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Pre-service K-8 Teachers' Professional Noticing and Strategy Evaluation Skills: An Exploratory Study

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ABSTRACT

This study sheds light on three teaching competencies: Pre-service teachers' (PSTs') professional noticing of student mathematical reasoning and strategies, their ability to assess the validity of student reasoning and strategies, and to select student strategy for class discussion. Our results reveal that PSTs with strong awareness of mathematically significant aspects of student reasoning and strategies (*focused* noticing) were better positioned to assess the validity of student reasoning and strategies. PSTs with higher strategy evaluation skills were more likely to choose the strategy to engage class in *justification* or to *advance students' conceptual understanding* compared to PSTs with low strategy evaluation skills.

Keywords: pre-service teacher education, professional noticing of student mathematical reasoning and strategies, assessment of the validity of student reasoning and strategies, pre-service teachers' pedagogical decisions

INTRODUCTION

Our study examines how pre-service teachers (PSTs) who are preparing to teach grades 1-8 mathematics attend to and interpret student thinking about numbers and operations and how they respond to student strategies. With a focus on mathematically significant aspects of student strategies, we examined the relationship between PSTs' professional noticing skills and their ability to evaluate students' arguments and strategies. We also explored how PSTs motivate their decisions of selecting student strategy for a class discussion planned to support students' reasoning skills.

It is widely accepted that mathematical reasoning provides the foundation for and supports learning of mathematics with understanding (Hiebert et al., 1997; Stiff, 1999; Thompson & Schultz-Farrel, 2008). However, research documents that giving mathematical reasons and justifying mathematical procedures is difficult for many students (e.g., Healy & Hoyles, 2000; Reiss, Klieme, & Heinze, 2001). To support the development of student reasoning skills by pressing students to formulate conjectures and provide explanations as to *why* their conjectures or strategies are valid, PSTs need the ability to critically examine mathematical thinking of their students and assess the validity of their students' reasoning and strategies. They need to be able to provide a rationale for whether or not a student-generated statement or strategy is true, recognize situations in which it is appropriate to use counterexamples to challenge a statement, and recognize whether or not a specific property or conjecture is generalizable. They also need the ability to make instructional decisions (i.e., select appropriate strategies, examples, tasks) that can effectively build on their students' thinking. Focusing on student reasoning and strategies is considered a key aspect of effective teaching practice (Steinberg, Empson, & Carpenter, 2004; Walshaw & Anthony, 2008).

Across the world, mathematics education standards place great emphasis on learning mathematics as a sense making discipline and emphasize the development of student reasoning skills–the ability to generate and critique mathematical arguments (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2011; Brodie, 2010; Li & Lappan, 2014; National Council of Teachers of Mathematics, [NCTM], 2000, 2014; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010). Focusing PSTs' attention on the validity of mathematical arguments and student-generated strategies is an important aspect

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Contribution of this paper to the literature

- With a focus on K-8 pre-service teachers this research examined the relationship among pre-service teachers' professional noticing skills, ability to evaluate student reasoning and strategies, and ability to select student strategy for class discussion.
- The results revealed a positive relationship between pre-service teachers' professional noticing of mathematically significant aspects of student reasoning and strategies, and their strategy evaluation skills.
- Pre-service teachers' pedagogical decisions related to selecting student strategy for class discussion were significantly correlated to their assessment of student strategies.

of teacher preparation. Noticing mathematically important features of student thinking and evaluating and making sense of student arguments and strategies, are important teaching competencies. Neither is easy for PSTs to develop as documented in the mathematics education literature (e.g., Fernández, Llinares, & Valls, 2013; Martin & Harel, 1989). Moreover, while there is an agreement in the field that teaching is a highly complex activity and teacher knowledge and practice are multifaceted (e.g., Ball, Thames, & Phelps, 2008; Ingersoll & Perda, 2008; Shulman, 1987), teaching competencies are frequently studied in isolation from one another. Teachers' competency in evaluating student arguments and strategies (e.g., Crespo, 2000; Monoyiou, Xistouri, & Philippou, 2006; Morris, 2007; Salinas, 2009), teacher noticing skills (e.g., Ding & Domínguez, 2016; Fernández et al., 2013; Roller, 2016; Sánchez-Matamoros, Fernández, & Llinares, 2015; Schack et al., 2013; van den Kieboom, Magiera, & Moyer, 2017), teacher skills and ability to plan classroom discussions linked to student reasoning and strategies (e.g., Larsson & Ryve, 2011; Meikle, 2014; Tyminski, Zambak, Drake, & Land, 2014).

Given the complexity of teacher knowledge and practice, central within the mathematics education community is the question of effective preparation of PSTs. Because teacher preparation programs cannot support every skill and competency PSTs might need, finding ways to integrate the essential knowledge and practice is indispensable in effort to help PSTs develop needed professional knowledge and skills (Castro, 2004). Effective teacher preparation requires then strong understanding of ways in which different forms of teacher knowledge and skills might connect and possibly support one another. While expert teachers might intuitively recognize opportunities for building on student thinking and reasoning (Berliner, 2001; Peterson & Leatham, 2009; Stockero & van Zoest, 2013), PSTs need more targeted opportunities for making sense of student thinking and for developing teaching competencies that support purposeful teaching. In this paper, we report on our research in which, with a focus on grades 1-8 PSTs, we examined the relationship between PSTs' competencies in (1) noticing mathematically significant aspects of student reasoning and strategies, (2) evaluating student strategies, and (3) selecting student strategies for class discussion. To provide more robust understanding of the relationship among these teaching competencies we situated our research in the same mathematical domain which includes reasoning about operations with fractions. The following questions guided our investigation:

1) Is there a relationship between PSTs' professional noticing skills and their ability to evaluate student strategies? If so, what is the nature of this relationship?
2) How do PSTs motivate their selection of student strategies for class discussion, and to what extent do their strategy selection relates to PSTs' strategy evaluation skills?

Professional Noticing of Students' Mathematical Thinking in the Context of Analyzing Student Reasoning and Strategies

In the mathematics teacher education literature, the competence of *teacher professional noticing* is broadly described. Teacher professional noticing skills draw on the ability to attend to relevant aspects of teaching situation (e.g., classroom events), to interpret observations to establish connections between teaching and learning, and to make instructional decisions on the basis of observations and interpretations (e.g., Jacobs, Lamb, & Philipp, 2010; Mason, 2002; Sherin & van Es, 2009; Star & Strickland, 2008). One specific aspect of teacher professional noticing is the ability to notice students' mathematical thinking. Llinares (2013) states that the skill of noticing students' mathematical thinking goes beyond the mere recognition of the correctness of students' responses. Teachers who notice students' mathematical thinking are able to recognize whether or not students' answers or reasoning are meaningful.

In our research we interpret PSTs' professional noticing of students' mathematical thinking drawing on Jacobs, Lamb, and Philipp's (2010) description of connected practices: (1) *attending* to mathematical details in students' thinking and strategies, (2) *interpreting* students' mathematical understanding and reasoning with a focus on details of the specific strategies, and (3) *responding* to students in a way that builds on students' thinking, understanding and strategies. Jacobs and colleagues use these practices to define the construct of *Teacher Professional Noticing of Students' Mathematical Thinking,* henceforth in this paper, *Professional Noticing*. Phillipp, Fredenberg and Hawthorn

(2017) described that the three practices are "highly interrelated and often occur seemingly simultaneously" (Phillipp et al., 2017, p. 114). Specifically, the practices of attending to and interpreting student thinking might be viewed as inseparable. Teachers might attend to (focus on) a specific aspect of student thinking and simultaneously interpret it as they prepare their response (Bautista, Brizuela, Glennie, & Caddle, 2014). In our work with PSTs then, we view *Professional Noticing* in terms of two, rather than three abilities: (a) attending to *and* interpreting mathematically significant aspects of student reasoning and strategies and (b) responding to students in a way that connects to students' mathematical thinking and ideas (e.g., conceptions, misconceptions, or strategies, as revealed in student work).

Drawing on Jacobs et al. (2010), we operationalize *attending* and *interpreting* in terms of descriptive accounts of specific aspects of student work which PSTs identify and highlight as mathematically significant, and ways in which they understand student reasoning and strategies. Interested in discerning *what* aspects of students' reasoning and strategies PSTs perceive as mathematically significant and *how* they understand student reasoning and strategies, like Jacobs et al. (2010) and Mason (2002), we exclude evaluative stances from our construct of *attending* and *interpreting*. We operationalize *responding* in terms of the specific strategies PSTs propose to *support* student reasoning.

Assessment of Mathematical Reasoning and Strategies

Generating and critiquing mathematical arguments and strategies constitutes an essential aspect of doing mathematics. The Common Core State Standards for Mathematics (NGA & CCSSO, 2010) set expectations that mathematically proficient students

> *…understand and use stated assumptions, definitions, and previously established results in constructing arguments… they justify their conclusions, communicate them to others, and respond to the arguments of others. They … distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. (NGA & CCSSO, 2010, pp. 6-7).*

Krummhauer (1995), drawing on Toulmin's (1958/2003) model of an argument, described three essential components of a mathematical argument: a conclusion—statement being argued for, an assertion made about an issue; data—which gives an evidence and provides the ground for the conclusion; and warrants with backing—which supply reasons and support for the conclusion by articulating the links from the data to the conclusion. When one evaluates a mathematical claim he or she needs to demonstrate awareness of inferences and assumptions being made, and needs to analyze and critically reflect on the provided information and evidence. We draw on this model to guide our assessment of PSTs' evaluations of student reasoning and strategies because in their assessment of the validity of student reasoning and strategies, PSTs need to make an argument for whether or not a specific strategy is valid, providing evidence and reasons for their assessment. Thus, we operationalize PSTs' ability to *assess* student reasoning and strategies in terms of PSTs' judgments (claims) about the *mathematical* soundness of student strategy and the quality of *evidence and reasons* they articulate to support their judgments. That is, we examine how well PSTs support their claims about student reasoning and strategies by analyzing the relevance of any evidence they identify in students' work and draw on as they formulate their arguments about student reasoning and strategies. We also examine the specific reasons they provide to link their evidence to stated claims.

Research shows that teachers (in-service and PSTs alike) often focus on surface rather than conceptual features of a given argument or strategy when asked to evaluate its validity. For example, they tend to validate mathematical arguments empirically by testing assertions and strategies with examples, rather than examining their logic and the validity of underlying evidence (Knuth, 2002; Knuth, Choppin, & Bieda; 2009; Martin & Harel, 1989; Monoyiou, Xistouri, & Philippou, 2006; Morris, 2007). Specifically, Knuth et al. (2009) observed that teachers often value empirical arguments as more convincing and easier for students to understand in contrast to general arguments. Thus, in their classroom practice, they frequently rely on the use of empirical over deductive arguments. Furthermore, Monoyiou et al. (2006) uncovered that when asked to evaluate student-generated arguments, elementary school teachers tend to favor arguments supported by *multiple* examples giving less significance to arguments in which students justify their assertions with a focus on a general case. Morris (2007) observed that when asked to evaluate mathematical arguments generated by students during class discussion PSTs preparing to teach elementary and middle school mathematics rarely used logical validity as a criterion of their assessment. Instead, they judged the validity of students' arguments and strategies on the basis of their own understanding of mathematical ideas articulated by students "filling in" the gaps in students' reasoning. PSTs also frequently used affective criteria (e.g., confidence with which a student presented his or her argument or strategy) as a basis for their assessment. Others (e.g., Crespo, 2000; Salinas, 2009; Son, 2013) described PSTs' difficulties in evaluating students' mathematical arguments more broadly, sharing that elementary school teachers often have difficulties to determine whether or not unconventional student-generated strategies are valid. The ability to understand and assess student-generated arguments and strategies is one of the essential teaching competencies that facilitates teaching mathematics with a focus on reasoning and sense making (Cengiz, Klein, & Grant, 2011; Crespo, Oslund, & Parks, 2011; Walshaw & Anthony, 2008).

METHOD

Participants and Study Context

Research presented in this paper was conducted in a large Midwestern university in the U.S. The data were collected in a mathematics course for prospective elementary and middle school teachers, Number Systems and Operations for Teachers. Thirty four PSTs enrolled in the course participated.

The course was the second in a 3-course mathematics sequence for elementary and middle school PSTs. It engaged PSTs in discussions and reasoning about mathematics concepts and ideas fundamental to elementary school mathematics curriculum. Course activities aimed to strengthen PSTs' conceptual understanding of mathematical ideas connected to elementary school mathematics (e.g., place value, operations with whole numbers and fractions, computational algorithms). During class discussions, great emphasis was placed on sharing reasoning and justifying mathematical ideas and procedures. While solving and explaining their solutions to discussed problems PSTs were engaged in generating and critiquing mathematical explanations with a specific focus on the quality of included evidence and reasons. Systematic efforts were also made to heighten PSTs' attention to elementary and middle school students' thinking about numbers and operations. Throughout the semester, PSTs analyzed and collectively discussed samples of student work (presented as written artifacts or video-records of elementary and middle school students' explanations).

Data and Data Collection

For this study we analyzed PSTs' written responses to four tasks which PSTs completed in the second half of the semester. Two of these tasks were designed to assess PSTs' noticing skills and two to assess their ability to evaluate student strategies and to explore pedagogical decisions PSTs make when asked to select student strategy for class discussion. The tasks addressed multiplication and division of fractions. Included in **Figure 1** are examples of tasks related to multiplication (Noticing, and Strategy Evaluation and Pedagogy). See the **Appendix** for tasks situated in the context of division of fractions.

We designed and implemented our study, providing PSTs with sufficient time to analyze students' responses. For the professional noticing tasks, we asked the participants to look out for specific evidence of students' understanding and strategies, and interpret what the evidence means with respect to students' understanding and strategies (indicators of what PSTs pay attention to and how they interpret their evidence in terms of student reasoning and strategies—attending *and* interpreting). We also asked PSTs to follow-up on students' ideas by proposing a strategy that could support students' thinking about a given problem (responding). For the strategy evaluation tasks, we asked PSTs to examine three or four (depending on the task) student strategies and assess the validity of each, providing a clear rationale for their assessment. By asking PSTs to analyze students' written work and respond in writing, we provided PSTs with opportunities to revisit aspects of student strategies as they tried to make sense of and reflect on students' reasoning and mathematical understanding.

Task 1: Vicky was asked to find the product of $3\frac{2}{3} \times 2\frac{1}{5}$.

Vicky's solution: $3\frac{2}{3} \times 2\frac{1}{5} = 3 \times 2 + \frac{2}{3} \times \frac{1}{5} = 6 + \frac{2}{15} = 6\frac{2}{15}$.

(a) Vicky has made a good attempt, but her solution is not correct. Analyze Vicky's work and describe and interpret her reasoning. Do not start from the beginning using different strategy, but identify specific flaws in Vicky's thinking about this problem situation.

(b) Rather than starting from scratch, work with Vicky and help her modify her way of thinking about $3\frac{2}{3} \times 2\frac{1}{5}$ to get the correct solution. In other words, make sure to propose a strategy that could support Vicky in her thinking about this problem. Briefly explain what Vicky needs to understand in the context of this problem.

Task 2: Following are Jessica's, Frank's, and Stacy's solutions for $\frac{9}{16}$ of 48. Closely examine each student's mathematical reasoning.

Jessica: $\frac{9}{16}$ is $\frac{1}{16}$ more than $\frac{1}{2}$. Half of 48 is 24, and $\frac{1}{16}$ of 48 is 3, so 27.

Frank: $\frac{9}{16} \times 48 = \frac{9}{\cancel{16}} \times \frac{\overset{3}{\cancel{48}}}{1} = \frac{27}{1} = 27$.

Stacy: $9 \times 16 = 144$; $144 \div 48 = 3$; and $9 \times 3 = 27$. So $\frac{9}{16}$ of 48 is 27.

(a) Make an argument for whether or not each student's reasoning and strategy is mathematically correct, flawed, or indicate that you cannot tell. Be specific explaining why you assessed each student's reasoning and strategy the way you did.

(b) Assume that Jessica, Frank, and Stacy are students in your class. You have just enough class time to invite one student to present and discuss his/her reasoning and strategy with the entire class. Which student's strategy would you select for classroom discussion for the purpose of advancing student' reasoning skills, and why? Clearly explain your choice.

Figure 1. Task examples. (1) Noticing Task, (2a) Strategy Evaluation and (2b) Pedagogy Task

Data Analysis

Assessing PSTs' Professional Noticing Skills. Consistent with our framework, to assess PSTs' professional noticing ability, we analyzed their responses with attention to mathematically significant aspects of student reasoning and strategies which PSTs addressed in their analyses of students' work. We focused on the relevance, clarity, completeness, and adequacy of interpretations PSTs shared about students' reasoning and strategies (*attending to* and *interpreting*) and the strategies they proposed as a follow-up on students' ideas (*responding*). Using our assessment of the strength of each of the two teacher noticing practices on a given task we assessed each PST's professional noticing ability on that task as *Focused* (score 2), *Mixed* (score 1) or *Superficial* (score 0). We then interpreted each PST's Overall Professional Noticing Ability as an average of their Professional Noticing Scores across the analyzed tasks.

Focused Noticing (score 2). On a given task, we assessed a PST's noticing ability as *Focused* if the PST *directly* referenced relevant mathematical ideas evident in student's thinking together with providing *meaningful interpretative comments*, and included *effective* (i.e. relevant and mathematically correct) response directly linked to the specific mathematical ideas, student difficulties, or observations made about student's thinking.

Mixed Noticing (score 1). On a given task, we assessed a PST's noticing ability as *Mixed* if the PST included some reference to relevant mathematical ideas evident in student's thinking but provided only *limited interpretative comments*. We also scored a PST's noticing ability as *Mixed* if the PST was able to recognize relevant aspects of student's thinking and provided their meaningful interpretation, however, was unable to propose an effective response to support student's thinking: the proposed response was either not linked to student's thinking or while attempting to build on student's thinking the response was ineffective.

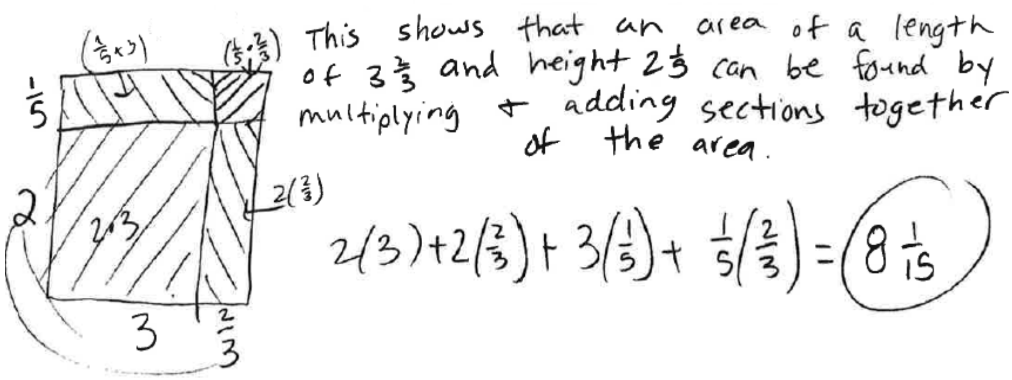

Figure 2. Responding offered by PST A32

Superficial Noticing (score 0). On a given task, we assessed a PST's noticing ability as *Superficial* if in his or her analysis, the PST shared *general* unclear impression of student's thinking without providing specific evidence, addressed *non-mathematical* aspects of student's work, and proposed *response* which was ineffective and unrelated to student's thinking.

Below, we use verbatim excerpts of PSTs' analyses of student's work presented in Task 1 (see **Figure 1** for task description) to further illustrate our scoring. Consider an excerpt of PST A32's analysis as an illustration of *Focused* noticing. PST A32 made a valid observation that the student ineffectively attempted to use the distributive property of multiplication over addition:

> *She [the student] started out right by using the idea of foiling, distributive property. She did 3·2 which is correct however she forgot to do* $3 \cdot \frac{1}{5}$ *which comes from 2 and* $\frac{1}{5}$*. She then did* $\frac{2}{3}$ *times* $\frac{1}{5}$ *which is correct, but she forgot to do* $\frac{2}{3} \cdot 2$*. She then would have* $(3 \cdot 2) + (3 \cdot \frac{1}{5}) + (\frac{2}{3} \cdot \frac{1}{5}) + (\frac{2}{3} \cdot 2)$.

In an effort to support student's reasoning, PST A32 proposed a strategy (**Figure 2**) which she directly linked to the difficulty she recognized while examining student's work. She suggested representing the problem with a diagram linked to the area model of multiplication.

Consistent with our rubric, we assessed PST A32's noticing ability in the context of this task as *Focused* (score 2) because he or she clearly identified the distributive property as a mathematically significant aspect of student strategy providing a meaningful interpretation of student's attempt to use the distributive property, and proposed an effective follow up response directly linked to her interpretation of student's thinking.

As an illustration of *Mixed* noticing ability, consider how PST A1 made sense of student's work for the same task:

> *I can see that she is attempting to use the distributive property by separating the whole from its [sic] fraction and then multiplying. She recognizes that* $3\frac{2}{3}$ *is* $(3+\frac{2}{3})$ *and* $2\frac{1}{5}$ *is* $(2+\frac{1}{5})$*. What she is trying to do is solve* $\left(3+\frac{2}{3}\right)\times\left(2+\frac{1}{5}\right)=(3\times 2)+\left(\frac{2}{3}\times\frac{1}{5}\right)+\left(3\times\frac{1}{5}\right)+(2\times\frac{2}{3})$*. She just does not recognize that each number needs to be multiplied by part of the second number. She needs to recognize that each whole [number] is also multiplied by the fraction as well.* $6+\frac{2}{15}+\frac{3}{5}+\frac{4}{3}=6+\frac{2}{15}+\frac{9}{15}+\frac{20}{15}=6\frac{31}{15}=8\frac{1}{15}$.

It is clear from her analysis that PST A1 identified student's attempt to use the distributive property of multiplication over addition and recognized student's ability to decompose fractions. However, as illustrated in **Figure 3** below, she was unable to propose a strategy which could effectively build on and support student's reasoning, helping the student to understand and correctly implement the distributive property in the context of this problem.

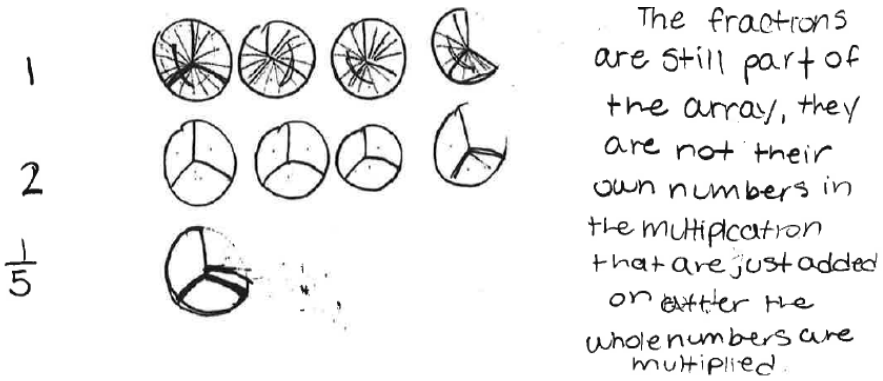

Figure 3. Responding offered by PST A1

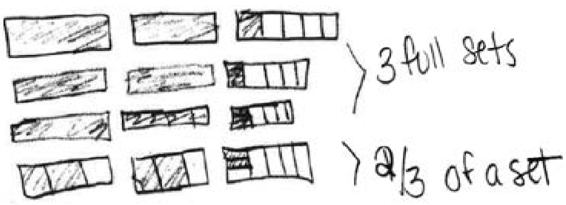

Figure 4. Responding offered by PST A21

Given that PST A1 was unable to effectively respond to the student by proposing a strategy that could support student's thinking about the distributive property in the analyzed problem situation, we assessed her professional noticing ability on this task as *Mixed* (score 1).

Finally, to illustrate *Superficial* noticing (score 0) we use an excerpt of PST A21's analysis of student's thinking and strategy on the same task. In her analysis, PST A21 shared unclear impression of student's reasoning:

> *Vicky went wrong in thinking that the 3 and the 2 were not out of the same bases (3 & 5). This was probably because it is how we normally multiply. So we know 3 sets of 3 but we need so there is 9 & then extra 2 is 11. Then for the other fraction 2 sets of 5 is ten plus the extra is 11 so 11·11 =121 & then she can find how many times 15 goes into 121 and get 8 with 1 remainder.*

As illustrated in the above excerpt, by alluding to changing mixed numbers into improper fractions and multiplying improper fractions to produce the result PST A21 reveals that he or she failed to attend to and make sense of *student's* thinking. PST A21 was also unable to propose a strategy that could effectively support student's reasoning. The diagram PST A21 offered as response (see **Figure 4**) neither supports thinking about the distributive property of multiplication over addition in the context of this problem, nor relates to the strategy PST A21 referenced in his or her analysis of student's work.

Assessing PSTs' Strategy Evaluation Skills. Across the two Strategy Evaluation Tasks (see **Figure 1 Task 2a** and **Appendix Task 4a**) we presented PSTs with a sample of seven student-generated strategies. In the context of each of these tasks, PSTs were asked to judge whether or not each student's reasoning and strategy is mathematically correct, and clearly support their judgment with a rationale. Drawing on our operational definition of PSTs' assessment of student reasoning and strategies (see the Assessment of Mathematical Reasoning and Strategies section) we focused our analysis on the validity of judgments PSTs made about student reasoning and strategies, the quality of *evidence* they provided in support of their assessment of students' reasoning and strategies, and *reasons* they articulated as a way of linking the provided evidence to their claims. We separately analyzed PSTs' assessment of each student's strategy and quantified their Overall Strategy Evaluation Skills as an average of their Strategy Evaluation Scores across the analyzed student-strategies. Below we describe a 3-point rubric we developed to guide our assessment of PSTs' strategy evaluation skills and illustrate our analysis with excerpts of PSTs' assessment of Stacy's strategy presented in **Task 2**, **Figure 1**.

Proficient Assessment (score 2). In reference to a given student strategy, we assessed a PST's strategy evaluation ability as *Proficient* if he or she accurately assessed whether or not student's reasoning and strategy are valid and supported their judgment with complete and valid *evidence* and *reasons.*

Limited Assessment (score 1). In reference to a given student strategy, we assessed a PST's strategy evaluation ability as *Limited* if the PST correctly assessed whether or not student's reasoning or strategy are valid without providing comprehensive support for his or her assessment. For example, the provided reasons were incomplete or primarily focused on evaluating outcomes of the strategy rather than the mathematical soundness of student's reasoning.

Insufficient Assessment (score 0). Finally, we assessed a PST's strategy evaluation skills as *Insufficient* if he or she incorrectly judged (e.g. accepted faulty reasoning and strategy as valid or accepted valid reasoning and strategy as faulty) or could not judge the validity of student's reasoning or strategy. We also assessed a PST's strategy evaluation skills as Insufficient if the PST did not provide *any* support for his or her correct assessment of student's reasoning and strategy.

To illustrate *Proficient* (score 2) strategy evaluation ability consider how PST A5 commented on Stacy's reasoning and her proposed strategy:

> *Stacy's strategy does not make sense because multiplying the numerator and denominator of a fraction does not give us a value that relates to what she is trying to find in the problem. This strategy would not work for other problems like $\frac{5}{8}$ of 16, for example. She multiplied the numerator and denominator in $\frac{9}{16}$ which doesn't have to do anything with solving a problem that is involving multiplying fractions. Multiplying the [number of] part[s] by the [number of] parts in the whole is irrelevant in this problem. Stacy's method, if used with other fractions, would not work.*

PST A5 assessed student's strategy by evaluating the logic and validity of the strategy. She recognized that while the strategy might produce the correct result for *this particular set of numbers*, it is not mathematically sound and generalizable. She provided a counterexample in support of her assessment *together* with discussion of the validity of mathematical steps of Stacy's procedure. Consistent with our rubric, we scored her evaluation of Stacy's response as *Proficient*.

Consider yet another example of an assessment of strategy presented by Stacy (see **Figure 1**, **Task 2**). We use an excerpt of PST B21's response as an example of *Limited* (score 1) strategy evaluation skills:

> *She gets the correct answer but does not show work with fractions. Again, mathematically she gets the correct answer. Stacy's work shows that she does not appear to understand multiplication of fractions. She does not use fractions in her method to find the answer nor will her methods work every single time to give her the correct answer. I checked to see if it would [work] by doing $\frac{3}{4}$ of 36 using her method which gives an answer of 1 if done her way which is not the correct answer, correct answer being 27.*

In his or her assessment of the validity of student's strategy, PST B21 exclusively focused on the outcomes of generated computations. While he or she correctly concluded that Stacy's strategy is not generalizable, PST B21 did not consider mathematical soundness of the steps of Stacy's strategy. Instead, throughout her analysis, she motivated her assessment of Stacy's reasoning and strategy by the fact that the student did not use fractions. For that reason, even though, the counterexample she provided ($\frac{3}{4}$ of 36) is sufficient to conclude that Stacy's strategy is not generalizable we assessed PST B21's strategy evaluation skills on this task as *Limited*.

Finally, consider an excerpt of PST A31's evaluation of Stacy's strategy as an illustration of *Insufficient* (score 0) strategy evaluation skills. PST A31 accepted Stacy's strategy solely on the basis of the correct final result the strategy produced for the given set of numbers, without considering whether or not the strategy is mathematically sound: "Stacy's work is correct. The processes that she does yield a correct answer. I do not know why she chooses to do what she does but she does get the correct answer."

Analyzing PSTs' Selection of Student Strategies. To determine how PSTs motivate their selection of student strategies for classroom discussion, we analyzed their responses to pedagogy prompts (see **Figure 1**, **Task 2b**) using qualitative methods and open coding (Corbin & Strauss, 1990). Our goal was to identify different ways in which PSTs motivated their choice of using student strategy in class discussion. The analysis comprised of multiple passes through the data during which each response was carefully annotated. We strived to refine and delineate characterizations of PSTs' responses and identify possible discrepancies and rival themes to assure the rigor of analysis. Emergent themes were grouped together to discern ways in which the PSTs motivated their decisions of using specific student-generated strategy for class discussion. Summarized in **Table 1** are the five categories identified through this analysis illustrated with examples of PSTs' responses.

Table 1. PSTs' Motivations for Strategy Selection

Categories (Motivations for Strategy Selection)	Sample PSTs' Responses
1. To engage students in justification	I would choose Jessica's [strategy] because other students could contribute and explain more in depth why $\frac{8}{16} = \frac{1}{2}$ & how $\frac{1}{16}$ of 48 = 3. They would explain what the 3 represents. Although Jessica does a good job, there is still more that needs to be explained [and justified.] [PST A28, Task 2b]
2. To advance students' conceptual understanding	If I choose one student, I would choose Jessica to share her answer. I like how she creatively realized $\frac{8}{16}$ as half and $\frac{9}{16}$ as $\frac{1}{16}$ more than half. This is an expedited mental math strategy for effectively solving $\frac{9}{16}$ of 48. [...] This would allow students to better understand multiplication of fractions and whole numbers. [PST A41, Task 2b]
3. To engage students in thinking about strategy generality	I would invite Jessica and encourage students to investigate if the strategy is accurate and works every time [PST A22, Task 2b]
4. To discuss possible misunderstandings or misconceptions	I would invite Mark to present and discuss his solution with the class. I think, Mark is on the right track, and the way he solved the problem is probably how a lot of other students would solve it too. Mark showed a good visual of what the 5 cups of flour look like & how each batch went in 2 full times. But what to do with that remaining $\frac{1}{2}$ will be a good class discussion. [PST A22, Task 4b]
5. Other (e.g., easy to follow, clear, unique, well organized)	I would invite Frank to share his solution because it is most straightforward example of how to multiply fractions. The majority of class should be able to follow along and compare their work to Frank's. [PST A44, Task 2b]

Quantitative Analysis. To answer Research Question 1 and explore a possible association between PSTs' Professional Noticing and Strategy Evaluation skills, we conducted the Spearman Rank Correlation analysis using Overall Professional Noticing and Strategy Evaluation Scores. To answer Research Question 2, we first assigned each of the five identified category variables (see **Table 1**) value "1" if a PST used this category to motivate his or her selection of student strategy or value "0" if he or she did not (Motivation Scores). To examine the extent to which PSTs' motivation for selecting student strategy might be related to PSTs' strategy evaluation skills, we then used the Motivation and the Strategy Evaluation Scores and conducted the Spearman Rank Correlation analysis.

Reliability. Cohen's Kappa was computed to determine the level of agreement between the two authors on all aspects of data analysis. The results were statistically significant at the 0.05 level. For the Professional Noticing Tasks the inter-rater reliability was $\kappa = 0.975$, $p = 0.000$; for the Strategy Evaluation Tasks $\kappa = 0.773$, $p = 0.015$; and for the analysis of Pedagogical Decisions, $\kappa = 0.808$, $p = 0.001$. Prior to conducting further analyses, we negotiated a 100% agreement on the discrepant cases.

RESULTS

Professional Noticing and Strategy Evaluation Skills

Our data revealed a positive relationship between PSTs' professional noticing skills ($\overline{M} = 0.794$, SD = 0.494) and their ability to evaluate student strategies ($\overline{M} = 0.713$, SD = 0.451). Spearman Rank Correlation between PSTs' Professional Noticing Scores and Strategy Evaluation Scores was statistically significant at the 0.05 level; $r = 0.414$, $p = 0.015$. We provide qualitative illustration of the uncovered relationship using, as a context for our discussion, responses of PSTs B16 (*Focused* Noticing) and A23 (*Superficial* Noticing) to tasks presented in **Figure 1**. Recall from **Figure 1**, that for the Professional Noticing Task (Task 1) PSTs were asked to identify mathematically significant aspects of student strategy and reasoning and propose a way of working with a student by building on student's mathematical ideas. The task presents PSTs with a situation where Vicky (the student) makes an attempt to apply the distributive property of multiplication over addition to multiply two mixed numbers. The student decomposes mixed numbers presenting each as a sum of a whole number and a proper fraction, however, considers only some of the partial products while executing her strategy.

The second task included in **Figure 1** (**Task 2a**, Strategy Evaluation Task) engaged PSTs in evaluating the validity of strategies presented by three students (Jessica, Frank, and Stacy) for a problem that addressed fraction multiplication. Jessica's strategy was based on a correct mental computation. Frank simplified fractions and applied the standard multiplication algorithm, and Stacy's strategy, was neither generalizable nor grounded in understanding of fractions or multiplication, however, she did generate a correct result for the given set of numbers.

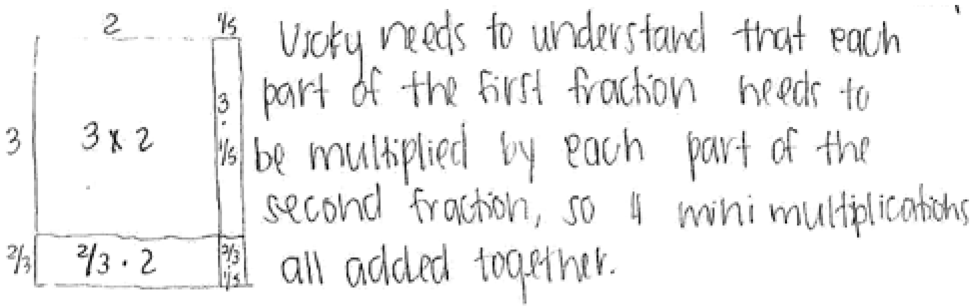

Figure 5. Responding, PST B16

PST B16

Professional Noticing. Consider the following response of PST B16 as an illustration of B16's high (*Focused*) professional noticing skills:

> *Her [Vicky's] first part is correct, but she's only half way there. Next, she needs to multiply the whole numbers by the fraction in the opposite number, so $\frac{2}{3}$ times 2 and 3 times $\frac{1}{5}$ [can be added]. Then she should add all of her quantities (four) together. Vicky was thinking about it [the multiplication problem] as two parts: 1) being $3\frac{2}{3}$ and 2) being $2\frac{1}{5}$, but really it has really separate parts being multiplied.*

While PST B16 does not state explicitly that Vicky is attempting to use the distributive property, her interpretation of student's thinking and strategy and her proposed response document that she noticed student's attempt to use the distributive property of multiplication over addition. Specifically, she proposed a visual model (see **Figure 5**) that builds on Vicky's reasoning. Her model can effectively help Vicky to think about this problem situation by highlighting the need for all four partial products. PST B16 recognized mathematically significant aspects of Vicky's strategy and provided a meaningful interpretation connecting her response to recognized flaws in Vicky's reasoning.

Strategy Evaluation. Now consider PST B16's (*Focused* professional noticing) evaluations of student strategies. As illustrated in excerpts that follow, PST B16 carefully assesses each student's strategy with a focus on the strategy validity and generality. She considers the claim (whether or not the strategy generates the correct result), analyzes the evidence included in student's work, and provides clear reasons for her assessment.

PST B16's assessment of Jessica's strategy

> *Jessica's answer is correct and so is her reasoning. She starts by taking $\frac{1}{16}$ out and is just working with $\frac{8}{16}$. She figures out that, since $\frac{8}{16}$ is half, she can take half of the total number, 48, which is 24. Then, she adds the $\frac{1}{16}$ back in by figuring out that $\frac{1}{16}$ of 48 is 3, which makes 27 when added to 24. Both Jessica's answer and reasoning are correct in this example because she is figuring out what $\frac{9}{16}$ of 48 is. She is just doing it in two steps: $\frac{8}{16}$ [of 48] plus $\frac{1}{16}$ of 48.*

In her evaluation of Jessica's strategy PST B16 provides the assessment of the correctness of Jessica's *claim* (i.e., the ultimate answer to the problem) and Jessica's reasoning. She carefully evaluates the *evidence* Jessica used (e.g., "$\frac{8}{16}$ is half" and $\frac{1}{16}$ of 48 is 3). Finally, PST B16 evaluates the *reasons* for the correctness of Jessica's strategy: $\frac{9}{16}$ of 48 can be found in two steps because $\frac{9}{16}$ equals "$\frac{8}{16}$ plus $\frac{1}{16}$".

PST B16's assessment of Frank's strategy

> *Frank's answer is correct and so is his reasoning in this particular example. But, I am not sure if his method [of simplification of fractions] will always work. If the numbers diagonal from each other do not have a common number [factor] that goes into both of them, he will not be able to reduce the*

fractions, and he will end up with another fraction. For example, if he was to do $\frac{2}{3}$ *of 16, he would end up with* $\frac{32}{3}$*. So I would say this method will work in some, but not all, instances.*

PST B16 also critically evaluates Frank's strategy. Although she does not explicitly consider *why* the standard multiplication algorithm gives the correct answer, her assessment includes significant insights about the generalizability of the simplification of fractions in Frank's work.

PST B16's assessment of Stacy's strategy

Stacy's answer is correct, but her reasoning is not. In this example, her method works, but it won't always. This is because the product of the numerator and denominator will not always be divisible by the number you are trying to find a portion of. For example, if you want to find $\frac{2}{3}$ *of 16 again, you would multiply 2×3 to get six. 6 cannot be divided evenly by 16, however, so the problem falls apart.*

As illustrated in the excerpt above, PST B16 evaluates Stacy's strategy with a focus on generated *claim* (i.e., the ultimate answer to the problem) and Stacy's *reasoning* "Stacy's answer is correct, but her reasoning is not". She assesses the generality of Stacy's strategy, stating that the strategy can work for some numbers but not for *all*, by providing a counterexample.

PST A23

We contrast our illustration of PST B16's (*Focused* professional noticing and *Proficient* strategy evaluation scores) responses with responses of PST A23 to the same two tasks (see **Figure 1**). Responses of PST A23 provide examples of low (*Superficial*) professional noticing and low (*Insufficient*) strategy evaluation skills.

Professional Noticing. As illustrated in the excerpt below, when asked to analyze Vicky's strategy (see **Figure 1**, **Task 1**), PST A23 failed to *attend and interpret* the mathematical aspects of student's reasoning and strategy:

Vicky understands that $3 \cdot 2 = 6$*. However, she may need help multiplying these mixed numbers with fractions. She could put* $3\frac{2}{3}$ *and* $2\frac{1}{5}$ *into* $\frac{11}{3}$ *and* $\frac{11}{5}$*, and then multiply straight across (like she had done).* $\frac{11}{3} \cdot \frac{11}{5} = \frac{121}{5} = 8.0666667$.

In his or her analysis PST A23 clearly diverts from *Vicky's* thinking and *Vicky's* attempt to use the distributive property, inferring that the student needs to learn how to rewrite mixed fractions into improper fractions and multiply the improper fractions. Likewise, response which PST A23 offered (see **Figure 6**) was neither connected to Vicky's attempt, nor could effectively support thinking about converting mixed numbers into improper fractions and multiplying fractions.

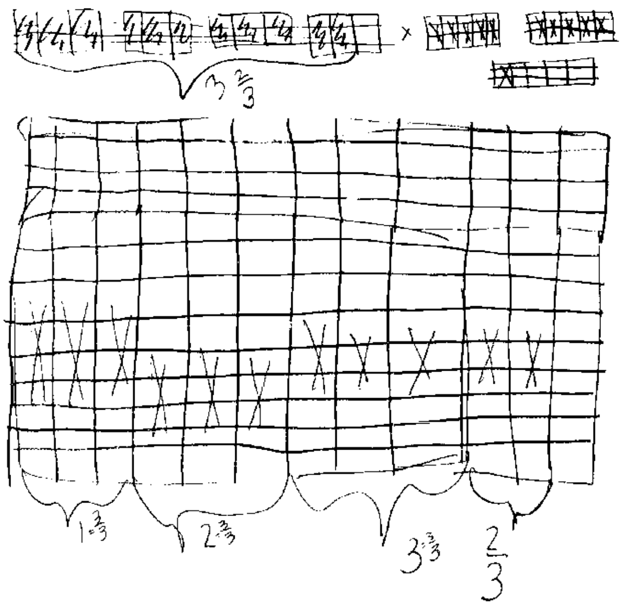

Figure 6. PST A23 proposed response

To motivate the above diagram PST A23 shared: "Vicky needs to look at $3\frac{2}{3}$ and $2\frac{1}{5}$ as 2 numbers, instead of just looking at the 3 and the 2, and ignoring the fractions. She needs to count how many times the $3\frac{2}{3}$ overlap with $2\frac{1}{5}$." PST A23 neither payed attention to the mathematically significant aspects of Vicky's reasoning nor provided their meaningful interpretation. The response he or she proposed was disconnected from *student's* (Vicky's) thinking and rather stemmed from PST A23's own idea about the strategy for solving the given multiplication problem. Even so, the proposed response was not effective. While PST A23 drew rectangular diagrams to represent each mixed number (see **Figure 6**), he or she was unable to provide a meaningful pictorial representation to support the student in making sense of the product of the two numbers.

Strategy Evaluation. Now consider PST A23's (low, *Superficial*, professional noticing ability) responses to strategy evaluation tasks. Unlike PST B16 (high, *Focused*, professional noticing) who in her assessment of students' strategies examined each strategy with a focus on its validity and generality, PST A23 limited her strategy assessment to the evaluation of the claim (i.e., a numerical result that the strategy generates for a given set of numbers). Below we illustrate and discuss PST A23's evaluation skills using her responses to Task 2 presented in **Figure 1**.

PST A23's assessment of Jessica's strategy

> *I think Jessica's work is very creative, and she receives a correct answer, but I am skeptical if these strategies would work for any problem. Jessica is correct when she says $\frac{9}{16}$ is $\frac{1}{16}$ more than $\frac{1}{2}$, because $\frac{8}{16}$ would be $\frac{1}{2}$. Then, she takes half of 48, to equalize the problem, but I am not sure how she knew to take $\frac{1}{16}$ of 48. Then, she adds the 3 to the 24 and gets the correct answer of 27.*

Even though PST A23 used some evidence to support her evaluation of Jessica's strategy, her assessment focused primarily on the correctness of generated result. She did not assess the logical validity and generality of analyzed strategy.

PST A23's assessment of Frank's strategy

> *I like how he did $\frac{9}{16} \times \frac{48}{1}$, and simplified the 16 and 48, so that he was left with $\frac{9}{1} \times \frac{3}{1} = \frac{27}{1} = 27$. I liked Frank's way the best because that is the way I learned, and it makes the most sense to me.*

PST A23's assessment of Frank's strategy was more descriptive than evaluative as she identified what Frank did (i.e., simplified fractions). Her assessment was based on her personal appreciation and familiarity with Frank's strategy.

PST A23's assessment of Stacy's strategy

> *Stacy's work is very creative, and she receives a correct answer. In Stacy's work, I am not sure why she multiplied $9 \cdot 16 = 144$. However, she gets the correct answer of 27. Her process seems that it will work for any situation.*

As illustrated in the excerpt above, PST A23 focused her assessment of student's strategy on the strategy's final outcome. Even though PST A23 appeared to be skeptical about the steps of Stacy's procedure, she did not question the rationale behind them. Limiting her evidence to the correctness of generated result she accepted student's strategy as valid and generalizable.

The two contrasting cases (PST B16—high, *Focused*, professional noticing, and PST A23—low, *Superficial*, professional noticing) provide qualitative illustration that well developed professional noticing skills positively relate to one's ability to critically evaluate and assess students' mathematical strategies.

Pedagogical Decisions and Strategy Evaluation Skills

As discussed in earlier sections, by engaging PSTs in mathematical activity from the perspective of their practice, our intent was to expose them to a variation of student reasoning and strategies. The analysis of PSTs' responses to the Pedagogy Tasks (see **Figure 1 Task 2b** and **Appendix Task 4b**) revealed several motivating factors our PSTs considered while planning to select student response for a class discussion: (1) to engage a student or a class in justifying result or strategy, (2) to facilitate and build students' conceptual understanding, (3) to bring students' attention to strategy generality, (4) to discuss common misconceptions, or (5) included other factors which did not directly link to the goal of selecting strategy to facilitate discussion focused on enhancing students'

Table 2. Reasons PSTs Provided for Selecting Student Strategy for Class Discussion

Motivation for selecting student strategy	***n* (%)***
To engage in justification	15 (44.12%)
To advance conceptual understanding	16 (47%)
To engage students in discussion about strategy generality	8 (23%)
To discuss misunderstandings and misconceptions	9 (26.5%)
Other	10 (29%)

* exceeds 100% because PSTs used more than one reason to motivate their strategy selection

reasoning skills (see the **Table 1** to view examples for each category). **Table 2** provides a summary of PSTs' responses to this group of tasks.

Our quantitative analysis revealed statistically significant positive correlation between PSTs' ability to evaluate students' strategies and their motivation for strategy selection at the 0.05 level: specifically, selecting a strategy to engage students in *justifying* (r = 0.522, p = 0.002) or for the purpose of *advancing students' conceptual understanding* (r = 0.371, p = 0.030). The analysis also revealed that PSTs with higher (*Proficient*) strategy evaluation scores were less likely to motivate their strategy selections in ways not directly linked to enhancing students' reasoning skills (category "Other"; r = - 0.443, p = 0.009).

Again, we use examples of PSTs B16 (high, *Proficient*, strategy evaluation skills) and A23 (low, *Insufficient*, evaluations skills) to illustrate this relationship. For example, in his or her response to Task 2b (see **Figure 1**) PST B16 shared that she would choose Jessica's strategy for class discussion and use Jessica's strategy to *engage students in justification* and to think about Jessica's *strategy generality*. PST B16 explained that by discussing Jessica's strategy, students could consider whether or not "her method will work universally for all other examples." She went on to explain that Jessica's strategy "would be a useful tool for the class to learn how to calculate these types of problems for all fractions with an even denominator," and continued saying that:

> *The students would have to justify whether and how Jessica's strategy could be used if the denominator is not even. The students would have to figure out how they would split the fractions up (because they would not be able to find "half' within this fraction [denominator]).*

PST B16's pedagogical decision of selecting Jessica's strategy draws on her careful evaluation of the strategy (as illustrated in the earlier section) because Jessica's strategy of using "half" cannot be used when the denominator is odd.

Now, consider how PST A23 (low, *Insufficient*, strategy evaluation scores) motivated her choice of selecting Stacy's strategy for class discussion in the context of the same task (**Task 2b**, **Figure 1**). PST A23 shared:

> *I would invite Stacy to share her way of thinking. I would invite her because it [Stacy's strategy] may make more sense to other students. It will also give them another way to look at the problem. Her process seems that it would work for any situation.*

As discussed in the Strategy Evaluation section, Stacy's strategy, while generates correct result for a given set of numbers, is not mathematically sound. Even though, as discussed earlier, PST A23 appeared to be initially sceptic about Stacy's strategy, he or she ultimately accepted it on the basis of the correct result that the strategy generates for the given set of numbers, and assessed the strategy as "very creative." PST A23's pedagogical decision to use Stacy's strategy for class discussion because "it [the strategy] may make more sense to other students," draws on PST A23's insufficient assessment and perceived view of strategy creativity.

The two presented examples, together with the results of our statistical analysis, demonstrate that PSTs' pedagogical decisions of selecting student strategy for class discussion relate to their strategy evaluation skills. It is unlikely to expect then that PSTs can effectively build on student strategies during class discussion without having robust strategy evaluation skills.

DISCUSSION

Our study provides an important window into PSTs' professional noticing skills, their ability to evaluate student strategies, and their thinking about using student strategies in class discussion for the purpose of advancing students' reasoning. We draw attention to (1) the relationship between PSTs' professional noticing and strategy evaluation skills and examine how (2) pedagogical decisions PSTs make about selecting specific student strategy for class discussion might relate to PSTs' ability to evaluate student strategies.

First, our data revealed a positive relationship between PSTs' professional noticing of mathematically significant aspects of student reasoning and strategies and their strategy evaluation skills. Attending to and

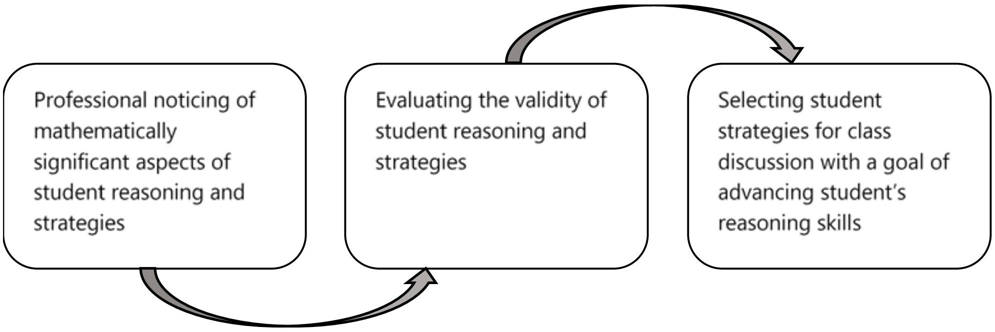

Figure 7. Intricacies among professional competencies: noticing mathematically significant aspects of student reasoning and strategies, evaluating, and building on student strategies

interpreting mathematically significant aspects of student strategies and responding to the student and the ability to evaluate student strategies with attention to the validity of generated outcomes (claims), evidence, and reasons for accepting or refuting student strategy constitute a set of closely related competencies. Our result suggests that supporting PSTs' professional noticing of mathematically significant aspects of student reasoning and strategies can positively enhance their competency in evaluating and making sense of strategies their students use and generate.

To answer our second research question we presented our PSTs with classroom scenarios of selecting student strategy for class discussion for the purpose of advancing students' reasoning skills. First, our data revealed that PSTs motivated their strategy selection by a desire to (a) engage students in providing justifications, (b) advance students' conceptual understanding, (c) engage students in discussion about the strategy generality, (d) discuss possible misconceptions or misunderstandings, or (e) other reasons, which did not support the specified purpose of class discussion. Our data also revealed that pedagogical decision choices of selecting student strategy for class discussion were positively related to PSTs' strategy evaluation skills. In our study, PSTs who proficiently assessed whether or not students' reasoning and strategies were valid and supported their judgments with comprehensive evidence and reasons, were more likely to make decisions about using student strategy as a springboard for class discussion with a goal of advancing students' reasoning skills. More specifically, to engage students in *justifications* or to *advance their conceptual understanding*. PSTs with lower strategy evaluation skills more frequently motivated their strategy selection without a clear focus on how the discussion around the selected strategy could potentially benefit the development of students' reasoning skills.

Our work extends prior research on teacher professional noticing by going beyond descriptive accounts of what teachers, including PSTs, do or do not notice as they observe mathematics classroom or make sense of student thinking and strategies (e.g., Dick, 2017; Lee & Choy, 2017; Roller, 2016; Sánchez-Matamoros et al., 2015; Sun & van Es, 2015; van den Kieboom et al., 2017). We draw attention to the relationship between teacher professional noticing of student thinking and other two complex teaching competencies: the ability to *evaluate* students' strategies and to *purposefully select* student strategies as a springboard for class discussion. As summarized in **Figure 7**, PSTs who notice, that is, are increasingly aware of mathematically significant aspects of student reasoning and strategies, are better positioned to evaluate the validity of student reasoning and strategies. In turn, they are also positioned to build on their understanding and evaluation of student reasoning and strategies and make pedagogical decisions informed by student work. The latter result supports the finding of Cengiz et al. (2011) who reported, based on their observations of classroom instruction, that teachers with weaker ability to proficiently evaluate students' arguments and strategies were less effective in building on student ideas during class discussions.

Noticing mathematically significant features of student reasoning and strategies and evaluating the validity of student reasoning and strategies are important teaching competencies. Neither is easy for PSTs to develop, as documented in the mathematics education literature (Fernández et al., 2013; Martin & Harel, 1989). Our work provides important direction for mathematics teacher educators by documenting that strengthening PSTs' competency in professional noticing mathematically significant aspects of student reasoning and strategies positions PSTs to better evaluate student strategies and reasoning. By drawing on stronger evaluation skills, PSTs can make informed instructional decisions about using student strategies to enhance students' reasoning skills in class discussion.

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APPENDIX

Tasks 3 and 4

Task 3: Marcus was asked to write a story problem for $1\frac{3}{4} \div \frac{1}{2}$. He wrote:

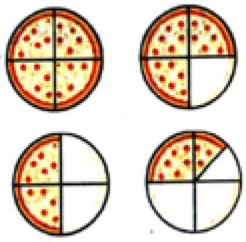
There are $1\frac{3}{4}$ pizzas left in the refrigerator. Marcus ate half of the pizza that was left in the refrigerator. How many pieces did Marcus eat?

Marcus used the following diagrams to solve his problem and explain that $1\frac{3}{4} \div \frac{1}{2} = 3\frac{1}{2}$.

Marcus's Explanation:

$1\frac{3}{4}$ *pizzas left in the refrigerator.*

Marcus eats $\frac{1}{2}$ of the left over pizza, which is $3\frac{1}{2}$ pieces.



(a) Marcus correctly says that $1\frac{3}{4} \div \frac{1}{2} = 3\frac{1}{2}$ but neither his story problem nor his diagram are consistent with $1\frac{3}{4} \div \frac{1}{2}$. Analyze Marcus's work and describe and interpret his reasoning. Do not start from the beginning using different strategy, but identify specific flaws in Marcus's thinking about this problem situation.

(b) Work with Marcus. Write a correct word problem for $1\frac{3}{4} \div \frac{1}{2}$ that is about pizza, and draw a diagram for your story problem. Clearly explain how your story problem and the diagram can help Marcus to understand $1\frac{3}{4} \div \frac{1}{2}$.

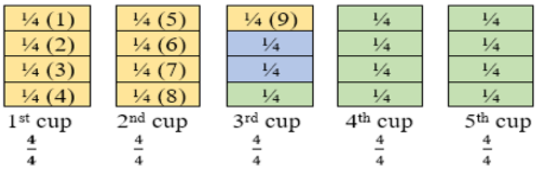
Task 4: Students are making cookies for a bake sale. They use a recipe that calls for $2\frac{1}{4}$ cups of flour for 1 batch of cookies. They have 5 cups of flour. Assuming that they have all the other ingredients and they want to use all flour how may batches can they make? Below you see how Mark, Jose, Edwin, and Sue worked out this problem.

(a) Make an argument for whether or not each student's reasoning and strategy is mathematically correct, flawed, or indicate that you cannot tell. Be specific explaining why you assessed each student's reasoning and strategy the way you did.

(b) Assume that Mark, Jose, Edwin, and Sue are students in your class. You have enough class time to invite one student to present and discuss his/her solution with the entire class. Which student's solution would you select for class discussion for the purpose of advancing students' reasoning skills, and why? Clearly explain your choice.

Mark: 2 batches of cookies will use up $4\frac{1}{2}$ cups of flour, because $2 \times 2\frac{1}{4}$ is $4\frac{1}{2}$. This leaves $\frac{1}{2}$ batches. So, they should be able to make $2\frac{1}{2}$ batches.

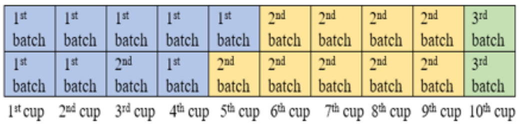
Edwin: $2\frac{1}{4}$ cup of flour needed for a batch of cookies that is the same as $\frac{9}{4}$ of a cup of flour. If you divide a cup of flour into fourths it has four fourths. So 5 cups of flour is the same as 20 fourths cups of flour.



I checked how many times $\frac{9}{4}$ goes into $\frac{20}{4}$ and it is $2\frac{2}{9}$.
So $2\frac{2}{9}$ batches are possible to make.

Jose: $5 \div 2\frac{1}{4} = 5 \div \frac{9}{4} = 5 \cdot \frac{4}{9} = \frac{20}{9} = 2\frac{2}{9}$. It will be 2 whole and $\frac{2}{9}$ of a batch. I basically inverted and multiplied.

Sue: If we had twice as many cups of flour, we would use twice as much for each batch: $2 \times 2\frac{1}{4} = 4\frac{1}{2}$ cups of flour for a batch of cookies. The picture shows how many batches of cookies we can make from 10 cups of flour; it shows how many times we can measure $4\frac{1}{2}$ cups given 10 cups.



From 10 cups of flour I can make $2\frac{2}{9}$ batches of cookies. Since I only have 5 cups of flour (which is half of 10 cups) I can make half of $2\frac{2}{9}$ batches which is $1\frac{1}{9}$ batch of cookies.

So $5 \div 2\frac{1}{4} = 1\frac{1}{9}$