is on generalization process that has been modified. Thus, specialization and conjecturing are pre- processes of generalization. The modified generalization strategies was then mapped onto the Tall's three worlds of mathematical thinking (embodied, symbolic and formal) to form MGSDI.

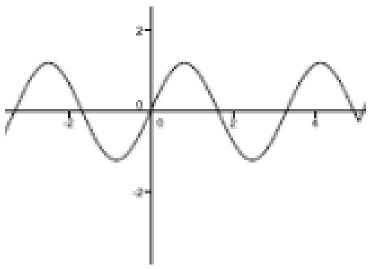
Figure 5 provides more details of the learning strategy. MGSDI is designed by considering problem solving, mathematical thinking and generalization. The quality of linkage between generalization strategies and mathematical thinking is illustrated in Figure 6. In addition, the relations between problem solving with mathematical thinking worlds are also shown.

MGSDI is proposed to rectify difficulties in the learning of these concepts. After designing MGSDI, it can be improved through dynamic activities. Based on

Table 3. Suitable questions for the strategy based on mathematical thinking

	Specializing	Conjecturing	Generalizing
Embodied	How can the specialization of derivative (Integral) in embodied world?	What are the guesses for derivative (Integral) property in embodied world?	What is the general form and definition of derivative (Integral) based on embodied world?
Symbolic	-Are there any special examples for derivative (Integral) in symbolic world? - Give more examples of derivative (Integral) in this world.	What are the similarities and differences of derivative (Integral) in the examples in the previous step in this world?	-What can you say about all of the examples? -Which one is the same for all? -What happens in general?

Table 4. Derivatives in embodied world with prompts and questions

Topic: Derivative **Activities: Specializing and Generalization in Embodied World**	**Prompts and Questions**
Example 1: Given  i) Sketch tangent line for x= $0, \pi$ ii) Sketch tangent line for x= $\frac{\pi}{2}, \frac{\pi}{4}$ and $\frac{\pi}{6}$ iii) Could you give other examples for negative x? iv) Give more general examples.	a- What do you see when you sketch tangent line from x= $\frac{\pi}{8}$ *to* π ? b- What can you say about the main property in question a? c- Give an example similar to example 3. d- Can you highlight the general rule for sketching the graph for derivative function in example 3? e- Compare examples 1, 2 and 3 and tell the similarities and differences. f- What is the general rule for derivative figure based on the original graph? g- Show it in general form graphically and algebraically.

MGSDI, many tasks and activities are designed by emphasizing both embodied and symbolic aspects of derivatives and integrals.

Designing and Developing MGSDI through Prompts And Questions

Prompts and questions are suitable methods to design strategies in mathematical thinking approach (Roselainy, 2008; Mason et al, 2010; Kashefi et *al*, 2013). Watson and Mason (1998) and Watson (2002) assert that prompts and questions can be used by teachers as guidance for developing mathematical thinking in the classroom within problem solving process.

Questions help students to focus on particular strategies and to see patterns and relationships (Mason et *al*, 2010). It will build the foundation of a strong perceptual network. In addition, questions can be used to prompt students when they become "stuck". Teachers are often tempted to turn these questions into prompts to encourage thinking and incorporate students' problem solving activities (Watson and Mason, 1998, 2006; Mason et *al*, 2010). Therefore, this study proposes that the contents of design strategies which depend on prompts and questions should cover generalization strategies, mathematical thinking process as problem solving framework and three worlds of mathematics.

Therefore, prompts and questions have been applied to improve the designed learning strategy (MGSDI). Table 3 presents the posture of appropriate prompts and questions based on mathematical thinking worlds.

The postures of using prompts and question should be shown in different mathematical worlds and different activities of mathematical thinking. It means that the prompts and questions activities are designed to develop

Table 5. Presenting derivatives with prompts and questions based on mathematical thinking

Topic: Derivative **Activities: Specializing and Generalization in Symbolic World**	**Prompts and Questions**
Example 2: Given $x = t^2$ i) Find the average velocity for $t_1 = 1$ and $t_2 = 3$. ii) Find the average velocity for t_1 and $t_3 = 2$. iii) Find the average velocity for t_1 and $t_4 = 1.5$. iv) Write more general example.	a- What is the similarity and difference in this example? b- Give exact rate for *t*. c- When the rate of t can be the best and what is the exact solution for velocity? d- Write the general form for this example. (what do you mean. e- What information do you need to write general forms from other topics such as function and limit? f- Give the general form of deprivation and describe it. g- Can you describe this example and the general form with the curve or graph of this example?

Table 6. Presenting integral with prompts and question based on mathematical thinking

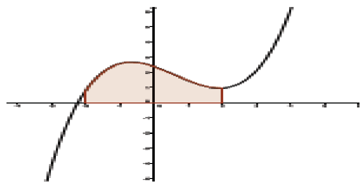
Topic: Integral **Activities: Specializing and Generalization in Embodied World**	**Prompts and Questions**
Example 3: Given  i) Try to find the area between the graph and *x*- axes from *x*=-2 to *x*=2. ii) Find the answer in several ways. iii) Describe how can find the exact solution of area in this example. iv) Please give another example. v) Please give a more general example.	a- Compare examples 1 and 2. b- What are the similarities and differences? c- What is the main property for both of them? d- What information in example 2 do you need to solve it? e- Sketch examples 1 and 2 in one coordinate axes and find the area between them and *x*- axes from *x*= -2 to *x*= 2. f- How can you relate the solution of these examples (1and 2) to algebraic aspect? g- Describe how you can connect this example to algebraic form.

Table 7. Prompts and question for infinite integral in symbolic world

Topic: Infinite Integral **Activities: Specializing and Generalization in Symbolic World**	**Prompts and Questions**
Example 4: Given $f'(x) = x$ i) Find $f(x)$ ii) If $g'(x) = x^2$ Then find $((x)$ iii) If $h'(x) = x^3 + 4x$, find *h(x)*. iv) Give a general example with suitable symbol.	a- What is similar? b- What is different? c- Describe the procedure to find the original function from derivative function. d- Give the general form of founded original function symbolically.

MGSDI in embodied and symbolic worlds of mathematical thinking. In addition, for each concept such as derivatives and integrals the activities of prompts and questions are designed through different worlds.

Table 4 presents prompts and questions which can be used for derivatives in embodied world. Students are guided to transit from embodied world to symbolic world. It should be noticed that Tall (2004a, 2008) believes that formal world of mathematical thinking is not applied for calculus. However, it can be used in mathematic analysis which is at a higher level than calculus.

Another example of derivatives using prompts and questions in the symbolic world is offered in Table 5.

Table 8. Presenting finite integrals with prompts and question based on mathematical thinking

Topic: Finite Integral **Activities: Specializing and Generalization in Symbolic World**	**Prompts and Questions**
Example 5: Given $\int_0^1 f'(x)\ dx$ when $f'(x) = x$. i) Find the answer. ii) Give another example which is like e.g. 2. iii) Give a general example iv) Find answers by changing boundary.	a- Compare e.g. 1 and 2. b- What is same, and what is different? c- Which property is similar in several examples of finite integral? d- What is the meaning of *dx*? e- Can you interpret example 2 by using a graph? f- Predict a general form for this example?

Most researchers (Tall, 1997; 2011; Roselainy, 2008, Kashefi et *al*, 2013) mention that derivative and integral are related with each other in calculus. According to Roselainy (2008), prompts and question can be used to teach integrals. Teaching integral in the embodied world via prompts and question is shown in Table 6.

The activities in Table 6 introduce and highlight the meaning and properties of integrals. The table shows another example of Integrals which is offered in the embodied world with specialization and generalization.

In addition, Table 7 illustrates the infinite integrals in the symbolic world based on specialization and generalization by using prompts and questions. This table also shows some activities to recall information from derivatives in integrals.

Table 8 shows the finite integrals in the symbolic world and some activities to make the connection between embodied and symbolic worlds.

CONCLUSION

Mathematical thinking by using both graphical and symbolic aspects of derivatives and integrals is an important method that can support students. Moreover, mathematical thinking process such as specialization, conjecturing and generalization can be a suitable framework in problem solving of derivatives and integrals. Using generalization strategies based on mathematical thinking can make the connection between these aspects. Thus, it seems that students' difficulties can be rectified by using mathematical thinking and generalization.

Different learning strategies can be designed to help students in the learning of derivatives and integrals based on generalization strategies. The MGSDI can help students to rectify their difficulties by modifying generalization strategies and combining them with three worlds of mathematics in the learning of these topics.

This study has presented the quality of using prompts and questions for MGSDI. Appropriate prompts and questions have been introduced for derivatives and integrals in embodied and symbolic worlds of mathematical thinking. Also, the activities of prompts and questions highlight specialization and generalization and their relationships through mathematical thinking worlds. Many researchers such as Yudariah (1995), Watson and Mason (2006), Roselainy (2008) have attempted to use mathematical thinking process for improving students' difficulties, but they did not use mathematical thinking worlds. They only used the framework of Mason and his colleagues (2010). However, one of the students' difficulties is the inability to connect embodied world and symbolic world. In addition, most researchers have used generalization in the learning of derivatives and integrals, but they used generalization strategies in symbolic world for problem solving (Karamzadeh, 2000; Tall, 2002a; Yudariah and Roselainy, 2004; Stacey, 2006; Roselainy 2008; Kashefi et *al*, 2013). Moreover, this study has proposed a modified generalization strategies for derivatives and integrals (MGSDI) to overcome the difficulties of problem solving through learning these concepts. It should be emphasized that the MGSDI supported the previous studies (Tall, 1992, 1997, 2011, 2012; Willcox and Bounova, 2004Stacey, 2006; Yazdanfar, 2006; Metaxas, 2007; Roknabadi, 2007; Metaxas, 2007; Javadi, 2008; Tarmizi, 2010; Rubio and Chacon, 2011; Pepper et *al*, 2012; Ghanbari, 2012).

Therefore, MGSDI is recommended to be used in real classrooms. This study also suggests that researchers have to design and develop suitable activities based on MGSDI to be implemented in the teaching process of derivative and integral. In addition, the impact of the MGSDI can be evaluated on students' problem solving.

ACKNOWLEDGMENTS

The authors acknowledge Universiti Teknologi Malaysia and the Ministry of Higher Education (MOHE) Malaysia for the financial support given in making this study possible through the research grant Vote No. Q.J130000.2631.10J54.

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