

Art and Geometry of Plants: Experience in Mathematical Modelling through Projects

Fabio Nelson Zapata-Grajales ^{1*}, Natalia Andrea Cano-Velásquez ², Jhony Alexander Villa-Ochoa ³

¹ Institución Educativa El Pedregal, COLOMBIA

² Institución Educativa Gilberto Alzate Avendaño, COLOMBIA

³ Universidad de Antioquia, COLOMBIA

Received 27 April 2017 • Revised 14 July 2017 • Accepted 7 September 2017

ABSTRACT

This article presents some results of a qualitative-case study carried out with students in last year of secondary education (15 - 17 years) into an official educational institution. The research looked at learning outcomes through the development of mathematical modelling projects related to studying the distribution of the leaves and the growth of some plants. According to the results, when modelling projects grounded in studying the form and geometric magnitudes are carried out, students are able to find out the meaning of certain mathematical concepts and participate in activities to reach other processes such as exploration, representation, formal construction, and validation of mathematical results. Some implications for teaching and learning in classroom derive from this study, particularly, the need for students to face the project development articulating research and mathematical modelling.

Keywords: geometric modelling, modelling projects, modelling of form, mathematical modelling

INTRODUCTION

One concern for teachers and researchers in mathematics education consists in achieving that students find meaning to mathematics and relate it to the contexts and extracurricular practices (Masingila, 2002; Masingila, Davidenko, & Prus-Wisniowska, 1996; Villa-Ochoa & López, 2011). Several ideas in the international literature argue the fact that school mathematics should transcend those approaches in which the knowledge transfer and playback procedures devoid of meaning have priority. Also, it is argued the need for practices in which students come into contact with various experiences of the “real world” and they can produce mathematical knowledge through them (Blum, Galbraith, Henn, & Niss, 2007; Borba & Villarreal, 2005).

To put students in research experiences through mathematical modelling is one of the options to attend the mentioned problem (Biembengut & Hein, 2004; Blum et al., 2007). When students are faced with this kind of experience, they can articulate the mathematical knowledge to other sciences and to the same application of mathematics to the “reality” (Barbosa, 2009; Biembengut & Hein, 2004; Blomhøj, 2004; Blum & Borromeo-Ferri, 2009; Trigueros, 2009; Villa-Ochoa, 2007).

There is no uniform understanding in international literature about the meaning of mathematical modelling and about the ways to integrate it into everyday school practices. Kaiser & Sriramam (2006) and Doerr & Pratt (2008) present different research perspectives about mathematical modelling in school. Each of these perspectives shows different purposes and emphasis on training which could be referred when mathematical modelling is integrated into everyday school life. Mathematical Modelling is organized in different ways into classrooms. These ways stand out: (a) as resolution of problems stated verbally (Bonotto, 2007; Verschaffel, Van Dooren, Greer, & Mukhopadhyay, 2010; Villa-Ochoa, 2015), (b) as a production of graphical representations of situations not necessarily linked to extra-mathematical phenomena (Arcavi, 2008), (c) as resolution of authentic task (Blum & Borromeo-Ferri, 2009; Kaiser & Schwarz, 2010; Muñoz, Londoño, Jaramillo, & Villa-Ochoa, 2014) and (d) as development through projects (Aravena, Caamaño, & Giménez, 2008; Borba & Villareal, 2005; Villa-Ochoa & Berrío, 2015).

Contribution of this paper to the literature

- This paper offers one way of tackling school geometry teaching by means of math modelling by using projects where students solve situations or phenomena of their daily lives. This is shown with an example, ways to study geometric objects, in one hand by studying the shapes and on the other hand by studying the magnitude. In the examples, the technology role is shown for each case. Besides, it helps to understand math modelling in geometry, what proposes two ways to act, as a part of mathematics where both intra and extra math situations can be given.
- The questions about the nature of geometric models, the contexts and situations which are modeled in geometry and their relation to other math areas and the other sciences, takes special sense for research in math education.

In the research from which this article is taken, the students were involved in the development of projects to study form and geometric magnitudes in plants. The question that routes the study was how the process of mathematical modelling of forms and magnitudes associated with the shape of the plants is given? In particular, this article highlights the design and validation of a strategy for project development of mathematical modelling as a tool for students to study situations or phenomena in their contexts.

MATHEMATICAL MODELLING IN SCHOOL GEOMETRY

Mathematical modelling is recognized as a tool to introduce and to develop ideas in different mathematical contents. For example, it is common to find researches about variation phenomena producing mathematical models associated to analytical expressions (functions, derivatives, integrals, differential equations, etc.); in geometry, a significant amount of works has focused on the modelling of certain systems from specialized software. Regarding phenomena modelling associated with society and culture, a smaller proportion of researches can be found in comparison with those found in other disciplines like algebra, calculus and even statistics (Girnat & Eichler, 2011).

Modelling in Geometry is an open research because although any form can be considered as an abstraction and a model to represent a noticeable form in the world of the experience, it is also true that the use and nature of representation are both different from the algebraic and analytic models; in regard, Girnat & Eichler (2011) have pointed out an academic debate about the model concept in elementary geometry as necessary. In their study, these researchers show that teacher ideas on modelling and school mathematics do not consider the existence of modelling processes in geometry. In the words of these two authors:

(...) In case of elementary geometry, model building is in conflict with aspects of traditional approaches to geometry and with educational goals of proving and problem-solving tasks. These oppositional requests have to be clarified in the academic debate and to be balanced for a realisable combination in practice. (p. 83)

Aspects, as presented above, allow argue the need for studies focused on modelling in elementary geometry. Questions in relation to the geometric model nature, contexts and situations modeled in this area and their relationship to other mathematics areas and sciences and art have special meaning to research in mathematics education.

In Colombia, as in other countries, it has been raising the need to articulate five transversal processes in mathematics education for levels of elementary/secondary education; mathematical modelling is one of these processes. It makes sense in mathematics study and, among them, issues and ideas related to spatial thinking. Through modelling, space's mathematical models can be built and a predominantly geometric reality can be interpreted. In this regard, the Ministry of National Education from Colombia (MEN, 1998) expresses:

Geometry, by its nature as a tool to interpret, understand and appreciate a world predominantly geometric, becomes an important modelling source and field for excellence to develop spatial thinking and high-level processes and, in particular, different ways of arguing. (p. 33)

An interpretation of MEN considerations allows to have a conception about geometry, and perhaps, about mathematics in general, as a set of models (theories) historically constituted as an abstraction of the human experience with the nature. In that sense, mathematical theories, or part of them, are considered as models resulting from a process to *mathematising of real it* (Israel, 1996). In this sense, the process of mathematical modelling plays an important role to study school geometry since, by its practical nature, it should allow students be closer to the study of their own reality and to the constitution of their own models and interpretations of that reality.

The interrelationships between the "real world" and mathematics have special meaning in works of mathematical modelling because these are where descriptions of what is mathematical modelling commonly are

founded. In literature about applied mathematics and mathematics education, diverse understandings of modelling and models can be found. There are understandings of modelling as a cycle/process (Blum et al., 2007; Fowler, 1998; Villa-Ochoa, 2007) to the construction a set of equations or mathematical expressions that make up the mathematical models (Biembengut & Hein, 2004; Fowler, 1998).

A common mathematical modelling understanding in the field of Mathematics Education is associated with a process that involves a series of phases or stages established cyclically and this process allows use mathematics as a tool to be close to the explanation, description, prescription or control of phenomena or real-world problems (Bassanezi, 2002; Biembengut y Bassanezi, 1997; Blum y Borromeo Ferri, 2009). This type of understandings could include algebraic and geometric denominations, and even statistical models. However, in literature about models and mathematical modelling in geometry, it is possible to observe the coexistence of diverse understandings orbiting in different directions. As an example, the term *mathematical model in geometry or geometric model* has a large development in design sciences (Rendón-Mesa & Esteban, 2013); in this field, this type of models seems to have meanings of engineering, architecture and related areas associated with physical and virtual constructions (models, plans and simulations) where the geometry is used. In Brisson's works (2000); Rodríguez (2007); Lopes, Martins, Campos & Pires (2007); Alsina (2008); Borda, Felix, Freitas Pires & Moraes (2008) and Alves, Pereira & Silva (2012), modelling in geometry can be considered as a process in which geometry knowledge is used to represent objects of reality. However, this relationship of "representation" between geometry and reality does not always emphasize the same characteristics of the modelling object.

In the works mentioned above, the relationships between the geometric domains and the "(extra-mathematical) real" seem to put more emphasis on the forms and the simulation of the apparent functioning of an object, than on the amounts and magnitudes involved in the phenomenon (e.g., simulate the movement of a fan using dynamic geometry software). Consistent with such emphasis, educational purposes in this type of modelling are focused on the development and implementation of skills and knowledge of geometry, and on the uses and knowledge of tools to achieve the objective. In contrast to other perspectives of modelling (e.g. socio-critical perspective) practical issues, contributions to the design model and its use in society and culture can be relegated to a second place or even not be present. Therefore, mathematical modelling has not a genesis in a problematic situation of social reality or other sciences; one behavior is imagined or observed in those contexts in order to rebuild it or simulate it using the geometric properties.

Based on the analysis of the works below, it is possible to conclude, at least, two uses and understandings of mathematical modelling in geometry:

- Uses to construct geometric mathematical models in order to represent objects. For instance, to build the form of objects such as windows, chairs, body parts, three-dimensional representations.
- Uses to do geometric analysis of models already built (application of models). It means, to use geometric mathematical models to study others of the same nature. For instance, to simulate the movement of a fan or another object in a dynamic geometry software.

As mentioned above, both types seem to have a different meaning to the classic speeches about modelling in Mathematics Education, since their purpose seems to have a perspective more intra-mathematic or mathematizing with purposes supported by their educational value rather than by understanding of quotidian reality or other areas of knowledge that they provide. On this last point, the modelling works would focus on "authentic contexts or situations" (Muñoz et al., 2014.), that is, situations resulting from quotidian activities, culture. From this perspective, Quiroz, Orrego & López (2015) developed a study in order to inquire the production of mathematical models produced by students from the study of flooding phenomenon in their locality because of the river overflow. In their study, they produced models associated with area and volume. To do this, these students, who often experienced the phenomenon, took measures, designed plans, diagrams, tables to organize information and produce functional expressions (cubic and quadratic) in order to represent the relationship between geometric magnitudes. According to Quiroz et al (2015), modelling was considered as a learning methodology to introduce geometric as well as algebraic concepts.

As noted above, geometry and mathematical modelling have different emphasis and purposes. Thus, development of modelling process in school geometry is not often articulated to the study or resolution of "real" problems, but also to the representation of one dimension to another, or the representation of objects in a virtual reality. About that, Blossier & Richard (2011, p. 3) have noted "(...) the geometry, in its interpretation and its treatments, behaves like a new reality, abstract and concrete at the same time in his logic, in its ways of representation." The considerations of Blossier & Richard (2011) indicate acting on intra/extra mathematical situations in geometry. Therefore, the reality can be assumed as elements perceived by the senses on experiences with the space environment or physical world.

STUDY

Methodology

To observe scopes and limitations in the classroom of mathematical modelling in geometry, the diversity of understandings suggests the need of establishing mechanisms focused on processes and other aspects seen through the processes that students do in their own performances. Thus, it is important to pay attention to the ways in which they develop projects, their interpretations and uses of models built by them. For this reason, the research was developed under the qualitative research guidance; and particularly, case study method was adopted, as Stake (1998, p. 11) argues, "case study is the study of the particularity and complexity of a singular case in order to understand its activity in relevant circumstances."

The study was conducted between June 2013 and June 2014, in an official educational institution with Preschool, Basic and Secondary Education levels. Data were obtained through a student practice in an ecological path or green area of the institution. For the educational community, the ecological path is an open classroom to make students get close to care for nature through processing, recovery and promotion of their plants.

Participants and Study Process

Six students were selected and named as Student 1, Student 2, Student 3, Student 4, Student 5 and Student 6 for anonymity and confidentiality reasons. They attended a call made to a wider group, 11th grade -in Colombia, the last year of Secondary School- (16-18 years). These students showed interest to work as a team, they were willing to extracurricular activities and they had a receptive attitude towards the recommendations offered.

The data collection occurred when the students conceived their first ideas about their own research project. Those ideas were discussed to consolidate the questions found in [Table 4](#). One of the researchers had a participant role as researcher and teacher; in that sense, he provided guidance on the project stages and problematized partial results that the students consolidated in order to provided them depth. The modelling project development had two stages: in the first stage, they built geometric models to explain, in mathematical terms, how the leaves or petals were distributed in their plants; in the second stage, they analyzed the growth of their plant, what mathematical function models its growth? Like a forest engineer, they did the same process to estimate the leaf area, to simplify magnitudes that are difficult to obtain like the area, and to determine the plant ability to take sunlight. Each of these stages had five phases. The main details of each of the phases at the stages 1 and 2 are presented in [Tables 1](#) and [2](#), respectively.

Table 1. Intervention phases. Stage 1: Model Obtaining of Leaf Distribution in Plants

Phases	Work environment	Purpose	Scope	Collection tool
Formulation of Question Research by the Students	Field trip through green area of the institution.	To explore student interest to determine study object.	Student interaction with physical environment (real context)	Learning guide. Unstructured interview
Data collection	Field trip through green area of the institution.	To analyze geometric aspects of the study object (plant) to formulate hypotheses.	Relationship between mathematical modelling and scientist method.	Learning guide. Pictures
Comparison	Working at computer's classroom with Photoshop and PaintTool Sai software.	To compare a geometric model established in geometric world with the shape of the study object (plant).	Recognizing mathematics as a useful tool.	Learning guide, student learning journal, student research report
Obtaining and interpreting the form model	Working at computer's classroom with Geogebra software.	To find a mathematic model adapted to distribution way of leaves or petals.	A model as part of reality and trying to explain it.	Learning guide, student learning journal, student research report, unstructured interviews
	Working at computer's classroom with Geogebra software. Handmade drawing in classroom.	To interpret the model and answer the research question To relate the algebraic representation and the geometric model.	Recognizing mathematics as structure.	
New contributions from the form model	Working at computer's classroom with Geogebra software.	To ask for geometric magnitudes like area, perimeter, volume and geometric features like angulation, remarkable lines...	Form modelling as method for learning of different geometric concepts.	Learning guide, student learning journal, student research report.

Table 2. Intervention phases. Stage 2: Obtaining processes of the plant growth model

Phase	Work environment	Purpose	Scope	Collection tool
Formulation of Question Research by the Students	Field trip through green area of the institution.	To explore students interest to determine study object.	Student interaction with physical environment (real context).	Learning guide, unstructured interviews.
Data collection	Field trip through green area of the institution. Leaf measurement inside classroom.	To analyze geometric aspects of the study object (plant).	Relationship between mathematical modelling and scientist method.	Learning guide, Pictures.
Production of table, Cartesian plane and mathematical regression function	Working at computer's classroom with Excel software.	To relate geometric magnitudes (length - leaf area by regression function).	Reading, interpretation and characterization of the model.	Learning guide, student learning journal, student research report.
Features of function graphic model	Working at computer's classroom with Excel software.	To obtain a cartesian graphic model and its algebraic representation.	A model as part of reality and try to explain it.	Learning guide, student learning journal, student research report, unstructured interviews.
Validation and conclusions	Exposition of study results.	To expose study results and propose new uses.	Use the model to the phenomenon interpretation and find new uses.	Learning guide, student learning journal, student research report, unstructured interviews.

Data Analyses and Study Validation

A study design based on the ideas of Villa-Ochoa (2011) was used for organization, systematization and analysis of data. A summary of this process appears in **Table 3**.

An emerging categorization was created to focus the attention on all those data that provided evidence of the processes and strategies used by students and the relationships that they established between geometry and real contexts throughout the development of their projects. For result validation, two triangulation types were carried out: the first one establishes a relationship between data collected through the different instruments, and the second one establishes a relationship between data, researchers and the theoretical framework.

Table 3. Registration form and systematization of information

Study design			
Students	Research question	Tools	Analysis
The units of analysis are the understandings of six students identified with the letter E. (E1, E2, E3...) all of them studying in 11 th grade. Establishing relationships between geometric models through modelling in a real context is explored. (Muñoz, Londoño, Jaramillo & Villa-Ochoa, 2014).	How do students perform modelling processes in the study of issues associated with spatial thinking and geometric systems through geometric analysis of some plants with regards to leaf distribution and growth?	The application of the learning activities that guided the research is registered by means of 13 field journals of the teacher, 312 learning journals of the students, photographic records, participant observation, unstructured interviews, 24 research reports of student and 14 learning guide. These tools allow for organizing information and detailing important aspects in modelling process of the form and growth.	Parallel analysis: when data were obtained, they were also analyzed to have a general sense and make decisions on how it should continue guiding student project development. In-depth analysis. All collected data were digitized. An encoding process was done to observe the ways of appearance of geometric model. Some categories emerged from data analysis.

MODELLING OF FORM AND RELATIONSHIPS BETWEEN MAGNITUDES. CONTRIBUTIONS TO THE SCHOOL GEOMETRY STUDY

Different categories emerged from the study which allowed obtain understandings about the dynamics involved in the (geometric) modelling process through projects. In particular, this article is based on analysis of those episodes that allowed characterize the processes and meanings of participants in the modelling projects.

In the first phase, the students took a walk around the green area (garden) of their institution in order to look for a plant of interest to make it a study object. Once they had a choice, they formulated a question that became the starting point of development of their project. As a result of this process, the students E1, E2, E3, E4, E5 and E6 asked several questions about biological, agricultural and mathematic aspects related to plants. [Table 4](#) shows the final questions asked by them:

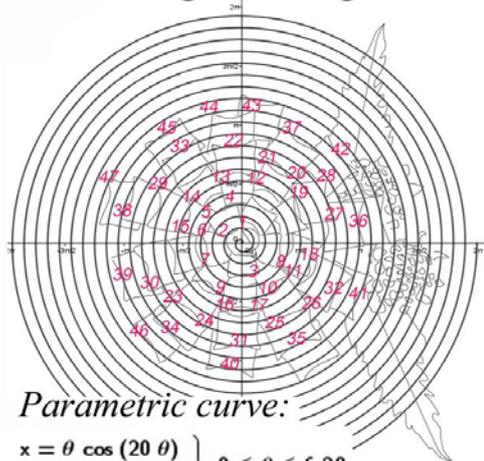
Table 4. Research questions asked by students

Research questions asked by students
How can it be deduced that the natural growth and development of plant and its leaves can be compared with the Fibonacci series? And how can the size of the Succulent plant be mathematically deduced? (Common name of Graptopetalum Paraguayense, in Colombia)
What mathematical procedures would be used to know how the plant grows and how the form of the flower Lantana Camara develops?
How can the growth and shape of Schefflera Enana plant be mathematically determined? (Common name of Schefflera Arboricola, in Colombia)
How can the plant area and the form in the figure produced by flower petals in Toscana plant be found? (Common name of Ruellia Simplex, in Colombia)
How do we apply geometry in the growth process and form of the Rosa de Francia? (Common name of Echeveria Elegans, in Colombia)
How can we identify the growth of the plant and the form of flower petals in San Joaquin plant? (Common name of Hibiscus Rosa Sinensis, in Colombia)

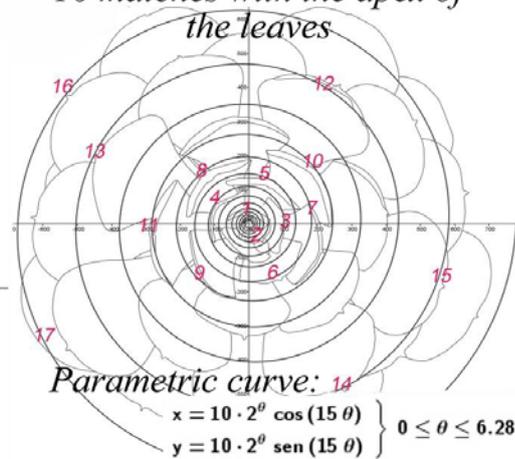
It is necessary to clarify that in the first question, the student E1 was already familiar with the concept of Fibonacci series because he belonged to the "Plantamáticos" School Research Group (Grupo de investigación Escolar Plantamáticos); a group where different mathematical concepts related to the nature and plants were studied; why he had a different vocabulary and he made his question regarding this issue. In question four, the plant area refers to the sum of leaf areas. Finally, in question six, identify growth refers how to determine the plant growth.

These questions ([Table 4](#)) are defined and focused on areas where mathematics, especially geometry, can support their solution. The questions mainly serve two interests, one oriented modelling forms perceived in plants (designs, patterns according to the leaves and their distribution); and another oriented modelling magnitudes and relations between them (e.g. length, leaf areas). With the exception of a student, it is noted that everyone else had a special interest in describing the plant growth; a special interest in the plant form and geometric magnitudes is also identified. Regarding this latter, the relevant magnitude was the plant growth. Focus on the growth was the result of the observation and the student contact with their respective plants. Also, they expressed: "it is easy to take and analyze data" (Students 2, 3 and 5).

47 points match with florets or concave irregular hexagons



16 matches with the apex of the leaves



Good angular and external coincidence on the petals

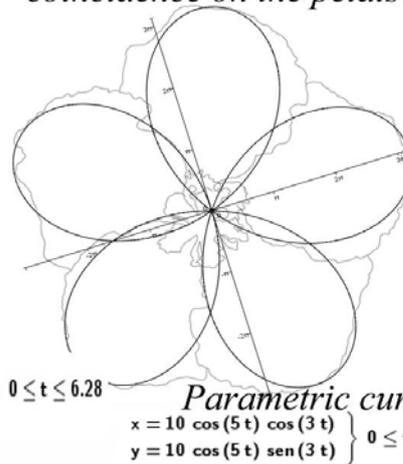
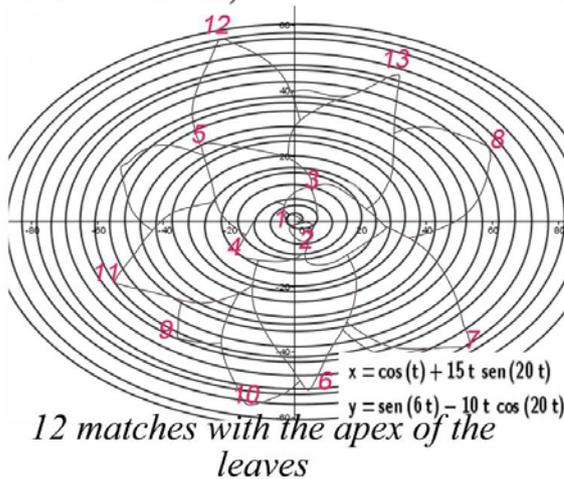


Figure 1. Distribution models of petals and leaves. Lantana Camara and Echeveria Elegans are at the top and Graptopetalum Paraguayense and Hibiscus Rosa Sinensis are below

At both stages (Table 1 and 2), the processes developed by the students made reference to the proposal on the process of mathematical modelling described by phases, stages or cycles (Berthelot & Salin, 1992; Laborde, 2002; Biembengut & Hein, 2004; Villa-Ochoa, 2007; Blum & Borromeo-Ferri, 2009; Trigueros, 2009). However, in each orientation type, modelling had different characteristics. More details of the processes developed by the students are provided below.

Geometric Modelling of Forms

The students articulated the need to determine geometric models of forms to the need to describe behaviors that they recognized in plants (Table 7). For instance, in Figure 1, the models produced by the students are presented; there, the spirals were constructed to describe how they observed the distribution of the leaves or petals in the plants studied. The students concentrated on the shape of the plants. They found in mathematics a way to model it. Modelling of forms was specifically a way to allow students to develop some artistic manifestations through replication.

In the Phase 2 of the first stage, the students explored through observation to choose a plant of interest; this allowed starting the process modelling in various topics. At this phase, they were empowered of their own research process; therefore, everyone came up with mechanisms and instruments to collect and analyze their own data. The process in this phase was contrary to other modelling experiences in which data are previously provided and, in many cases, already idealized; this fact interferes with the direct approach of the students to the problem or phenomenon to solve.

In the *Phase 3*, called *Abstraction*, the students made a theoretical model and then, they made a comparison with data obtained in experiments and observations on plants chosen for their project. This comparison was based not only on the geometric representation, but also in algebraic representation; to do that, the use of the Geogebra software was suggested to the students. Thus, they created an idea about how leaves or petals of plants studied were mathematically distributed. In this phase, they were able to identify what geometric models could be built (**Figure 1**). As it will be argued later, also the Excel software allowed the participants to analyze collected numerical data and determine aspects involved in the plant growth.

Phase 4, obtaining and interpreting of the model of form. This process not only included the model production (graphical representations of the forms), but also the reciprocal process of model interpretation to generate the descriptions of how the petals and leaves of the respective plant are distributed. In this phase, three components were involved, namely:

- *Geometric Component*: Although this component was present at the two stages in the project development (e.g. form and magnitudes modelling), it had special meaning in this Phase; because it is when geometry-reality connections were constituted (perceived form of plants). Graphic process is pointed out as a key aspect in the process of geometric modelling. The corresponding analysis of form and magnitudes relationships derives from that process.
- *Arithmetic Component*: The count was used in this component as a strategy for comparing the spiral shapes of plants (**Figure 1**). Also, converting units was present as a way to obtain the leaf areas. An important aspect of this component was the construction of the table and measurement comparison used to study the geometric model. The study of the irrational number Phi (golden number) also emerged in this component. These issues supported the identification of regularities and establishing relationships that subsequently led to other geometric and algebraic representations.
- *Algebraic Component*: Another way to represent the spiral shape or rosette (**Figure 1**). It is also understood as a ratio of two variables or a function (see **Figure 2**). Here was really important the use of both Geogebra and Excel software. This component appeared not as the only way to represent mathematical models, but as an alternative way to do it.
- *The artistic component*: This component illustrates a link feature between Mathematics and Arts. So that, some geometric models that were analyzed can be visualized and enjoyed in visual and aesthetic terms. Then, they have been developed through three software: Sai, Photoshop and GeoGebra (see **Figure 1**) where geometric - algebraic knowledge has been applied. Now, Geometry is a concept that is usually associated with calculations, formulas, numbers. Unfortunately, it is associated with beauty, creativity and harmony rarely. In brief, geometry (in this case) has been basic in the conception, design, development and even execution on physical form and external structure of plants, which it is referred bluntly as artistic drawings.

It was asked the students to respond partially the research question that themselves had formulated, based on their models. So, they used models to make descriptions about the distribution of petals and leaves on their plants.

The first student ideas about the models were associated with formulas or equations to replace numbers; aspects such as the recognition of magnitudes and variables and their relationships were hidden in those expressions; it means that the polar coordinate only worked as a formula. To illustrate this situation, one production of the student 1 is presented:

Table 5. Arguments provided by student E1 about software use

It was frustrating at the time we entered data because when we did it, instead of giving us a spiral, it gave us a dispersion of points, a total madness. We tried everywhere, with all kinds of numbers but nothing, there was a worse dispersion becoming a heavy duty ... when we were almost half hour analyzing the entire formula in all possible ways, we realized that the formula was wrong, because a piece of information was missing and because of this, we had that dispersion of numbers and we didn't have what we needed. (idea of student 1)

To overcome the above problem, an academic advice was offered to each working group to explain the operation of the Geogebra software. They had to write the meaning of every change in their graphic changing their algebraic representation and, then, to find the best model of distribution of the petals or leaves in each plant. This strategy worked because the students recognized the direct relationship between the algebraic representation and the geometric representation.

To build geometric patterns, the students used Photoshop and Geogebra software; the idea of using spirals and rosettes as geometric models to compare how the petals or leaves were distributed in plants emerged from their

direct observation, they expressed: "Ah... that plant looks like a spiral". Because of that, the comparative study started in Photoshop software, a tool where students, with already built models taken from internet, compared these curves with the distribution of petals or leaves. They used three types of procedures to find the model:

- Count matches between the spiral curve and the apex of the leaves or the edges of the ray florets. Method used by groups with *Echeveria elegans*, *Graptopetalum Paraguayense* and *Lantana Camara* plants (Figure 1).
- Observe matches between edges and angular distribution of petals with a rose curve. Method used by groups with *Hibiscus Rosa Sinencis* and *Roella Simplex* plants (Figure 1).
- Count matches between the spiral curve and the base of the leaves. Method used by the group with *Schefflera arboricola* plant (Figure 1).

These procedures allowed the students generate hypotheses about what type of spiral or rose was closer to the way of distribution of the leaves or petals of their plants. Each working group had 10 drawings of plants to compare each of the spirals or mathematic roses, making sure a reiterative match in other flowers or plants in order to have a good generalization. The models of the forms were determined by the conception of what the students wanted to represent, also by the technique and the mathematical tools that were used for its elaboration. These models, as artistic representation of the forms, not only reproduced a drawing but also allowed to communicate a relation between the algebraic representations and the coincidence of the drawing of the plant with its geometric form.

This process has a better explanation in research reports by the students 3, 4 and 5 in his research report:

Table 6. Arguments provided by students E3, E4 and E5 about the form

*At first, I thought it was circular, its leaves grow from the center to the outside in a rosette, the plant is small and it has a soft, fleshy texture, according to research realized to its geometric form, two hypotheses emerged: the logarithmic spiral and the Archimedean spiral. A study with Fibonacci series and its spiral was also made; we proposed these hypotheses because the number of matches between the spiral line and the apex or around is high. (idea of student 3 referring to *Graptopetalum Paraguayense* plant)*

*At the end of the study in Photoshop, we should answer the question about which spiral was closer to the distribution way of leaves in our plant; this decision was taken by comparing all spirals and observing which of them had more approximations from the apex of leave to the spiral curve. In our case, we had several hypotheses, Archimedean with 5 points or 5 apices, rosette with 9 points. (Idea of student 4 referring to *Lantana Camara* Plant)*

*Photoshop is a program that allows people to play and modify images, to create new and applicable things. And that's what I did; I used the Photoshop software to study my plant using the image before outlined in the Sai program. What we did was basically superimposing the image of a mathematical concept like: Archimedean spiral, Fibonacci spiral, golden number, rosette, polygons, etc., in order to observe which of these concepts was closer to the distribution way of petals in our plant and its growth. (Idea of student 5 referring to *Echeveria Elegans*)*

The hypotheses proposed by the students were studied using an algebraic representation produced by themselves; thus, they had a model of the form. With this experience, the students had an approach to polar coordinates, issue necessary to use but it is not part of the proposed issues in the curriculum of the institution. Due to this situation, they consulted in different media and they used the Geogebra and Photoshop to study their hypotheses (Table 7).

Table 7. Arguments provided by the student E5, about his model of form

*Idea of the student 5: We have not yet studied the plant growth, but according to studies of plant form, two geometric shapes can be used to study our *Echeveria Elegans* plant: the Logarithmic Spiral! Several leaf apices touch the curve. The Fibonacci Spiral was also selected to study the plant, but it coincided with the apex of the leaves in some sections. The first spiral was chosen to continue our research, it is corresponding to the polar coordinate:*

$$\begin{aligned}x &= 2 \cdot 2^\theta \cos(7\theta) \\y &= 2 \cdot 2^\theta \sin(7\theta) \\ \text{where } 0 \leq \theta &\leq 6.28\end{aligned}$$

The students were able to draw the curves and the roses up in an easy and quick way, at the same time, they could try different algebraic representations to impact on the polar graph and get the polar graph model and the algebraic model associated with it. They had to realize the meaning of parametric form, the value of θ or directed angle, the travel of the curve, the amplitude of the curve, the number of turns of the spiral curve or the number of petals, to the rose case. This identification allowed them to know what changed and what remained constant.

The geometric models (Figure 1) allowed describe, in mathematic way, how the leaves and petals of the plants are distributed. In this case, the students found several geometric models as Archimedean spiral, the hyperbolic spiral, Pythagorean spiral, logarithmic spiral and five-petals mathematical rose with their corresponding algebraic model. These geometric models obtained by the students E3, E4, E5 and E6 are presented in Figure 1.

The construction of the models in Figure 1 requires the students to develop skills to perform geometric and analytical procedures and to identify relationships between variables. Some skills were: *observe, conjecture, relate, refine assumptions, and develop a problem into smaller problems* (Gravina & Contiero, 2011). Discussions on the scope of the model and its connections to other areas of knowledge were also promoted (Table 8).

It important to mention that the students were having their first experience using geometric software and, of course, the productions (Figure 1) show their progress in recognizing patterns or invariants; in their exploration and experimentation; in their thoughts and demonstrations, in making conjectures and description of forms. These are reasons to continue doing experiences in the process of geometric modelling.

Beyond a definition, it is important to mention the use of such representations; in particular, the students E3 and E6 are highlighted. The first student proposed one application descriptive, but he made a new research question: Solar panels of plants are the leaves and they are adapted to the light conditions to receive it optimally according to the place where they are born, so, if solar panels are distributed in the same way as one of these plants, could they have better performance and charge a battery faster? The second student shows how useful was the geometric process for his research. In their own words:

Table 8. Arguments provided by the students E3 and E6, about the new use for model of form

I realized it serves to recognize that mathematics and especially geometry is used for describing the world and getting new architectural forms, for example: Succulent could be a model to make solar panels due to the distribution of its leaves; hypothetically, it could be a model to generate more energy. This research, that I and my partners developed, produce more questions which could change the world in the future. (Idea of Student 3)

(...) It was worth because making the draw of this spiral made me feel like a scientist developing equations, formulas and all that, it was another successful day in our research process. (Idea of Student 6)

In addition, the study serves for the initial purpose of thinking about the form of the object (Table 7) and thus, the students explained and described the distribution of leaves or petals on the plant chosen by themselves; why they did not abandon the initial idea of continuing with the purpose of answering the question they made; using necessary mathematic concepts, but not imposed on. In happened, in part, due to the research question was always highlighted as the core of the activities.

Table 9. Arguments provided by the student E1, about the form analysis in the object or plant

Our study keeps going forward, becoming into something much more structured, something much more concrete and much clearer, of course, to continue this study, we should ask ourselves or be questioned and it was what Professor Fabio made with the fourth guide, he made us to question about What form could have our plant? Or do you think Succulent could be studied through regular polygons? And What forms could be useful to study the distribution of petals or leaves in our plant? All these questions allowed us to analyze our process, if it was based on its real objective since the question was not only responding but having awareness and clarity about what we wanted to get.

The study of form in the objects from the perceptible world through the senses is one of the components to be proposed in the cycle of mathematical modelling in geometry. It allows mathematically describing geometric characteristics of form and magnitudes associated with the object. This process has various applications in science especially those related to design and architecture.

Stage 5 was focused on new contributions from model form. At that time, the possibility to continue working on the *model of form* was conceived either from its geometric magnitudes such as area, perimeter and volume or from its geometric features as angulation, notable lines (bisector, median, perpendicular bisector, etc.)

Some research questions made by students refer both to the form and to the growth and size of the plant, so modelling of form is not enough but also modelling growth. Growth can be studied from the analysis of geometric magnitudes such as length, width and area of the leaves in the plant and their relationship; process used in forestry and known as Allometric model (Galindo & Clavijo, 2007) (Table 3 and Figure 2).

The modeling of form was understood as a process involving the study of an object, space or phenomenon. This process involved two characteristics:

- *Produce an algebraic representation of the form:* It is the case of spiral curves, roses and related curves. The experience was focused on the study of form to obtain an algebraic model.
- *Study magnitudes associated with that form:* The form can be studied according to different geometric features such as object movement, angles, lengths, diagonals, area, perimeter or volume. Regarding this aspect, one of the groups compared the separation of leaves with the angles of Pythagorean spiral. This is known as working with geometric magnitudes.

The Modelling of the Relation of Geometric Magnitudes

As mentioned before, the concern of students about the study of plants originated the study of form, but also the study of magnitudes that they could recognize. In phase 3, in the second stage, they built a table, Cartesian graph and a regression function. There, the attention was focused on the length magnitudes (long and width) and leaf area.

In the study, the students used graphical models (form modelling) to establish a mathematical function in order to interpolate data collected on plant growth; these mathematical expressions became mathematical models (algebraic) of such magnitudes. Through these models, they focused their observations on the proportional growth of the plant and analyzed the place conditions for planting (Table 10).

Table 10. Arguments provided by the Student E3, about the graphic model analysis

We did the process into Excel program to study in depth how the leaves grew, based on width-area and long-area. This procedure is called Correlation Coefficient. This showed us if our plant was growing in good condition or not.

In this case, the highest correlation coefficient was found in a plant located in front of the teacher's room since, apparently, the conditions were optimal receiving sunlight, water and fertilizer.

*Instead, in the study of the students "E6 and E2" (real names were deleted), correlation coefficient was really low; it allowed us to realize that the growth of Succulents (*Graptopetalum Paraguayense*) located on the path were not the most optimal. They were having problems with sunlight, water or fertilizer. It could happen because the tree in front of them was blocking the sun causing, perhaps, Succulent to receive little sunlight.*

For the construction of the models, the students took data from the magnitudes of interest, subsequently; they organized them in tables and tried to identify a pattern. Unable to find it, they used the Excel software; they built two tables; the first table contained records on the *leaf length* (independent variable) and the *area* thereof (dependent variable), the second table contained records on the magnitudes *leaf maximum-width* (independent variable) and *area* (dependent variable). With software tools, the students made a dot chart linking these two magnitudes and obtained a regression model (Figure 2).

The models obtained by the students are part of what is known as direct measurement of growth (Melgarejo, 2010). This process consisted of estimating the leaf area of the plant from the direct measurement of the leaf dimensions (length and width). They made their own regression models and established the best relationship between the area and the dimensions of the leaf (Figure 2). These models are descriptive and, according to Melgarejo (2010), have not explicitly into account the effects of the environment, but they serve to establish the ability of plants to capture sunlight, do photosynthesis and produce agricultural goods. (Galindo & Clavijo, 2007) The growth model interpretations were directed to the mentioned idea (Interpretations of Students E2, E2 and E4, in Table 11); although initial models belong to biology, they could already report on the proportional growth of the plant and its conditions to capture sunlight.

In Phase 4, models provided by the software were analyzed, interpretations derived in relation to the phenomenon of modelling were looked for, and the validity of such models was analyzed (Table 10). In this process, data of some groups of students had a little dispersion evidenced through the correlation coefficient. In the analysis, some models had to be refined, for example, they added quadratic terms in order to adjust better the curve to the set of points.

Table 11. Argument provided by the students E1, E2 and E4, about graphic model analysis

(...) precisely in this leaf width-area ratio is the best growth because the place where my plant was located has better correlation coefficient, this one measures the dispersion of the points with regard to the straight line or curved line, if they are away from the trend 1 is less possible to predict the plant growth and if they are close to the trend 1 is possible to predict the future plant growth. The chosen formula came of this width-area ratio because it is the closest between the four resultants at the trend 1. (Idea of Student E1)

(...)At the end of this research, we realized that the plant inside the school was in bad condition and it needed sunlight because the tree in front of it was blocking sunlight, while the outside plant got good amount of sunlight because it had a better location and it received sun and air in good way having a good growth. (Idea of Student E2)

From the chart above, it is possible to say that the plant, studied by "x" and me, is the most proportional and in better condition, maybe, because it gets more sunlight. This chart shows the correlation coefficient $r^2=0.9329$ long-area is closer to 1. It is indicating a possible future optimal growth.

On the other hand, the plant studied by my partners "w" and "y" is not so accurate; it is disproportional because R^2 is lower, between 0.4 and 0.5. It indicates that its growth is not well. Lack of sunlight, water and fertilizer could be the reason. (Idea of Student E4)

The students E4 and E1 are close to the concept of function explaining some facts about their plant according to the graphical representation of its proportional growth and analyzing the behavior of the function through the correlation coefficient value. It shows student is learning during the project. On the other hand, the student E3 presents an interpretation of the importance and utility of the study, supporting his ideas on the chart results, but without direct relationship.

In the study, the students agree with the student E4 (Table 11) in reference to the imprecise models; they suggested the following assumptions:

- Data were taken from sick plants or they were just growing, because of that, the correlation did not generate good results.
- The closest model was a power function with constant decimal exponent and correlation index (R^2) below 0.5. The students said they could have neglected accuracy aspects in data collection or, even, they have committed significant mistakes making their estimates; because they thought a good relationship was focus around a correlation index greater than 0.8. (See argument of student E4 in Table 11)

In this phase, graph analysis is relevant; those who used software presented some inferences to variations, constant periods, growth, continuity, concavity, maximum and minimum, inflexion points, periodicity, intercepts, symmetries, asymptotes. These inferences are an application of the produced model.

Phase 5 was the final stretch of the projects. The students had to provide arguments for validating results and establishing some general conclusions. Most of them provided interpretations about the usefulness of geometry to explain the growth phenomenon and the plant form (Table 11). The model construction generated different student interpretations based on the entire process of the table and the Cartesian or geometric graphic model (Example in Figure 2).

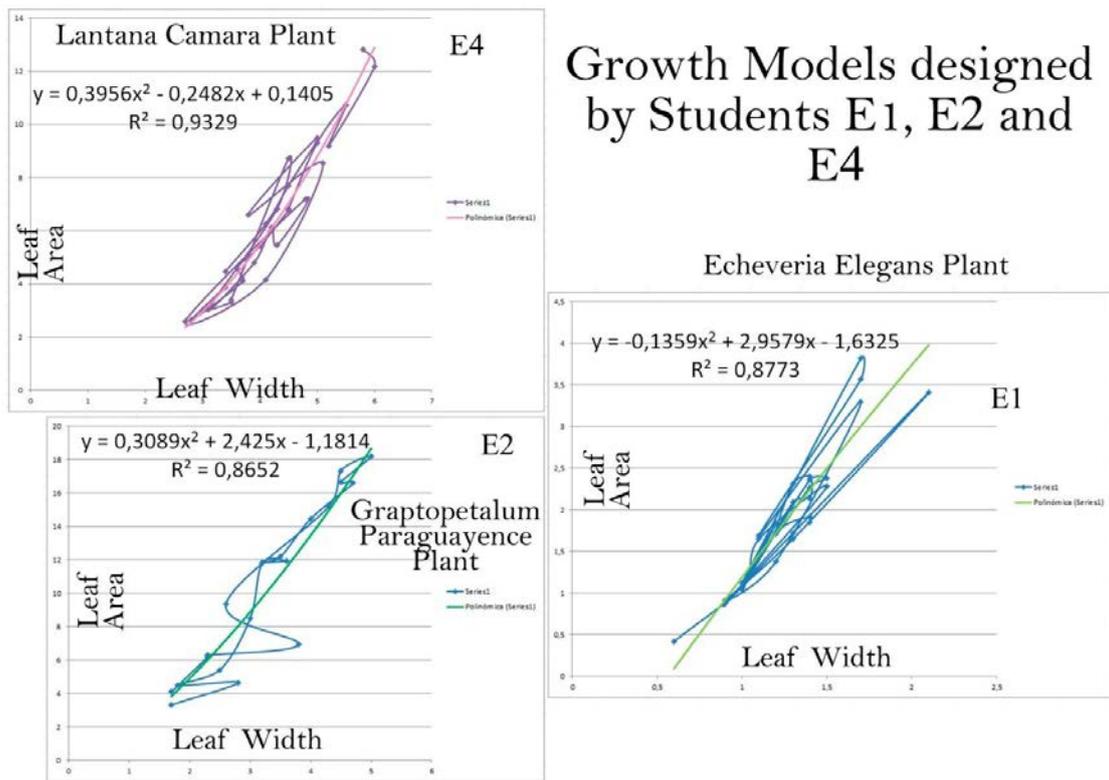


Figure 2. Growth Model designed by the students E1, E2 and E4

In the present work, it was observed that the students achieved to interpret their graphical model and they explained how the variables are related when the correlation coefficient was close to one and what could be inferred according to the information (See interpretation of students E1, E2 and E4, phase 4 in [Table 11](#), interpretation of student E3 in [Table 10](#) and interpretation of students E1, E2 and E5 in [Table 12](#)). This treatment was observed and compared with the numerical result, reason for this double association between the geometrical and numerical. (De la Torre 2003).

Table 12. Conclusions of the students E1, E4 and E5 at the end of their project

Students	Conclusions
E1	<i>Geometry allows us to study more thoroughly the features of all things around us. In this project, I learned a lot of geometry, like the spirals which I had no more knowledge, I learned that joining long-area or width-area measures we can discover how a plant grows. This new experience motivated me to study what I want (Environmental Engineering).</i>
E4	<i>I mainly learned about the relationship between mathematics and nature. I also realized that mathematics is useful to study different forms, not only studying numbers in a classroom. The most innovative and the most fascinating is how to measure a plant according to measures of its petals, I always wondered myself how calculate the approximate size of a plant and I had the opportunity to do it. We can calculate the size of a plant using the measures of each petal. It is not a complicated process because it is not hard; we just have to take good care of the plants.</i>
E5	<i>It is very curious, you go along the street and you do not see the plants in the same way. Before this research, a leaf was a single leaf, but now, when I go along the street and I see plants, I can have a concept about them, about their area or if they are growing in the right way. The most important is that I learned to take care of nature because this project allowed us to have conscious of the reality around us.</i>

DISCUSSION

This article deals with a mathematical modelling experience where a student group, through the research project development, researched several aspects involved in the growth and form of plants. During this process, two mathematical modelling approaches in geometry (geometric modelling) were characterized; the first approach was called form modelling and it was focused on the production of geometric figures representing the characteristics of objects observed in the environment; the second approach is focused on the magnitudes and produces models for attempting to describe, understand and, in many cases, determine the phenomenon behavior. Both approaches are detailed below.

Form modelling. As mentioned above, elementary geometry has common representations of the forms that people perceive in their contact with the environment. In this respect, two orientations are recognized:

- *Centered on the root cause:* This orientation was evident when the students were able to recognize how the plant form could be explained as a geometric model. This orientation is consistent with ideas by Berthelot and Salin (1992). They argue that spatial geometric modelling is a process in order to model the space through knowledge emerged from geometric knowledge. Therefore, the sensitive world (reality) is the starting point for identifying situations, problems or forms modeled through geometric objects. Also in the international literature, characteristics associated with this concept could be found, for example, in works by De la Torre (2003); Biembengut, (2004); Salin, (2004); Cardoso, Franco, Oliveira Abreu, Santos, Carvalho & Sacramento, (2007); Tramontina, (2012); Zanette, Lima, Lazzari & Schulz (2013); Menezes, (2013); Rocha & Santana (2013). The researchers propose situations related to the form modelling processes and use modelling, in turn, as way of teaching some geometric concepts.
- *Centered on construction process:* The students could associate a mathematical model to a perceptible form in the environment to study it through the multiple possibilities that software offer (PaintTool Sai, Geogebra and Excel), observing its behavior and its algebraic representation and, as a result, they could get closer to the polar coordinate concept. These tools, software for example, become important to build models for describing or simulating forms or functioning of reality objects. In the above approach, are found works by authors such as: Laborde (1997), Gonzalez Lopez (2000), Laborde (2002); Richard (2010); Blossier & Richard, (2011); Gravina & Contiero (2011). In their researches, geometric modelling based on intra-mathematics processes is accepted. In the works, it is important to mention.
- They are involving the observation of an object interacting with the student environment. The process performed by software contributes to obtaining a model representing (simulating) the observed object.
- They are presenting a new understanding way of mathematical modelling in Mathematics Education. It moved in the field of the phenomena representation or extra-mathematics problems, because reality is not associated with experiences and everyday needs but an own reality can be created through the software.

Modelling of the relationship between geometric magnitudes. This type of modelling also has its genesis in phenomena or objects of reality (intra and extra mathematical) where understanding of forms, figures and space takes part; however, it differs from the previous because the emphasis is on the magnitudes, quantities and variables involved in the phenomenon or object. In the development of student projects, this type of modelling was materialized through the process that they made to interpret the growth of plants considering the mathematical regression concept, using the Excel software to achieve the objective. The model obtained by the students allowed to question how would grow their plant and whether it would keep its leaf area. On that basis, they would determine whether the sunlight conditions and soil were ideal.

In correspondence with the above, the mathematical models constructed in this type of modelling make use of analytic representations (algebraic expressions, Cartesian planes). In these models, the interpretation given to the mathematical result in relation to the phenomenon or situation that gave it a rise becomes important. In literature, the works by some authors are recognized: Azcarate & Deulofeu (1996); Cordero (2006); Suárez & Cordero (2008) Quiroz et al. (2015).

As it is argued in this article, mathematical modelling acquires other nuances and action modes in the classroom corresponding to the notion of a mathematical model, the kind of phenomenon and relationship to model and purpose for which the model is built. Focusing on forms and magnitudes offers different dynamics to school mathematical activity; the notion of model is not exhausted in equations and standardized modelling that trying to close mathematical activity developed through modelling. Consistent with these positions, considerations about model and mathematical modelling in geometry are discussed:

A mathematical model in geometry (geometric model) can be understood as a (mental and physical/semiotic) representation of an object or phenomenon of the "real world." This representation has geometric characteristics that correspond either to magnitudes associated with the geometry (e.g.

co-variation phenomena in length, area and volume) or to the ability to visualize geometric forms, their respective operations and relationships. Mathematical Modelling in Geometry is the process in which the model production is involved. This process does not end with the production of a representation, it takes account the contexts in which models and subprocesses to its production appear: simplification, abstraction, mathematization, representation, validation, among others. A mathematical model in geometry plays a role of representation and significance of magnitudes and form, and also, of relationships between components of the figure and the figure with others.

In the projects developed by students, the concept of model and modelling fulfilled descriptive and explanatory features of phenomena; plus, the possibility to find new uses to produce knowledge (Table 7). In any case, a geometric treatment was always present.

Other important aspects of this modelling experience to highlight are:

Exploration: This aspect was present when the students began their study process and they identified some characteristics likely to be the most studied. Exploration in this case served as an activity of immersion and situational awareness, which allowed the students to empower themselves of their own research projects. The exploration also encouraged them to collect their own data, study a plant and make a modelling process taking into account the form and growth of their plant. This allowed them to discover a new way of relating to nature, to build the research question; it is linked to what Blum & Borromeo-Ferri (2009) mention in their cycle of modelling as a first approach to the modelling process.

Representation: This process allowed the students to observe a model in different representations, always starting with the geometric representation to get an algebraic representation. The transition from one representation to another made possible the study of model in a comprehensive manner, observing different information of each representation. Also, it made possible the association of these representations as a great contribution in order to answer the research question.

Formal construction: The students were able to collect data, obtaining magnitudes and relationships between them. This promoted the production of models of both forms and magnitudes. Then, they had to work with their model, to interpret creating an abstract language related to the activities of the modelling process. The reasoning processes are consolidated as proposed by Duval (2001) an important process for the relationship with discursive processes, for extending the knowledge, for demonstrations, for the explanation.

The use of mathematics: Using geometry as a tool to study the form and growth of plants, allowed students to recognize that mathematics is a science related to real aspects and it has many uses. So they could answer the question: what is mathematics for? (Table 10). In turn, they also recognized mathematics as a useful and applicable tool to explain natural phenomena as the form and growth of plants, in this case.

The data reported in this article become an evidence of obtaining an analytical graph, initially outlined by students through a qualitative perception of variation, allowed them to define the function and to perform an algebraic analysis. This type of situations coincides with others reported in the literature (Suárez & Cordero, 2008; Azcárate & Deulofeu, 1996); where it is reported that the graphs have argumentative functions and provide diverse information.

CONCLUSIONS

This article introduced two types of mathematical modelling when phenomena are studied from elementary geometry; these two types were named Form Modelling and Modelling of the Relationship between Geometric Magnitudes. Both modelling types were presented in the project development by students and also discussed according to the international literature from which they can be too gathered.

The mathematical modelling reported in this article was articulated to the research project development. In this kind of projects, students were invited to study a phenomenon associated with the plant growth. The actions of participants become evidence when they are involved in research project development, they carry out processes of exploration, representation, formal construction of a (mathematical) representation and other actions articulated to scientific activity. By the nature of the studied phenomena, geometry was part of the foundations in mathematical treatments, in some cases as a figure representing forms and other characteristics of objects in the environment; in other cases, such figures were built with the support of a Dynamic Geometry Software. At other times, the study of the phenomenon made the students to focus their attention on the behavior of some magnitudes; other software were also used for supporting the mathematical model construction (e.g. linear regression); from which the modelling activity involved a model validation and its confrontation with data and the phenomenon that gave it rise.

The mathematical modelling articulation to project methodology recreates “research activities” where the students develop projects on topics chosen by themselves. Criteria to organize and analyze data through

mathematical tools supported with software were able to establish. (Thus, the students used the observation to study the form and growth of their plant as it is shown in reality, they could make understandable relevant aspects and propose hypothesis emerged from their own observation.) In addition, they formulated questions to be validated and studied in order to answer their research question. Many studies emphasize the relevance of mathematical modelling in learning mathematics (Blum et al., 2007) and learning of other aspects from the contexts and culture (Villa-Ochoa & Berrio, 2015). The present study has highlighted the importance of modelling forms as scaffolding to the students to practice their artistic abilities.

This research suggests fourth implications for the classroom. The first is related to the modelling and project development where mathematics becomes a tool to study different phenomena. In this way, the closeness of the students to the scientific method becomes a priority. It was essential to offer the students research experiences and make them feel like researchers; therefore, the research question guided all the actions carried out.

The second implication is related to teaching and learning of geometry through modelling. This type of proposal shows how elementary geometry could be a tool to analyze the reality with other lenses, attracting the student attention. This approach favors the development of creative mathematical thinking. Furthermore, Euclidian Geometry and Dynamic Geometry became a tool to represent natural objects with models coming from the mathematical world that explain the real world. Consequently, mathematical modelling in geometry becomes a tool for the student to study, transform, mathematize and think about the "reality".

The third implication is related to the uses, advantages and limitations of doing modelling in this way. Therefore, some proposals to apply modelling processes in the classroom from elementary geometry emerged from this implication; the doors are open to new researches in order to continue contributing to this process, for instance in basic education, with ideas about how to carry this process out.

Finally, the fourth implication concerns the relationship between Arts and Mathematics, for instance, in geometric models is possible to observe both mathematical and abstract concepts. Modelling of form involved the use of mathematical expressions to produce specific effects. It helped students not only replicate but also reconsider the geometric models as an expression of their knowledge.

The plurality of plant forms and structures shows multiple applications of Mathematics to Arts; not only concepts of Geometry have influence in the latter, also concepts of Calculus, Algebra, Graphic Design, and drawing. It invites students to this unique experience of feeling the beauty and aesthetics of Mathematics in geometric models.

REFERENCES

- Alsina, C. (2008). Geometría y realidad. *Revista Sigma*, (33), 165-179. Retrieved from http://www.hezkuntza.ejgv.euskadi.net/r43573/es/contenidos/informacion/dia6_sigma/es_sigma/adjuntos/sigma_33/10_geometria_realidad_33.pdf
- Alves, L. B. S., Pereira, A. C. G. R., & Silva, J. C. (2012). Modelagem Geométrica das Instalações do IST-Rio. *Revista de Publicação técnico-científica do IST-Rio*, (5), 50-62. Retrieved from <http://revista.ist-rio.net/ojs/index.php/revISTA/article/viewArticle/50>
- Aravena, M., Caamaño, C., & Giménez, J. (2008). Modelos matemáticos a través de proyectos. *Revista Latinoamericana De Investigacion En Matematica Educativa-Relime*, 11(1), 49-92.
- Arcavi, A. (2008). Modelling with graphical representations. *For the learning Mathematics*, 28(2), 2-10. Retrieved from http://srvcnpbs.xtec.cat/creamat/joomla/images/stories/documents/visualizacio/arcavi_modeling.pdf
- Azcárate, C., & Deulofeu, J. (1990). *Funciones y gráficas*. Madrid: Editorial Síntesis.
- Barbosa, J. C. (2009). Modelagem e Modelos Matemáticos na Educação Científica. *Revista de Educação em Ciência e Tecnologia*, 2(2), 69-85.
- Bassanezi, R. C. (1999). *Modelagem matemática uma disciplina emergente nos programas de formacao de professors*. Universidade Estadual de Campinas. Retrieved from http://www.ime.unicamp.br/~biomat/bio9art_1.pdf
- Bassanezi, R. C. (2002). *Ensino-aprendizagem com modelagem matemática*. São Paulo: Contexto.
- Bassanezi, R. C., & Biembengut, M. S. (1997). Modelación matemática: Una Antigua forma de investigación - un Nuevo método de enseñanza. *Números-Revista de didáctica de las matemáticas*, (32), 13-25.
- Berthelot, R., & Salin M. H. (1992). *L'enseignement de l'espace et de la géométrie dans la scolarité obligatoire* (Doctoral dissertation). Université de Bordeaux. Francia. Retrieved from http://hal.archives-ouvertes.fr/docs/00/41/40/65/PDF/these_Berthelot_Salin.pdf
- Biembengut, M. S. (2004). *Modelagem Matemática & implicações no ensino e na aprendizagem de matemática*. Blumenau, Brasil: Edifurb.

- Biembengut, M. S., & Hein, N. (2004). Modelación matemática y los desafíos para enseñar matemática. *Educación Matemática*, 16(2), 105-125.
- Blomhøj, M. (2004). Mathematical modelling- a theory for practice. In Clarke, B. et al. (Ed.), *International perspective on learning and teaching mathematics* (pp. 145-159). Göteborg: National Center for Mathematics Education.
- Blossier, M., & Richard, P. R. (2011). Modélisation instrumentée et conceptions a priori dans un espace de travail géométrique en évolution. *Journées mathématiques IFÉ*, 1-24. Retrieved from <http://mathieblossier.free.fr/irem/geogebra/lyon/presentation/Archive-GGB-3D/Modelisation-%20instrumentee-v.1.1.pdf>
- Blum, W., & Borromeo-Ferri, R. (2009). Mathematical Modelling: Can It Be Taught And Learnt? *Journal of Mathematical Modelling and Application*, 1(1), 45-58.
- Blum, W., Galbraith, P., Henn, H., & Niss, M. (Eds.). (2007). *Modelling and Applications in Mathematics Education. The 14th ICMI Study* (Vol. 10). New York: Springer.
- Bonotto, C. (2007). How to replace word problem with activities of realistic mathematical modelling. In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Modelling and application in Mathematics Educations. The 14th ICMI Study* (pp. 185-192). New York: Springer.
- Borba, M. C., & Villareal, M. E. (2005). *Humans-with-Media and the Reorganization of Mathematical Thinking: Information and Communication Technologies, Modeling, Visualization and Experimentation* (Vol. 39). New York: Springer.
- Borda, A., Félix, N., Freitas Pires, J., & Moraes, A. (2008). Modelagem Geométrica nos estágios iniciais de aprendizagem da prática projetual em Arquitetura. *14 convención científica de ingeniería y arquitectura*, La Habana, Cuba (pp. 1-10). Retrieved from <http://cumincad.scix.net/data/works/att/5d77.content.05083.pdf>
- Brisson Lopes, J. M. (2000). *Computação gráfica: Modelação geométrica*. Licenciatura em engenharia informática, Universidade Técnica de Lisboa. Retrieved from <http://disciplinas.ist.utl.pt/leic-cg/textos/livro/Modelacao%20Geometrica.pdf>
- Cardoso, A., Franco, B. S. C., Oliveira, M. J. R., Abreu, K. K., Santos, N. P., Carvalho, J. R., . . . Santos, D. O. (2007). *Modelando uma caixa de sabao em po*. Programa Institucional de Bolsa de Iniciação a Docência. (PIBID), da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), Brasil. Retrieved from <http://www.sbmec.org.br/cmecs/cmec-se/2011/trabalhos/PDF/281.pdf>
- Cordero Osorio, F. (2006). La modellazione e la rappresentazione grafica nell'insegnamento apprendimento de lla matematica. *La Matematica e la sua Didattica*, 20, 1, 59-79. Retrieved from http://webcache.googleusercontent.com/search?q=cache:http://www.tbu.uan.edu.mx/_Lib_Art_En/_Arts/F.Cordero2006b.pdf
- De la Torre, A. (2003). *Modelización del espacio y del tiempo*. Medellín: Universidad de Antioquia.
- Doerr, H. M., & Pratt, D. (2008). The learning of mathematics and mathematical modeling. In M. Kathleen Heid & G. W. Blum (Eds.), *Research on technology and the teaching and learning of mathematics: Research syntheses*. (pp. 259-285). USA: Information Age Publishing.
- Duval, R. (2001). La geometría desde un punto de vista cognitivo (Página de Internet). Retrieved from <http://fractus.uson.mx/Papers/ICMI/LaGeometria.htm>
- Fowler, A. C. (1998). *Mathematical models in the applied science*. Cambridge: Cambridge University Press.
- Galindo, J. R., & Clavijo, J. (2007). Modelos alométricos para estimar el área de los folíolos de arveja (*Pisum sativum* L.). *Revista Corpoica Ciencia y Tecnología Agropecuaria*, 8, 1, 37-43. Retrieved from <http://www.corpoica.org.co/SitioWeb/Archivos/Revista/5.Modelosalomtricos.pdf>
- Girnat, B., & Eichler, A. (2011). Secondary Teachers' Beliefs on Modelling in Geometry and Stochastics. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in the teaching and learning of mathematical modelling - ICTMA14*, 1, 75 - 84. Netherlands: Springer.
- González López, M. j. (2000). La gestión de clase de geometría utilizando sistemas de geometría dinámica. In Gómez P. y Rico, L. (eds). *Iniciación a la investigación en didáctica de las matemáticas*. Granada: Editorial Universidad de Granada.
- Gravina, M. A., & Contiero, L. de O. (2011), Modelagem com o GeoGebra: uma possibilidade para a educação interdisciplinar? *Novas Tecnologias na Educação*, 9(1), 1-10. Retrieved from <http://seer.ufrgs.br/renote/article/view/21917/12717>
- Hein, N. (Marzo, 2006). Modelaje matemático en la enseñanza un tratado teórico y un ejemplo didáctico. *V Festival internacional de matemática de costa a costa matemática como lenguaje para interpretar nuestro entorno*. Universidad

- de Regional de Blumenau, Santa Catarina - Brasil. Retrieved from <http://www.cientec.or.cr/matematica/pdf/P-Hein.pdf>
- Israel, G. (1996). *La mathématisation du réel*. Paris: Editions du Seuil.
- Kaiser, G., & Schwarz, B. (2010). Authentic Modelling Problems in Mathematics Education—Examples and Experiences. *Journal für Mathematik-Didaktik*, 31(1), 51-76. doi:10.1007/s13138-010-0001-3
- Kaiser, G., & Sriramam, B. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM*, 38(3), 302-310.
- Laborde, C. (1997). Cabri-geómetra o una nueva relación con la geometría. L. Puig (Ed). *Investigar y enseñar variedades de la educación matemática* (pp. 33-48). Retrieved from https://campusvirtual.univalle.edu.co/moodle/pluginfile.php/316457/mod_resource/content/0/Laborde_Cabri_una_Nueva_Relacion_con_la_Geometria.pdf
- Laborde, C. (2002). Basar la enseñanza y el aprendizaje de las matemáticas en la noción de variación con geometría dinámica. *Tecnologías computacionales en el currículo de matemáticas: Memorias del Congreso Internacional*. Ministerio de Educación Nacional, Bogotá.
- Lopes, D. S., Martins, J., Campos, J., & Pires, E. (2007). Modelação geométrica de estruturas humanas baseada em imagens de tomografia computadorizada. *Proceedings of the CMNE/CILAMCE, Porto, Portugal*. Retrieved from <https://dspace.ist.utl.pt/bitstream/2295/639602/2/p1083.pdf>
- Masingila, J. O. (2002). Examining Students' Perceptions of Their Everyday Mathematics Practice. In M. E. Brenner & J. N. Moschkovich (Eds.), *Journal for Research in Mathematics Education. Monograph, Everyday and Academic Mathematics in the Classroom*, 30-39.
- Masingila, J. O., Davidenko, S., & Prus-Wisniowska, E. (1996). Mathematics learning and practice in and out of school: A framework for connecting these experiences. *Educational Studies in Mathematics*, 31(1-2), 175-200. doi:10.1007/BF00143931
- Melgarejo, L. M. (2010). *Experimentos en fisiología vegetal*. Colombia: Universidad Nacional. Retrieved from http://ciencias.bogota.unal.edu.co/fileadmin/content/laboratorios/fisiologiavegetal/documentos/Libro_experimentos_en_fisiologia_y_bioquimica_vegetal_Reparado_.pdf
- MEN-Ministerio de Educación Nacional. (1998). *Lineamientos Curriculares: Matemáticas*. Bogotá: Magisterio.
- Muñoz, L. M., Londoño, S. M., Jaramillo, C. M., & Villa-Ochoa, J. A. (2014). Contextos Auténticos y la producción de modelos matemáticos escolares. *Revista Virtual Universidad Católica del Norte*, (42), 48-67.
- Quiroz, S. M. R., Orrego, S. M. L., & López, C. M. J. (2015). Measurement of Area and Volume in an Authentic Context: An Alternative Learning Experience through Mathematical Modelling. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical Modelling in Education Research and Practice. Cultural, Social and Cognitive Influences* (pp. 229-240). Switzerland: Springer International Publishing.
- Rendón-Mesa & Esteban, (2013). La modelación matemática en la ingeniería de diseño. *1 Congreso de Educación Matemática de América Central y El Caribe*. República Dominicana. Retrieved from <http://www.centroedumatematica.com/memorias-icemacyc/387-483-1-DR.pdf>
- Richard, P. R. (2010). La geometría dinámica como herramienta para desarrollar competencias de modelización en el Bachillerato. In Gómez Chacón, I. M. (ed.) *Competencia matemáticas instrumentos para las Ciencias sociales y naturales* (pp. 89-115). España: Ministerio de Educación.
- Rocha, J., & Sant'ana, M. (2013). Modelagem matemática a partir de uma fotografia. *VIII CNMEM Conferencia Nacional sobre modelagem na educao matemática: pesquisas, práticas e implicacoes para an educao matemática*, Centro Universitário Franciscano, Santa María Rio Grande do Sul, Brasil (pp. 1-15).
- Rodriguez, C. A. (2007). Los Sistemas CAD/CAM/CAE y su aplicación para la formación de competencias profesionales en estudiantes de Ingeniería. *5th Latin American and Caribbean Conference for Engineering and Technology*, Tampico, Mexico (pp. 1-12). Retrieved from <http://www.laccei.org/LACCEI2007-Mexico/p89.doc>
- Salin, M. H. (2004). La enseñanza del espacio y la geometría en la enseñanza elemental. In L. Puig; J. Calderon (Eds.) *Investigación y didáctica de las matemáticas* (pp. 7 - 309). Madrid: Ministerio de Educación y Ciencia.
- Stake, R. E. (1998). *Investigación con estudio de casos*. Madrid: Ediciones Morata.
- Suarez, L. y Cordero, F. (2008). Modelación-Graficación. Una categoría en cálculo para resignificar la variación en una situación de modelación del movimiento. *ICME 11, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, México*.

- Suárez, L. y Cordero, F. (Diciembre 2010). Modelación - graficación, una categoría para la matemática escolar. Resultados de un estudio socioepistemológico. *Revista Latinoamericana de Investigación en Matemática Educativa*, 13(4), 319-333. Retrieved from <http://www.scielo.org.mx/pdf/relime/v10n1/v10n1a2.pdf>
- Tramontina Zukauskas, N. S. (2012). Modelação matemática no ensino fundamental: motivação dos estudantes em aprender geometria (Master's thesis). Pontifícia Universidade Católica do Rio Grande do Sul, Brasil. Retrieved from <http://repositorio.pucrs.br/dspace/bitstream/10923/3125/1/000438484-Texto%2bCompleto-0.pdf>
- Trigueros, M. (2009). El uso de la Modelación en la Enseñanza de las Matemáticas. *Innovación Educativa*, 9(46), 75-87.
- Verschaffel, L., Van Dooren, W., Greer, B., & Mukhopadhyay, S. (2010). Reconceptualising word problems as exercises in mathematical modelling. *Journal für Mathematik-Didaktik*, 31(1), 9-29.
- Villa-Ochoa, J. A. (2007). La Modelación como Proceso en el Aula de Matemáticas: Un Marco de Referencia y un Ejemplo. *Tecno Lógicas*, 0(19), 63-86.
- Villa-Ochoa, J. A. (2011). *La comprensión de la tasa de variación como una manera de aproximarse al concepto de derivada*. Tesis doctoral no publicada, Facultad de Educación-Universidad de Antioquia, Medellín.
- Villa-Ochoa, J. A. (2015). Modelación matemática a partir de problemas de enunciados verbales: un estudio de caso con profesores de matemáticas magís, *Revista Internacional de Investigación en Educación*, 8(16), 133-148. doi:10.11144/Javeriana.M8-16.MMPE
- Villa-Ochoa, J. A., & López, C. M. J. (2011). Sense of Reality through Mathematical Modelling. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in Teaching and Learning of Mathematical Modelling* (pp. 701-711). Netherlands: Springer.
- Villa-Ochoa, J., & Berrío, M. J. (2015). Mathematical Modelling and Culture: An Empirical Study. In G. A. Stillman, W. Blum, & M. Salett Biembengut (Eds.), *Mathematical Modelling in Education Research and Practice* (pp. 241-250). Switzerland: Springer International Publishing.
- Zanette, Lima, Lazzari & Schulz (2013). Modelagem matemática no ensino superior: variação do volume de um cilindro em posição inclinada. *VIII CNMEM Conferencia Nacional sobre modelagem na educao matemática: pesquisas, práticas e implicacoes para an educao matemática*. Centro Universitário Franciscano. Santa María Rio Grande do Sul, Brasil (pp. 1-11).

<http://www.ejmste.com>