MODESTUM
OPEN ACCESS
EURASIA Journal of Mathematics, Science and Technology Education
ISSN: 1305-8223 (online) 1305-8215 (print)
2018 14(2):657-669
DOI: 10.12973/ejmste/80629

Shaping Mathematics into Stories by means of Propp's Narratemes

G. Soto y Koelemeijer [1*], F. J. J. M. Janssen [2], P. M. G. M. Kop [2]
[1] Postdoc ICLON/ Secondary school math teacher, Stedelijk Gymnasium Leiden, 2316 ES Leiden, THE NETHERLANDS
[2] ICLON Graduate School of Teaching, Leiden University, P.O. Box 905, 2300 AX Leiden, THE NETHERLANDS

Received 4 August 2017 ▪ Revised 2 September 2017 ▪ Accepted 1 March 2017

ABSTRACT

For a few decades now, scholars have tried to make mathematics education more meaningful and motivating by using narratives. However, there still remains a gap between the theory of narratives and classroom practice. In this paper we provide two design heuristics for teachers by which to bridge this gap. We discuss Dietiker's mathematical story framework, based on narratology, and present a design heuristic based on this framework by which to design mathematical stories. We extend this heuristic on the basis of the work of the Russian formalist Vladimir Propp, who analyzed Russian folktales. He found that 31 irreducible narrative elements, called *functions* or *narratemes*, can be distinguished. We exemplify our heuristics by means of two mathematical stories, one of which concerns Kepler's conjecture. The value of this approach is that it provides mathematics teachers with a heuristic story framework that is suitable for designing or redesigning mathematical stories.

Keywords: mathematics, mathematics education, narratemes, narratology, Propp

INTRODUCTION

For a few decades now, scholars have been trying to make mathematics and science education more meaningful and more engaging, by using narratives. Since narratology (the theory of narratives and narrative structures) covers a large research field, it seems a good idea to distinguish different ways of combining the two. According to Boström (2006) narratives can contribute to education from three different perspectives: the *teaching* perspective, the *inquiring* perspective, and the *epistemological* perspective. All narratives used by teachers and students, such as textbooks, YouTube, auxiliary materials, inside and outside the classroom in order to make the subject meaningful and engaging, are to be considered of the first type (Zazkis & Liljedahl, 2009; Albool, 2013a, 2013b; Balakrishnan, 2008). When narratives are used as a methodology in educational research, for example, when a mathematical textbook is analyzed (Dietiker, 2012, 2015a, 2015b; Soto y Koelemeijer, 2008), we speak of the *inquiring* perspective. When narratives are used as a way of observing the world, we speak of the *epistemological* perspective (Boström, 2006; Kaasila, 2006; Burton, 2001; Gade, 2010; Drake, 2006; Healy & Sinclair, 2007).

This paper is concerned with the teaching perspective. More specifically, we focus on narratives told by teachers in a classroom setting. Since narrativity and storytelling are closely related, one obvious reason to use narratives in mathematics education is by telling stories in class: stories about the history and philosophy of mathematics, about mathematicians, about the interplay of mathematics and other disciplines, and stories about the applications of mathematics and its place in human history and thought. Stories are an excellent tool to 'humanize' abstract mathematics. This is in line with Freudenthal's and Hersh's view of mathematics as a human activity (Freudenthal, 1991; Hersh, 1997). This humanization of mathematics contributes to making the subject matter more meaningful "by relating it to students' lives and interests" (Hobbs, 2013).

If narratives can be used to make the subject more meaningful and more engaging, the question arises how mathematics teachers can tell interesting and effective stories. Not every teacher is a born storyteller. A second problem is how the story is linked to the mathematical content. Egan (1988) presents a design heuristic consisting of five steps, in which the first step is to identify importance. This can be done by answering three questions: What

✉ g.soto@gymnasiumleiden.nl **(*Correspondence)** ✉ fjanssen@iclon.leidenuniv.nl
✉ koppmgm@iclon.leidenuniv.nl

Contribution of this paper to the literature

- Narratives can contribute to education from three different perspectives: the teaching perspective, the inquiring perspective, and the epistemological perspective. This paper presents a fourth perspective of how narratives can contribute to (mathematics) education: the ontological perspective.
- This paper presents a design heuristic to design a mathematical story based on Dietiker's framework. We exemplified this heuristic by means of an example containing five story elements: character, action, actor, setting, and plot.
- This paper presents a second design heuristic to design or redesign a mathematical story. This heuristic is based on the work of the Russian folklorist Vladimir Propp, more specifically his The Morphology of the Folktale. With this theoretical framework the mathematical story becomes much more detailed and complex. We also illustrate the heuristic by means of an example.

is most important about this topic? Why should it matter to students? And last but not least: What is affectively engaging about it? The second step consists of finding a binary opposite that best catches the importance of the topic. The third step consists of organizing content into story form. Although Egan prescribes some steps for using narratives in classroom practice, it is still not clear how to organize the content into story form. In this paper we aim to bridge this gap by presenting two design heuristics. The value of this approach is that it provides mathematical teachers with a detailed story framework that is suitable for designing or redesigning mathematical stories.

The first section begins with a brief overview of the interplay between narrativity, storytelling, and (mathematics) education. In section 3 we briefly discuss Bruner's two modes of thought. In section 4 some examples are given of different perspectives on how narratives can be used in mathematical education research and practice. In section 5 we focus on the work of Dietiker, more specifically on Dietiker's theoretical framework (2015a, 2015b), in which five story elements are used to improve the internal mathematical story. We will use these story elements to offer a design heuristic, and demonstrate this heuristic by means of an example of a mathematical story in section 6. Ideally, teachers have more than one heuristic at their disposal, so that different kind of stories can be told. This first design heuristic is very basic: it consists of only five elements which can be used to design a story. In order to expand the heuristic basis, in section 7 the Russian formalist Vladimir Propp is introduced, and in section 8, a second design heuristic is given which is much more detailed. It is based on the work of Propp, who analyzed over 100 Russian folktales. He found that 31 irreducible narrative elements, called *functions* or *narratemes*, can be distinguished. We have used the reverse path, that is, we do not analyse an existing story, but show by means of an example how a mathematical story can be designed or redesigned with the help of these narratemes. This is the subject of section 9. The last section contains a conclusion and some ideas for further research.

NARRATIVITY, STORYTELLING AND EDUCATION

Narrativity and storytelling are closely related, since "a narrative text is a text in which an agent relates ('tells') a story in a particular medium, such as language, imagery, sound, buildings, or a combination thereof" (Bal, 2009). The introduction of storytelling, or narratives, into education is not a recent development. Authors such as Aristotle, Barthes, and Bruner have all placed narrative in the centre of human cognition. Over the past few decades there has been an increasing emphasis on the relationship between stories and knowledge construction. Egan (1988) writes:

> *The story form is a cultural universal; everyone everywhere enjoys stories. The story, then, is not just some casual entertainment; it reflects a basic and powerful form in which we make sense of the world and experience. Indeed, some people claim that the story form reflects a fundamental structure of the mind.*

His goal is to show how to use the power of the story form in order to teach any content engagingly and meaningfully. Therefore, he advises to ensure that a conflict or sense of dramatic tension is set up at the beginning of the lesson to create some expectation, just as a storyteller, or narrator, does. Human beings make sense of a story affectively and cognitively, which makes it a powerful tool for learning. According to Egan (1988), human emotions and intentions are important in making things meaningful. "To present knowledge cut off from human emotions and intentions is to reduce its affective meaning." Hobbs (2013) writes that "Essentially, the stories serve to situate the subject matter historically, culturally, socially or personally, that is, they essentially humanize the content in order to make it meaningful."

THE PARADIGMATIC MODE VERSUS THE NARRATIVE MODE

On a cognitive level, Bruner (1986) distinguishes two modes of thought:

> *... one mode, the paradigmatic or logico-scientific one, attempts to fulfill the ideal of a formal, mathematical system of description and explanation. It employs categorization or conceptualization and the operations by which categories are established, instantiated, idealized, and related to one another to form a system.*

The paradigmatic mode is the one used in science, in which we logically try to categorize the world around us. The narrative mode, on the other hand, is concerned with the meaning that is ascribed to experiences through stories. Adler (2008) explains that:

> *... thought grounded in the paradigmatic mode seeks to explain the underlying relationships between sets of observable variables while thought grounded in the narrative mode seeks to explain the storied meaning people make of these relationships. Each mode of thought has significant strengths. The paradigmatic mode offers the power of prediction in that it sets up and tests hypotheses about the nature of reality. In contrast, the narrative mode organizes the complex and often ambiguous world of human intention and action into a meaningful structure.*

The difference between the two modes is summarized by Brendel (2000) as the distinction between causal relationships (the paradigmatic mode) and meaningful explanation (narrative mode). One could argue that in mathematics education both modes play a role. Of course, mathematics itself is about patterns, structure and causal relations, but the interpretation of a mathematical text, whether a textbook or a teacher's introduction or explanation, is part of the narrative mode. The same holds for how meaning is derived.

The central issue in the *paradigmatic* mode is to know truth, that is, its objective is truth. The central issue in the *narrative* mode is to endow experiences with meaning. The objective in the narrative mode is verisimilitude rather than truth. Different people tell different stories about a similar event. There is not one truth, but several, depending on who tells, feels, or sees. Where the method of the paradigmatic mode is concerned with sound argument, reason, logic, and proof, the method of the narrative mode is concerned with a good story, association, aesthetics, and intuition. This does not imply that in the narrative mode, everything is allowed. A good story for example is a coherent story, it contains a clear inner structure, and there must be some logic in the sequence of events.

We can also distinguish some characteristics for each of the two modes. Where the paradigmatic mode is theory-driven, general, abstract, de-contextualized, a-historical, and non-contradictory, the narrative modes are meaning-centered, particular, concrete, context-sensitive, historical, and contradictory. Thus, the narrative mode allows us to present mathematics in a different way, that is, as a human activity. Narratives can be used to place abstract mathematics in a context, historical or not.

According to Bruner there are four characteristics of narratives (see also Healy & Sinclair, 2007): they have an inherent sequentiality, they can be about real or imaginary events, they forge connections between the exceptional and the ordinary, and they have some kind of dramatic quality. These characteristics are part of the two design heuristics that will be presented in this paper.

NARRATIVITY AND MATHEMATICS

Narratives in mathematics education have received increasing attention over the last thirty years. Also in this research field the three perspectives mentioned above can be distinguished. We would like to add a different perspective on how narratives can contribute to education, that is, the *ontological* perspective. By this we mean the study of the interplay of the essence of both disciplines. We will give two examples.

The book *Circles Disturbed*: *The Interplay of Mathematics and Narrative* (Doxiadis & Mazur, 2012) contains 15 fascinating essays in which the interplay between mathematics and narratives is examined. One of the most interesting essays is that by Greek author and mathematician Doxiadis. In it, he searches for the origin of mathematical proof, which he relates to rhetoric. He shows in this lengthy essay that mathematics and narrativity are interrelated from the beginning. The essay by Gowers, called *Vividness in Mathematics and Narrative*, also belongs to this category. He argues that, like narratives, "mathematics explanations can have or lack the quality of vividness as well, and that the causes are similar."

Concerning the teacher perspective in mathematics education, Doxiadis is also an advocate for telling stories in math class. He is the author of the famous graphic novels *Logicomix, an epic search for truth*, and *Uncle Petros and Goldbach's Conjecture,* and claims (Doxiadis, 2003) that "mathematical narrative must enter the school curriculum, in both primary and secondary education." The challenge is to make sure that mathematical narrative supplements and interacts with technical mathematics teaching. The aim of introducing narratives is threefold:

- to increase the appeal of the subject,
- to give it a sense of intellectual, historical and social relevance, and a place in our culture,
- to give students a better sense of the scope of the field, beyond the necessarily limited technical mathematics that can be taught within the constraints of the school system.

Balakrishnan (2008) shows how stories can be used in various ways to add emotional value and meaning to facts, rules, and algorithms. Zazkis & Liljedahl (2009) explore the function of storytelling in classroom to achieve an environment of imagination, emotion and thinking, to make mathematics more enjoyable and more memorable, to engage students in a mathematical activity, to make them think and explore, and to help them understand concepts and ideas. Albool (2013a, 2013b) concludes that using storytelling in teaching mathematics increases students' ability to understand fraction concepts, increases their ability to solve mathematics problems, and increases students' motivation toward learning mathematics.

The *epistemological* perspective can be found in for example Burton (2001) and Drake (2006). Burton focuses on children's narratives. She concludes that "children are natural storytellers and they practice this in *all* aspects of their learning, including mathematics." Drake uses "teachers' narrative descriptions of themselves as learners and teachers of mathematics to understand teachers' interpretations and implementations of a reform-oriented mathematics curriculum." The focus here is on turning-point stories, that is, "stories in which the teachers initially experienced significant failures in mathematics, but, as the result of a turning-point experience, now view themselves positively as both learners and teachers of mathematics."

Narratology is "the theory of narratives, narrative texts, images, spectacles, events; cultural artefacts that 'tell a story'" (Bal, 2009). We believe that not only narratology (such aspects as different kind of story forms, focalization, the difference between text/fabula and story, etc.), but also more general literary theory (for example close reading and poetry analysis) can in a broad sense be used to improve mathematics education. If narratology is used to analyse a mathematical text, we speak of the *inquiring* perspective.

A mathematical textbook, or a lesson, can be seen as a narrative text, and therefore narratology can be used to analyse the text. Differentiating between fabula and story can shed light on the difficulties that weak students experience doing mathematics. One difference between story and fabula is that the sequence in the fabula is chronological, whereas the sequence in the story does not have to be. This can be related to, for example, the order of solving a mathematical problem, and to the question of where to begin. Dialogues and other story forms can be used to identify oneself with a character. Dietiker (2012, 2013, 2015a, 2015b) is mainly concerned with the inquiring perspective.

To sum up: storytelling or, more in general, narratives and narratology have been found to be interesting and powerful tools in mathematics education.

A DESIGN HEURISTIC BASED ON DIETIKER'S FRAMEWORK

Dietiker (2012, 2013, 2015a, 2015b) draws from literary theory, and more specifically from narratology (Bal, 2009), to theorize the mathematical story framework. This framework can be used "for interpreting sequences of mathematical content found in the curriculum." Dietiker interprets these mathematical sequences as stories. The idea is to offer some conceptual tools, which helps educators to make sense of curriculum materials. This framework can also be fruitful in that it adds an aesthetic dimension to mathematical sequences, and rearranges the sequences in such a way that a surprise or suspense is obtained. Dietiker presents some questions that should be asked before one starts to design a new curriculum based on this framework, such as

- *How does this sequence enable anticipation of what is to come?*
- *How can this mathematical story be designed so that the students want to know how it ends?*
- *Does this story make sense?*

Hence, this framework is not only used to motivate students, but also ensures that mathematics becomes more meaningful.

We shall not go into narratology in much detail, except for some basic concepts of a story that Dietiker (2015a, 2015b) uses to create a design heuristic. For more about narratology we refer to Bal (2009), and for more about applying narratology in the context of mathematics education to Dietiker (2012).

First, a story needs some characters and a plot. The *plot* can be seen as a sequence of events which are all interrelated and build up tension. A beautiful or attractive story makes you curious about the development and the end. Authors use different tricks to keep the reader going. In a well-written and coherent story, you are dying to know what happens to the characters and what conflicts may be introduced. Is the conflict solved in the end, and

Story Element	Definition	Examples
Mathematical Character	The mathematical objects that are discussed and used in the mathematical story.	$y = x + 2$, Δ**ABC**, $0.\overline{9}$
Mathematical Action	Something done in a mathematical story that results in a change in the mathematical content.	The manipulation of a mathematical character (e.g., scaling a triangle or factoring a polynomial) or multiple characters (e.g., adding numbers).
Mathematical Actor	The people or narrator who performs the action of the mathematical story.	Textbook author (e.g., a worked example), students (e.g., solving a task or problem), teacher (e.g., advancing the mathematical story through a statement or demonstration)
Mathematical Setting	The mathematical representation in which the mathematical story is set.	Physical tiles on a table, dynamic geometry tool, symbols drawn on a paper
Mathematical Plot	The way in which the sequence of interrelated events reveals and withholds information to increase or release tension.	A sequence of tasks and activities that supports inquiry from the generation of a mathematical question to the revelation of its answer.

Figure 1. Mathematical Story Elements (Dietiker, 2015b)

if so, how? *Characters* are important because they make it possible for readers to identify with one of them. This identification can result in feeling that you yourself have become this character, you are driven into his or her world. The character's worries and luck become your worries and your luck. Furthermore, the *action* within and the *setting* of a story are important. They take the reader to different epochs and other contexts, which can contribute to the meaning of the story and the understanding of the action of the characters. Last but not least, there is the question of who performs the action of the story (*actor*). From whose perspective is the story told? A movie about the Second World War will be completely different if told through the eyes of a German from the story told through those of a Dutchman, and this of course immediately affects many of the story elements.

To tell or write a good story is a very complex task, but it forces us to think about the content, the coherence, and the action in which the story must evolve. Narratology not only distinguishes and analyses the different elements of a story, but it may also be used to create better and more effective stories.

Dietiker (2015b) argues that every story element (character, action, actor, setting, plot) from the story has its mathematical counterpart (see **Figure 1**), thus that we are able to speak of mathematical characters, mathematical action, mathematical actor, mathematical setting and mathematical plot. These elements may now be used to create a mathematical story.

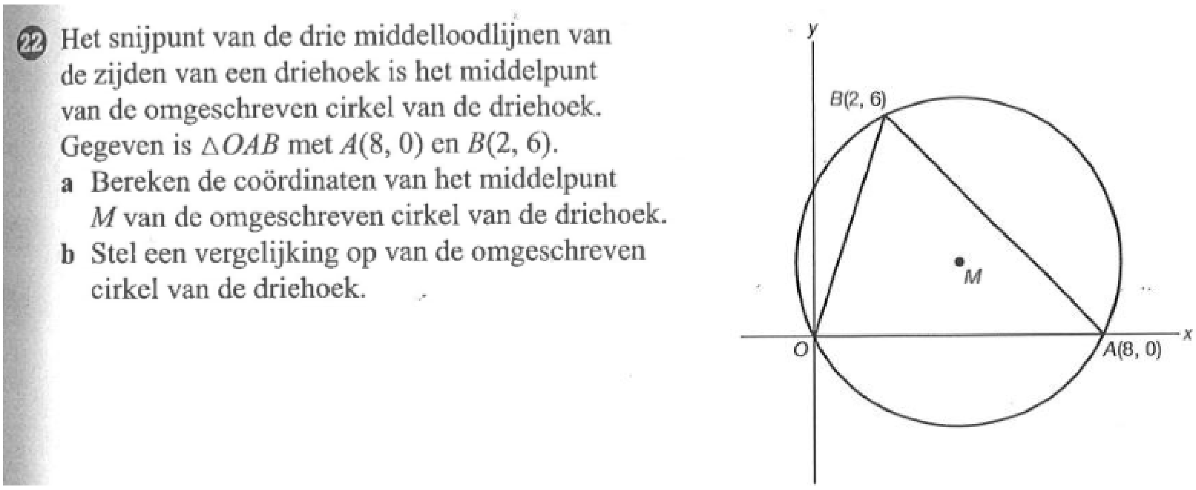
Dietiker uses narratology in order to create a mathematical sequence of events (a mathematical story) "that focuses attention to [*sic*] how the content is slowly (or quickly) 'revealed' or 'obscured' for students." To do this, a mathematical object is seen as a character. This implies that Dietiker's story stays *within* mathematics itself. By that we mean, for example, that a number (which is considered a mathematical character) remains a number. This number itself will be part of the mathematical action. It is very hard, however, to identify oneself with the number 9 for example.

We believe that the story elements as described in **Figure 1** can be used as a design heuristic by which to create a different kind of mathematical story that is meaningful and engaging for students. To do so, our mathematical characters become real characters. Our story, therefore, becomes a story *outside* mathematics. The mathematical characters, the setting, the actors, the actions, and the plot are all taken into the narrative mode. We do still call these elements mathematical because every character, every action in the narrative mode, represents a character or action in the mathematical mode.

Like *The Story of Tan* (Schiro, 2004), our story is about a mathematical problem, which is "more meaningful when placed in the context of a fantasy situation" (Balakrishnan, 2008). After the story has been told the students encounter the same problem as the protagonist. This implies that the success of the protagonist is directly linked to their own success.

MATHEMATICAL STORY 1

As an example of how an ordinary exercise can be transformed into a story, consider the following exercise, which is taken from a Mathematics B textbook (12th grade). It says that the intersection point of the three bisectors of AB, BC, and AC is the midpoint M of the circumcircle of the triangle ABC.



22 Het snijpunt van de drie middelloodlijnen van de zijden van een driehoek is het middelpunt van de omgeschreven cirkel van de driehoek.
Gegeven is $\triangle OAB$ met $A(8, 0)$ en $B(2, 6)$.

a Bereken de coördinaten van het middelpunt M van de omgeschreven cirkel van de driehoek.

b Stel een vergelijking op van de omgeschreven cirkel van de driehoek.

Figure 2. Getal en Ruimte, vwo B part 4, 11th edition 2017

The circle with midpoint M is given, but the coordinates of M and the radius of the circle are unknown (see **Figure 2**). Three points, A, B, and C, which are on the circle and whose coordinates are known, are given. Students are first asked to calculate the coordinates of midpoint M, and secondly to give the equation of the circle, which can be written as

$$(x - a)^2 + (y - b)^2 = r^2$$

where a and b are taken from the first question and the radius r can be derived by applying the Pythagorean theorem.

First, note that since the circle is the circumscribed circle of the triangle ABC, M is the intersection point of the bisectors of the line segments AB, BC and AC. The problem can be solved via vector representation, or via the representation $y = ax + b$ (or the form $ax + by = c$). After either the vectors or the formulas of two of the three lines have been derived, the corresponding bisectors can easily be found because the inner product of two straight lines must be equal to zero (and in case of the formulas, the multiplication of the slopes must be equal to -1). By calculating the intersection point from two bisectors, the coordinates of M are found. If M is known, the radius of the circle can be derived with the Pythagorean theorem, since we can easily construct a right-angled triangle MM'A where M' is the projection of M on the x axis.

Of course, this problem can be presented as has been done in the textbook. It is a properly abstract problem, which takes only a few steps to solve. Instead, one can also consider the following story, in which mathematical objects are personified and some action or events are being used in order to design a plot. If also a setting is introduced and a narrator (in this case the teacher) then a mathematical story can be created. Note that we used the same elements as Dietiker, but since our characters are not mathematical objects but real persons, the context of our story lies *outside* mathematics (in our case a dating show) instead of *inside* mathematics, as in the case of Dietiker.

> *I have a cousin named John (a few pictures of John can be shown on the digiboard). John is around my age, a very introverted but intelligent guy with a very good job. John is very happy, (he loves his job, he likes doing sports and is keen on challenges) except for one thing: he would like to be in a relationship. Every family party his aunts and uncles ask him if he 'has met someone yet'. And every time he must say no. This makes him a bit sad. All his friends have children, and he does not even have a partner. The problem is in fact that he does not know well how to talk to strangers, even though he is very social.*
>
> *Lately I heard of a new television dating show in which someone is dropped in the middle of a tropical island. Three other participants, potential future partners who have been chosen carefully to make a good fit, will arrive at the island by boat. The last 100 meters they have to swim to the shore. They are not bringing don't bring a lot of luggage, but they each do have a global positioning system. The island is a dense jungle. When they arrive at the beach, their coordinates are known and will be briefed to the candidate, who is in the middle. The candidate in the middle of the island doesn't know his or her exact position, only that it is the exact middle. The island is a perfect circle as you can see on the map. Before the candidates can meet and spend two days together to get to know each other, they must find the person in the middle. Therefore, this person needs to calculate his exact coordinates, which are then phoned to the three participants at the beach. They are only allowed to enter the jungle when his exact position is known, otherwise it is too dangerous -- one can easily get lost in the dense jungle.*

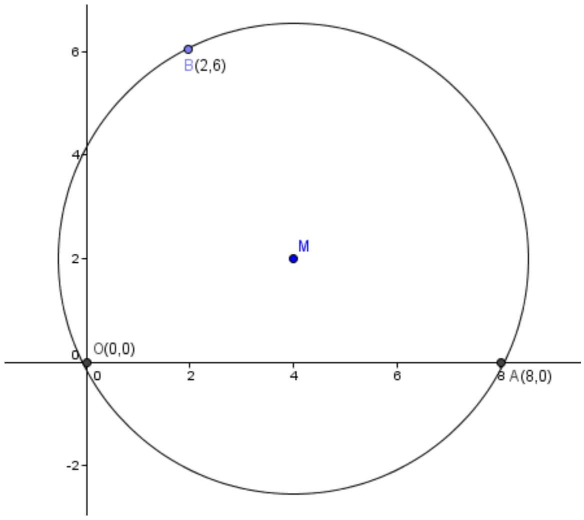


Figure 3. Island

> *Last month, I discussed the program with my cousin. He was really excited and agreed to join, although he was bit nervous about being on national television. So, I wrote a letter to the television show and after some interviews it is actually going to happen. John will be the first candidate. This implies that he will meet three women of his own age. Only one problem remains, that is, how to calculate the exact coordinates of the middle of a circle if the coordinates of the others are known. Can you help us? It seems a challenging mathematical problem. As an example, we try to calculate the midpoint M of the circle in the following situation (see* **Figure 3***).*

In this case mathematical meaning is given via the dating show context, which is introduced as a story. Because this subject relates to their social lives, students will probably be more motivated to solve the problem. Because the mathematical objects are personified, as well as the circle, now an island with a dense jungle, which can easily be imagined, students understand the emotions involved and are able to talk about actual persons and actual dangers on the island instead of about mathematical objects. The plot in this example is about cousin John, who wants to be in a relationship. Instead of a bare mathematical problem, students are more affected because they feel sorry for him and want him to meet the three women to see if it clicks between him and one of them.

The conflict in this story is that until now John has not succeeded in getting into a steady relationship. But this problem is now in the hands of the students -- they themselves can influence the end of the story. In order to do so, they can discuss the problem outside mathematics. For example, they understand that person A is able to walk in the direction of person B, in a straight line, or on the beach. The midpoint of the straight line can also be derived easily.

Note that our story possesses Bruner's four characteristics of narratives, as given in section 3. The connection between the exceptional and the ordinary, for example, can be explained by the difference between regular dating (ordinary) and dating on a television show (exceptional). The fact that John is still single can be interpreted as the dramatic quality of the story.

THE MODERN BEGINNING OF NARRATIVE STUDIES: VLADIMIR PROPP

We have given a design heuristic by which to create a mathematical story based on the five elements of a story (character, action, actor, setting, and plot). The main idea here is to personify mathematical objects and create a context as a setting that relates the story to students' lives and interests. However, as in Egan's heuristic, as presented in the introduction, it is still not clear how to organise the content into story form. That is, only some basic elements are given, but that does not tell one how to create a corresponding story -- which therefore still depends on the creativity and the imagination of the educator.

In the following we will give a second design heuristic. This heuristic can be interpreted as an extension of the previous one, that is, it also contains the five story elements depicted in **Figure 1**, but also much more. Hence, the story that will be designed or redesigned with this heuristic becomes much more detailed and complex.

According to Bruner (2002), the modern beginning of narrative studies is in the work of the Russian folklorist Vladimir Propp, more specifically his *The Morphology of the Folktale*. Propp (1895-1970) was a Russian scholar who broke Russian fairytales and folktales into little sections, just as has been done in linguistics, where a similar process is called morphology. Morphology, which comes from Greek and means 'study of shape', analyzes the structure of words and parts of words, such as stems, root words, prefixes, and suffixes, and dates back to the ancient Indian linguist Pāṇini, who formulated the 3,959 rules of Sanskrit morphology. In the same vein, Propp (Propp, 1928) analyzed over 100 Russian folktales. He found that 31 irreducible narrative elements, called *functions* or *narratemes*, can be distinguished. Every single tale consists of a number of narratemes. It must be noted that not every narrateme has to be used in every tale, and sometimes a narrateme can be used several times. If every story is made up from a number of narratemes, one could easily invert this condition and use the narratemes as the structure for an engaging story.

The 31 narratemes Propp came up with are divided into seven groups: preparation, complication, transference, struggle, return, and recognition. In Appendix A only the first and the last group are presented. Of course this is just a summary[1]; see Propp (1928) for the detailed version.[2]

DESIGN HEURISTIC BASED ON PROPP'S MORPHOLOGY OF THE FOLKTALE

Propp has given each function a letter. The first function, absentation - one of the members of a family absents himself from home - is designated by β. There are three possible options that satisfy this function, indicated by a superscript, β^i, where $i = 1,2,3$. In this specific example of absentation, β^1 means that the person absenting himself can be a member of the older generation; β^2represents an intensified form of absentation, i.e., the death of parents; and finally β^3 implies that a member of the younger generation leaves home. In the same vein, if the villain causes harm or injury to a member of a family, this is designated by the letter A. Of course there are several ways to harm or injure someone. Therefore, again several possibilities are distinguished: A^1 indicates the villain abducting a person, and A^7 the villain causing a sudden disappearance. Propp now argues that every tale can be written as a sequence of events, or symbols, for example:

$$B^4C \uparrow D^1E^1H^1J^1K^4 \downarrow F^2$$

where ↑ represents a departure and ↓ a return. This sequence of symbols reminds us of the formalist view in the philosophy of mathematics. According to Hilbert, a proof is nothing more than a sequence of symbols, manipulated according to some rules. In the case of Propp, narrativity, or the narrative mode, is transformed into the paradigmatical mode. It is this sequence of events that is important in both mathematics and narratives (Dietiker, 2015a). These functions can now be used in order to design a mathematical story which is both meaningful and engaging.

In the tales analyzed by Propp, eight common characters are distinguished: the *hero*, the *false hero*, the *villain*, the *helper*, the *donor* (who gives the hero something special to help with the quest), the *dispatcher* (the character who sends the hero on a mission), the *princess* (object sought by hero, or reward), and the *princess*'s father (who sets the hero a task, and can be protective of daughter). Since every story contains the basic story elements listed in **Figure 1**, they are also a key element in our invented story (as all folktales).

Before we exemplify Propp's formalist analysis by a mathematical story, we would first like to stress three points:

- Not every function appears in each tale. This implies that by constructing a story we can select some of them.
- Not everything in the tale must be true. It is allowed to use imaginary characters, for example, if they make the story more credible, or more beautiful.
- Not every mathematical story has to end with a wedding. Instead, something comparable to such an event can be chosen, in the mathematical setting this could for example be the final recognition of a proof.

The beautiful part of telling stories is that we are able use our imagination. An important feature of a good story is a certain amount of vividness (Gowers, 2012). As far as we are concerned, everything is allowed if it helps to create a vivid story.

It may sound strange to design or redesign a mathematical story based on the morphology of folktales. However, we believe that the stories of Kepler's conjecture, as well as that of Fermat's last theorem, which was proven by Andrew Wiles, can be interpreted as modern fairytales. The same holds for Perelman's proof of the Poincaré conjecture, and so on. Therefore we think that Propp's narratemes can be fruitful in this respect.

In order to demonstrate how Propp's narratemes can be used, let us reconstruct our thoughts. Last year we encountered the following example in the mathematics textbook.

Five circles with equal radius are drawn, and the corner points of the square are the exact midpoints of the four outer circles (see **Figure 4**). The question is: which part of the square is coloured blue?

[1] This is the best summary we could find for our purpose. Taken from the English Wikipedia lemma 'Vladimir Propp'.
[2] The functions can also be found at: http://homes.di.unimi.it/~alberti/Mm10/doc/propp.pdf

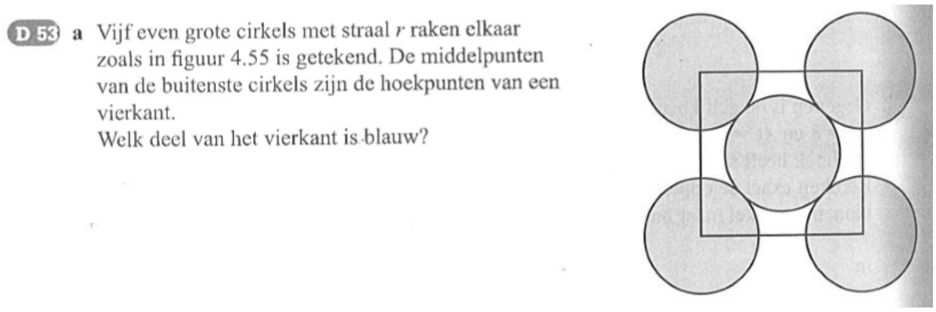

D 53 a Vijf even grote cirkels met straal r raken elkaar zoals in figuur 4.55 is getekend. De middelpunten van de buitenste cirkels zijn de hoekpunten van een vierkant.
Welk deel van het vierkant is blauw?

Figure 4. Getal en Ruimte, vwo B part 2, 11th edition 2017

Our direct association with this exercise was Kepler's conjecture. Kepler saw a stack of oranges in a market, and published a conjecture in 1611 about sphere packing in three-dimensional Euclidean space. It says that no arrangement of equally sized spheres filling space has a greater average density than that of the cubic (face-centered cubic) and hexagonal close packing arrangements. The density of these arrangements is around 74.05%.

Of course, the above exercise is not about three-dimensional space. But the story about Kepler's conjecture can be deployed in several ways. First of all, we can tell the students something about the history of mathematics. Second, we can show what kind of problems mathematicians are currently working on. Third, we can focus on the discipline and the inventiveness needed to obtain a result (in this case a mathematical proof). Fourth, the process of how to get an article published becomes clear. Finally, we can show how exciting mathematics and the mathematical community can be.

We will explain below how we invented an engaging and meaningful story. It would be even better to devote an entire lesson to these kinds of mathematical problems, or a series of lessons, in which attention could also be paid to some three-dimensional problems in order to make sense of Kepler's conjecture. Propp's narratemes make it easy to vary, so that different stories can be told. The details and the vividness of the stories can be filled in as one likes.

The story about the proof of Kepler's conjecture has two protagonists: Kepler himself and the man who succeeded in proving the conjecture: Thomas Hales. Note that Kepler, besides being a mathematician, astronomer, physicist, and astrologist, also wrote science fiction stories. This fact can be used to evolve a story about machines that can count faster than human beings, which appeared in one of his dreams. The story would then be the story that Kepler himself wrote after the revelation in his dream. Although there are obviously good reasons to tell the story from the perspective of Kepler himself, we have chosen to tell this story from the perspective of the mathematician that eventually, more than 400 years later, proved the long-standing conjecture.

MATHEMATICAL STORY 2

In this section we describe a story on Kepler, in which 17 narratemes are being used.

Initial situation, absentation (1): Thomas Hales (1958), an American mathematician in his mid-thirties, is walking in a market with his wife. He sees a stack of oranges and tries to remember whether something mathematical was involved.

Interdiction (2): At home he looks at internet and sees that sphere packing is in fact Kepler's conjecture, stated in 1611. He gets really excited: he found a very old, unproven conjecture and dreams about solving it. He tells his wife. She asks him why he thinks he will be able to solve a 400-year old conjecture. She forbids him to work on the problem, because she is afraid that he is going to be distracted. On the internet he finds an article by László Fejes Tóth (1915–2005) from 1953, in which Tóth shows that Kepler's conjecture can be reduced to a finite number of calculations, instead of an infinite number (Wiedijk, 2016).

Violation of interdiction (3): Secretly Hales starts to work on this big project, and after a few weeks he believes his goal on earth was to prove the conjecture. He tells his wife that he will continue. The first hurdle is taken.

Villainy (8): After some minor results Hales explains to the Dean of his faculty that he intends to continue his research on Kepler. But the dean doesn't agree; in fact, he forbids Hales to continue, because other things are more urgent. Hales does not listen and reads everything he can lay his hands on concerning the proof. Because he's doing this openly and gives lectures about the subject at the faculty, the Dean calls Hales's wife and argues that she must convince him to stop his research. Even some of his colleagues want him to focus on other things. They are also a bit jealous -- what if he eventually does solve the problem?

Beginning counteraction (10): Hales continues his research and develops some ideas that can help him to solve the problem. He writes them down and systemizes the ideas. Then he organizes a meeting with his Ph.D. students and explains what he is up to.

First function of the donor (12): A few days later, one of his Ph.D. students comes to see him and tells him that he could help. The name of this student is Samuel Ferguson. He explains to Thomas that if Thomas is able to solve a specific problem, Samuel will be able to write a program to check a lot of different cases.

Hero's reaction (13): His Ph.D. student's suggestion looks very hard to Thomas. Now he also begins to understand what kind of trouble he has got himself into. For the first time he sees that this is going to be a huge project, which is going to take a lot of time. He thinks of all the complaints of his boss and his colleagues. After a short depression, he decides to continue the work.

Receipt of a mathematical agent (14): Thomas has to review a paper. This paper gives him an idea, and he is working all day and night to transfer the ideas in the paper to his own problem.

Struggle (16): The Dean has heard that Thomas is continuing with Kepler's conjecture and makes an appointment, in which he makes clear that Thomas has to give up his work.

Victory (17): Together with his Ph.D. student he finds a proof of Kepler's conjecture in August 1998. It is based on a lot of computer computations. The results are split over two articles, which he sends to the Annals of Mathematics (the theoretical part) and to Discrete and Computational Geometry (computer part). They are probably going to be accepted, and he shows a preprint to the Dean -- who does not know what to say. There is a good chance that one of his employees solved a 400-year old problem.

Unfounded claims (24): Since the proof is a 300-page monster, it takes 12 reviewers four years to check for errors. Years later, they finally accept it, that is, it is going to be published, but in 2005 the reviewers can say only that they were "99 per cent certain" the proof was correct. That is, they believe it to be true, but they are not fully sure.

Difficult task (25): Because it has taken so long Thomas thinks there was a sort of mistake in the proof. So, in 2003 he starts another project: an effort to vindicate his proof through formal verification by computer.

Solution (25): Then, finally, in August 2014, Thomas claims that with the help of others, he has verified that his 2005 proof is indeed correct. There is no longer any doubt. He and his team have solved the 400-year-old conjecture.

Exposure (28): Hales gets into a conflict with the Dean after writing a devastating article about his Kepler research and the role of the Dean.

Transfiguration (29): Thomas is invited all over the world to speak about his solution and about computer proofs in general.

Punishment (30): The Dean gets fired.

Wedding (31): In March 2106, Thomas receives an email from a young, female Ukrainian mathematician, Maryna Viazovska (1984), who claims to have found a proof for the stacking problem in dimension 8. Only a week later, together with some others she published a proof in dimension 24.

Several variants of this story can now be told. For example, the initial situation (1) can also be the death of one the protagonist's parents, or an accident. Because of this tragic happening, the second function, the interdiction (2), can be inverted: Thomas receives an order, a suggestion, or a sign to solve this particular conjecture, because he now understands that life is short and he should focus on important matters.

Stories about mathematics, such as the story of the proof of Kepler's conjecture, can be found everywhere, and these kinds of stories can easily be redesigned based on Propp's narratemes.

These narratemes can also be used to design a story from scratch. In that case, one could for example start with the first step of Egan's heuristic, namely with questions such as: What is most important about this topic? Why should it matter to children? And what is affectively engaging about it? One could then invent some characters and a setting. Then the narratemes come in.

After the story has been told, several options are possible. For example, since the story is about how balls fit into a certain shape, it would be nice to give some 2D and 3D exercises that calculate the percentages filled. For example, given a quarter of a circle with radius equal to 8. First calculate the radius of the smallest circle, and second, calculate the percentages of the area of the purple circle compared to that of the blue quarter circle (see **Figure 5**).

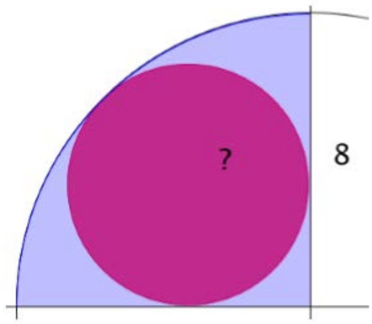


Figure 5. Calculate the radius of the inner circle

Of course, one can also use this story to introduce the Riemann sum, which basically is about how many rectangles fit within a certain area. Another idea is to discuss what a mathematical proof is, and whether or not a computer proof can be interpreted as a real formal proof.

CONCLUSIONS

Stories in (mathematics) educations can be used to engage students and to make the subject more meaningful, by relating it to students' lives and interests, by situating the subject matter historically, culturally, socially or personally, that is, to humanize the content, and by increasing affective meaning. However, there still remains a gap between the theory and how to apply this theory in the classroom. It probably will not be easy for most mathematics teachers to invent an interesting story that corresponds to the mathematical content. In this paper we have tried to bridge this gap by presenting two design heuristics. The value of this approach is that it provides mathematical teachers with a detailed story framework that is suitable for designing or redesigning mathematical stories.

The first design heuristic is based on Dietiker's mathematical framework (2015b). Dietiker uses story elements to improve the mathematical story *within* mathematics, i.e., the mathematical sequences. "While carefully crafting sequences of mathematical tasks, the mathematical story framework focuses attention to how the content is slowly (or quickly) 'revealed' or 'obscured' for students." We present a design heuristic in which the mathematical objects, or characters, are personified in an imaginative setting. Since a plot, actors, and action are also added, we are able to speak of a mathematical story, one which occurs, unlike Dietiker's, outside mathematics. We demonstrated the design heuristic by means of a mathematical story (12th grade).

Because we do not want to tell the same kind of stories over and over again, it is necessary to obtain a range of different design heuristics. Therefore, we have presented a second heuristic design based on the work of Russian formalist Vladimir Propp, who analyzed over 100 Russian folktales. He found that 31 irreducible narrative elements, called functions or narratemes, can be distinguished. We have used the reverse path, that is, we did not analyse an existing story, but have tried to show by means of an example how a mathematical story can be designed or redesigned with the help of these narratemes. The story, concerning the proof of Kepler's conjecture, sheds some light on how the mathematical community works and what is needed to solve a long-standing problem.

Future research should focus on finding more design heuristics for transforming mathematical content into mathematical stories. The first results are promising, but more research should be done to explore whether mathematics teachers are sufficiently capable of designing effective stories with the help of the design heuristics a presented in this paper. Are they capable of redesigning their classroom practice? What kind of difficulties do they experience, and how can these be solved? Furthermore, it would be necessary to explore the effects of storytelling on students' performance and motivation for learning mathematics. This can be achieved by means of an educational design research, which we will carry out in the coming two years. We will design and test two lesson series, measuring students' motivation by means of a questionnaire, and their understanding of mathematics by means of interviews and thinking out loud. Two central research questions in this study will be: what are the characteristics of meaningful and engaging mathematics education based on storytelling, and how can mathematics teachers transform their classroom practice by using narratives?

REFERENCES

Adler, J. M. (2008). Two Modes of Thought: The Narrative/Paradigmatic Disconnect in the Bailey Book Controversy, *Arch. Sex. Behav., 37*, 422–425. doi:10.1007/s10508-008-9318-0

Albool, R. M. (2013a). The storytelling strategy in teaching mathematics may increase students' abilities to solve high order questions and word problems. *5th International Conference on Education and New Learning Technologies, Barcelona, Spain 1-3 July, 2013, EDULEARN13 Proceedings*

Albool, R. M. (2013b). The Effect of Utilizing Storytelling Strategy in Teaching Mathematics on Grade Four Students' Achievement and Motivation towards Learning Mathematics. *International Conference the future of education second edition. New perspective in science education, 14-15 march 2013.*

Bal, M. (2009). *Narratology: Introduction to the theory of narrative (3rd ed.)*. Toronto: University of Toronto Press.

Balakrishnan, C. (2008). *Teaching secondary school mathematics through storytelling* (Doctoral dissertation). Fraser University.

Boström, A. (2006). *Sharing lived experience: How upper secondary school chemistry teachers and students use narratives to make chemistry more meaningful* (Doctoral dissertation). Stockholm University, The Stockholm Institute of Education, Department of Curriculum Studies and Communication (LHS).

Brendel, D. H. (2000). Philosophy of mind in the clinic: The relation between causal and meaningful explanation in psychiatry. *Harvard Review of Psychiatry, 8*, 184–191.

Bruner, J. (1986). *Actual minds, possible worlds*. Cambridge, MA: Harvard University Press.

Bruner, J. S. (2002). *Making stories, law, literature, life*. New York: Farrer, Straus and Giroux.

Burton, L. (2003). Children's Mathematical Narratives as Learning Stories. In: B. van Oers (Eds). Narratives of childhood. *Theoretical and practical explorations for the innovation of early childhood education* (pp. 51-67). Amsterdam: Vrije Universiteit.

Dietiker, L. (2015a). A Mathematical story: a metaphor for mathematics curriculum. *Educational Studies in Mathematics November 2015, 90*(3), 285–302. doi:10.1007/s10649-015-9627-x

Dietiker, L. (2012). *The Mathematics Textbook as a Story: A Novel Approach to the Interrogation of Mathematics Curriculum* (Doctoral Dissertation). Michigan State University

Dietiker, L. (2013). Mathematical texts as narrative: Rethinking curriculum. *For the Learning of Mathematics, 33*(3), 14-19.

Dietiker, L. (2015b). Shaping Mathematics into Compelling Stories, Educational Designer. *Journal of the international society for design and development in education, 2*(8).

Doxiadis, A. (2003). Embedding Mathematics in the Soul: Narrative As a Force in Mathematics Education. Opening address to the *Third Mediterranean Conference of Mathematics Education, Athens, Greece, January 3, 2003*. Retrieved from www.apostolosdioxiadis.com

Doxiadis, A., & Mazur, B. (2012). *Circles Disturbed: The Interplay of Mathematics and Narrative*. New Jersey, USA: Princeton University Press.

Drake, C. (2006). Using mathematics life stories to understand the implementation of mathematics education reform. *Journal of Mathematics Teacher Education, 9*, 579-608. doi:10.1007/s10857-006-9021-9

Egan, K. (1988). *Teaching as storytelling: An alternative approach to teaching and curriculum*. London, UK: Routledge.

Freudenthal, H. (1991). *Revisiting Mathematics Education, China Lectures*. Dordrecht: Kluwer Academic Publishers.

Gowers, T. (2012). Vividness in Mathematics and Narrative. In Doxiadis, A., Mazur B., *Circles Disturbed: The Interplay of Mathematics and Narrative*, New Jersey, USA: Princeton University Press.

Healy, L., & Sinclair, N. (2007. If this is our mathematics, what are our stories? *Int. J. Comput. Math. Learning, 12*, 3-21, doi:10.1007/s10758-006-9109-4

Hersh, R. (1997). *What is mathematics really?* Oxford, UK: Oxford University Press.

Hobbs, L., & Davis, R. (2013). Narrative pedagogies in science, mathematics and technology. *Research in Science Education, 13*(3), 1289-1305. doi:10.1007/s11165-012-9302-5

Kaasila, R., Hannula, M. S., Laine, A., & Pekhonen, E. (2006). Autobiographical Narratives, Identity and View of Mathematics, in European research in Mathematics Education IV. *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education*, Universitat Ramon Llull, Barcelona, p. 215-224.

Propp, V. (1928). *Morphology of the Folktale*. Trans., Laurence Scott. 2nd ed. Austin: University of Texas Press, 1968.

Schiro, M. (2004). *Oral storytelling and teaching mathematics: Pedagogical and multicultural perspectives*. Thousand Oaks, CA: Sage Publications.

Soto y Koelemeijer, G. (2008). *Welke problemen kunnen zwakke leerlingen ondervinden als gevolg van de structuur van het wiskundeboek?* (Unpublished master thesis). ICLON-Leiden

Weinberg, A., Wiesner, E., & Fukawa-Connelly, T. (2016). Mathematics lectures as narratives: Insights from network graph methodology. *Educational Studies in Mathematics*, *91*(2), 203-226. doi:10.1007/s10649-015-9663-6

Wiedijk, F., Geuvers, H., & Urban, J. (2016). Een wiskundig bewijs correct bewezen, De meest efficiënte manier om bollen op te stapelen. *NAW*, *5*(17), nr. 3.

Zazkis, R., & Liljedahl, P. (2009). *Teaching Mathematics as Storytelling*. Rotterdam, The Netherlands: Sense Publishers.

APPENDIX A: SOME OF PROPP'S NARRATEMES

Preparation	Recognition
1. ABSENTATION: A member of the hero's community or family leaves the security of the home environment. This may be the hero himself [*sic*], or some other relation that the hero must later rescue. This division of the cohesive family injects initial tension into the storyline. This may serve as the hero's introduction, typically portraying them as an ordinary person. 2. INTERDICTION: A forbidding edict or command is passed upon the hero ('don't go there', 'don't do this'). The hero is warned against some action. 3. VIOLATION of INTERDICTION. The prior rule is violated. Therefore, the hero did not listen to the command or forbidding edict. Whether committed by the Hero by accident or temper, a third party or a foe, this generally leads to negative consequences. The villain enters the story via this event, although not necessarily confronting the hero. They may be a lurking and manipulative presence, or might act against the hero's family in his absence. 4. RECONNAISSANCE: The villain makes [*sic*] an effort to attain knowledge needed to fulfill their plot. Disguises are often invoked as the villain actively probes for information, perhaps for a valuable item or to abduct someone. They may speak with a family member who innocently divulges a crucial insight. The villain may also seek out the hero in their reconnaissance, perhaps to gauge their strengths in response to learning of their special nature. 5. DELIVERY: The villain succeeds at recon and gains a lead on their intended victim. A map is often involved in some level of the event. 6. TRICKERY: The villain attempts to deceive the victim to acquire something valuable. They press further, aiming to con the protagonists and earn their trust. Sometimes the villain makes little or no deception and instead ransoms one valuable thing for another. 7. COMPLICITY: The victim is fooled or forced to concede and unwittingly or unwillingly helps the villain, who is now free to access somewhere previously off-limits, like the privacy of the hero's home or a treasure vault, acting without restraint in their ploy.	27. RECOGNITION: The hero is given due recognition – usually by means of their prior branding. 28. EXPOSURE: The false hero and/or villain is exposed to all and sundry. 29. TRANSFIGURATION: The hero gains a new appearance. This may reflect aging and/or the benefits of labor [*sic*] and health, or it may constitute a magical remembering after a limb or digit was lost (as a part of the branding or from failing a trial). Regardless, it serves to improve their looks. 30. PUNISHMENT: The villain suffers the consequences of their actions, perhaps at the hands of the hero, the avenged victims, or as a direct result of their own ploy. 31. WEDDING: The hero marries and is rewarded or promoted by the family or community, typically ascending to a throne.