MODESTUM
OPEN ACCESS
EURASIA Journal of Mathematics, Science and Technology Education, 2018, 14(9), em1587
ISSN:1305-8223 (online)
Research Paper https://doi.org/10.29333/ejmste/92557

How Can Mathematical Modeling Facilitate Mathematical Inquiries? Focusing on the Abductive Nature of Modeling

JinHyeong Park [1], Kyeong-Hwa Lee [2*]
[1] Gongju National University of Education, KOREA
[2] Seoul National University, KOREA

Received 9 March 2018 ▪ Revised 18 May 2018 ▪ Accepted 25 May 2018

ABSTRACT
The purpose of this study is to investigate the nature of mathematical modeling and identify characteristics of mathematical inquiries triggered by mathematical modeling. We investigated three cases of mathematical inquiries facilitated by mathematical modeling. As a result of this study, we revealed the abductive nature of mathematical modeling. We also determined that mathematical inquiries triggered by mathematical modeling have abductive, recursive, analogical, and context-dependent aspects.

Keywords: abduction, Euler, Napier, nature of modeling

INTRODUCTION

Mathematical modeling has been discussed as a means of fostering students' mathematical inquiries. Because mathematical modeling can both support students' reinvention of mathematical contents and in itself be a goal of mathematical inquiries (Ottesen, 2001), researchers have been attempting to build theoretical frameworks to design and analyze mathematical modeling activities that aim to accomplish these educational goals.

Nevertheless, it has been pointed out that the key characteristics of mathematical modeling need to be identified (Ärlebäck & Doerr, 2015). To be specific, we still have no clear answers to the following questions: What is it about the nature of mathematical modeling that enables it to trigger mathematical inquiries? What are the aspects or characteristics of mathematical modeling that enable it to trigger mathematical inquiries?

Peirce emphasized the key role of abductive reasoning in inquiries and knowledge creation (Prawat, 1999). In this study, we aim to answer the above questions from Peircean perspective. To be more specific, we aim to investigate the nature of mathematical modeling and identify characteristics of mathematical inquiries triggered by mathematical modeling. To accomplish the aims of this study, we first examine students' use of abduction observed in a mathematics classroom that focused on mathematical modeling activities. We then analyze historical episodes that show the creation of mathematical objects. We then identify the nature of mathematical modeling and characteristics of mathematical inquiries fostered by mathematical modeling.

MATHEMATICAL MODELING AND ABDUCTION

What is Mathematical Modeling?

Peirce declared,

> *mathematics to be the science which draws necessary conclusions … essence of mathematics lies in its making pure hypotheses, and in the character of the hypotheses which it makes. What the mathematicians mean by a hypothesis is a proposition imagined to be strictly true of an ideal state of things (C.P. 3.558).*

In other words, hypotheses can be true of an ideal state of things since a mathematical inquiry is done in the form of a simplified problem that can be dealt with mathematically (C.P. 3.559). In addition, because a simplified

✉ demxas@gjue.ac.kr ✉ khmath@snu.ac.kr **(*Correspondence)**

Contribution of this paper to the literature

- The abductive nature of mathematical modeling, which enables triggering of mathematical inquiries, is identified, and this contributes to establishing theoretical grounds for utilizing mathematical modeling in the mathematics classroom.
- The key characteristics of mathematical inquiries triggered by mathematical modeling identified in this study makes contributions to local learning theories about teaching and learning mathematics using mathematical modeling.
- This study presented further key issues and research questions regarding mathematics teaching and learning using mathematical modeling.

problem is represented by mathematical language such as diagrams, characters, equations, and so on (C.P. 3.560), representing a problematic situation by means of mathematical language is a *model* (Lenhard, 2005).

The first business of the mathematician, often a most difficult task, is to frame another simpler but quite fictitious problem (C.P. 3.559).

> *Kant is entirely right in saying that, in drawing those consequences, the mathematician uses what, in geometry, is called a "construction," or in general a diagram, or visual array of characters or lines. Such a construction is formed according to a precept furnished by the hypothesis. Being formed, the construction is submitted to the scrutiny of observation, and new relations are discovered among its parts, not stated in the precept by which it was formed, and are found, by a little mental experimentation, to be such that they will always be present in such a construction. Thus, the necessary reasoning of mathematics is performed by means of observation and experiment, and its necessary character is due simply to the circumstance that the subject of this observation and experiment is a diagram of our own creation, the conditions of whose being we know all about (C.P. 3.560).*

Rothenberg (1989) noted that the basic use of a model is to signify something else. Thus, if we would like to clarify the meaning of a model, we need to identify what it signifies and how it signifies it. Fischbein (1987) and Lesh and Doerr (2003) both pointed out that mathematical models are systems that signifying other systems. In other words, models explain the relations and mechanisms of objects rather than signifying a single object. Fischbein (1987) noted that relations between originals and models are structural isomorphism while Lesh and Doerr (2003) considered these relations structural similarity.

In this study, we interpret relationships between originals and models as structural similarity. We expect that models can have more potential meanings than originals, and new meanings supported by *semiosis* can be arise by modifying and revising mathematical models (Park et. al., 2013). Based on the above discussions, we define a mathematical model as follows:

Mathematical model: A system described by mathematical language that signifies another system focusing on structural similarity.

We define a mathematical model as a system described by mathematical language because we expect that the usage of mathematical objects and procedures can be enriched through building and revising mathematical models from a *commognitive perspective* on the creation and development of mathematical knowledge (see Sfard, 2008).

Mathematical modeling is usually considered a cyclic process that includes building, validating, and revising models (Bailer-Jones, 1999; Lesh & Doerr, 2003). The explanatory power of mathematical models can be validated by inductively applying models to similar problematic situations and by deductively examining the internal consistency of the models (Bailer-Jones, 1999). Given that, we define mathematical modeling as follows:

Mathematical modeling: The processes that involves building a mathematical model, examining its appropriateness, validity, and consistency by using it inductively or deductively and then revising it.

Inquiries via Abductions

Abduction is the process of forming an explanatory hypothesis based on observed results (C.P. 5.171). According to Peirce, abduction, which draws a case from a rule and a result (C.P. 2.623). Since a case drawn from a rule and a result is a plausible hypothesis, Peirce called a hypothesis abduction.

Peirce emphasized the abduction is the only way of knowledge creation (Prawat, 1999). According to Peirce, "[abduction] is the first step of scientific reasoning, as induction is the concluding step" (C.P. 7.218). Thus, we first utilize abduction to hypothesize a provisional general rule that explains particular observed results, then inductively and deductively verify and revise the initial abduction and hypothesis (C.P. 5.171).

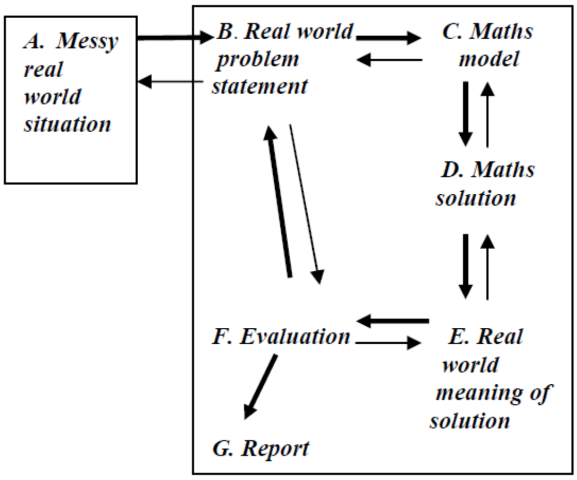

Figure 1. Modeling cycle (Galbraith & Stillman, 2006, p. 144)

Eco (1983) categorized abduction into three types: overcoded, undercoded, and creative. This classification is related to the number of rules that can explain the observed results. Overcoded abduction is deriving a rule that explains the observed result automatically or semi-automatically. If there are a series of equiprobable rules to explain an observed result, undercoded abduction is utilized. If there are no rules to explain observed result, we have to create a rule to explain the result. That is the moment that creative abduction occurs.

Except for overcoded abduction, undercoded and creative abduction usually involve three steps (Peng & Reggia, 1990). The first step is generating provisional hypotheses to explain observed results and problematic situations. This step is implemented when we simplify given complex situations into more manageable situations, or models (Lenhard, 2005). The second step is updating hypotheses based on newly available information. We construct abductions based on the observed results of experiments on simplified problem situations. Thus, we can update or generate new abductions based on additional experiments on simplified situations. The third step is verifying a hypothesis utilizing abductions inductively and deductively. These three steps are usually cyclic and non-linear and involve generation, refutation and revision of abductions (Peng & Reggia, 1990).

We adopt Eco's categorization of abductions as a framework for analyzing mathematical inquiries that have been facilitated by mathematical modeling. To be more specific, we analyze how abductions can be constructed using mathematical modeling and how these abductions are related to the students' mathematical inquiries using mathematical modeling. We also adapt three steps of constructing abductions as a framework to analyze mathematical inquiries triggered by mathematical modeling.

Modeling and Abduction

Mathematical modeling usually proceeds cyclically as shown in **Figure 1**.

Building a model of a problem situation (or mathematizing B->C in **Figure 1**) involves searching mathematical rules to explain the observation results of a problem situation and interpreting a given situation using mathematical objects or procedures (Park & Lee, 2016). Given that, we can consider that abductive reasoning intervenes in building models.

In the following, we mainly focus on undercoded abductions and creative abductions triggered by the mathematizing phase (B->C) of modeling in order to identify how modeling and abductive reasoning are related. As reviewed above, overcoded abduction involves automatic or semi-automatic establishment of a hypothesis. Given that, it is unlikely that overcoded abduction can support productive mathematical inquiries. Therefore, investigating undercoded and creative abductions facilitated by modeling is enough to identify a relationship between abductions and mathematical inquiries using modeling.

To analyze undercoded abductions triggered by mathematical modeling, we mainly focus on the results of Park and Lee (2013). McClain and Cobb (2001) noted that analyzing the results of prior research can be an appropriate research method to initiate discussions and draw implications about educational issues. As Park and Lee (2013) reported the detailed processes of students' attempts to discover the mathematical rules that explain a problem situation, we considered the results of this study to be proper research material for analyzing undercoded abductions triggered by mathematical modeling.

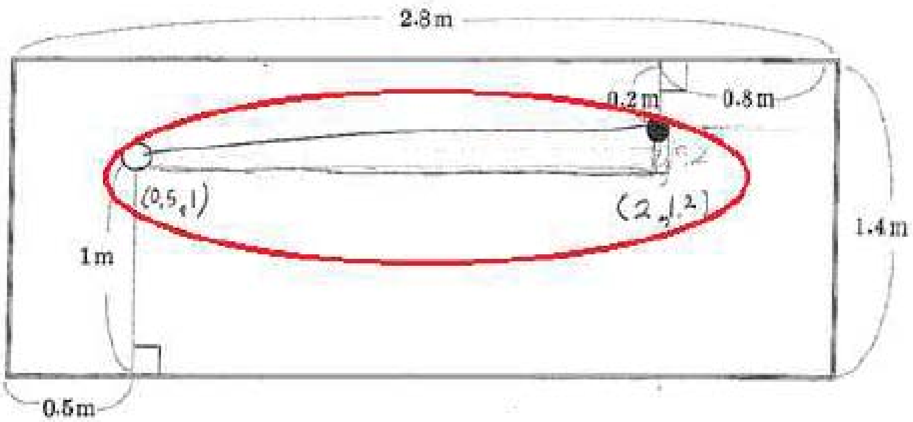

Figure 2. The students' modeling of rectilinear motion (Park and Lee, 2013, p. 102)

To analyze creative abductions triggered by mathematical modeling, we focus on two historic episodes involving Euler and Napier, who created mathematical objects and procedures while modeling. Analysis of key historic episodes can be used in identifying how mathematical inquiries progress (Lee, 2009). Thus, we considered analysis of these two historic episodes to be proper research material for analyzing creative abductions triggered by mathematical modeling. In the following section, we analyze cases involving undercoded and creative abductions triggered by modeling, and identify characteristics of mathematical inquiries supported by mathematical modeling.

CASES: MODELING WITH UNDERCODED AND CREATIVE ABDUCTION

Modeling with Undercoded Abduction

Park and Lee (2013) reported how high school students model the trace of a billiard ball. In this study, the students tried to build mathematical models of the rectilinear motion of balls by interpreting a given problem situation using mathematical objects such as right triangles, lengths, and equations. They considered the given problem situation to be a case in which a general mathematical rule, the Pythagorean theorem (B->C), is applicable. The students then applied the Pythagorean theoream to the given situation and calculated the length between the starting point and end point of the ball (C->D).

We can summarize the students' modeling as follows:

Observing the problem situation
⇓
Searching for mathematical objects and procedures related to the problem situation
⇓
Selecting a mathematical rule: the Pythagorean theorem
⇓
Interpreting the given problem situation to be a case in which the Pythagorean theorem is applicable
⇓
Calculating the distance between two points

The students selected the Pythagorean theorem as the mathematical rule based on the observation results of given problem situation and interpreted the given situation to be a case in which the Pythagorean theorem is applicable. In this case, we can identify the following abductive reasoning:

Abductive reasoning	Modeling
Observation results (Result)	Billiard balls at two points
⇓	⇓
Rule is determined based on observation results (Rule)	Pythagorean theorem
⇓	⇓
Consider the observation results to be a case in which the rule is applicable (Case)	Calculate the distance between two points

As Prawat (1999) pointed out, abductive reasoning is establishing an explanatory hypothesis for a problematic situation. In the above modeling process, the students' abductive reasoning was involved in determining the appropriate mathematical objects or procedures to describe the problem situation. Because the students selected the Pythagorean theorem in particular among alternative mathematical objects, their abductive reasoning was undercoded.

Although the students calculated the distance between two points, this was not the proper way of modeling the trace of the ball (D->E). They then discovered that a linear function is a more appropriate mathematical object to interpret the given problem situation. In this process of interpreting the given problem situation as a case in which a linear function is applicable, the following abductive reasoning was identified.

Abductive reasoning	Modeling
Observation results (Result)	Billiard balls at two points
⇓	⇓
Rule is determined based on observation results (Rule)	Linear function
⇓	⇓
Consider observation results to be a case in which the rule is applicable (Case)	Find a linear function passing through two points

These two rounds of abductions also accord with the arguments of Peng and Reggia (1990):

Peng & Reggia (1990)	Park & Lee (2013)
Initial abduction	Interpret the given situation as a case in which the Pythagorean theorem is applicable
⇓	⇓
Refute the initial abduction/ Revised abduction	Interpret the given situation as a case in which a linear function is applicable
⇓	⇓
Adopt the second abduction	Find a linear function passing through two points

As we identified, the students' use of abduction was undercoded because they selected mathematical rules to model the given problem situation among several available mathematical rules.

Sfard (2008) noted that mathematical discourse develops in two ways. First, endogenous expansion of discourse is "what we observe when discourses grow in volume simply because of their being in constant use" (p. 119). On the other hand, exogenous expansion of discourse is the conflation of several different discourses into one subsuming discourse (p. 122). In Park and Lee's (2013) results above, the students' mathematical achievements were related to endogenous expansion of mathematical discourse. To be more specific, the students tried to apply the mathematical rules (that they already had available them) to unfamiliar new problem situation while modeling. As a result, the students found that a linear function is appropriate to describe rectilinear motion. In other words, the students' mathematical narratives related to the use of linear functions were increased. As such, the students' use of undercoded abduction expanded their use of familiar mathematical language to an unfamiliar context. This was also an attempt to interpret and describe the given problem situation and led to endogenous expansion of mathematical discourse.

Although the students' use of undercoded abduction led to an endogenous expansion of mathematical discourse in the above case, it is difficult to consider the use of undercoded abduction unrelated to exogenous expansion of discourse. The relationship and order of undercoded abduction and creative abduction is not clear, but initial inquiries usually begin with attempts to apply preexisting familiar rules to establish an explanatory hypothesis for a given situation. As Prawat (1999) pointed out, transplanting discursive constructs to new contexts

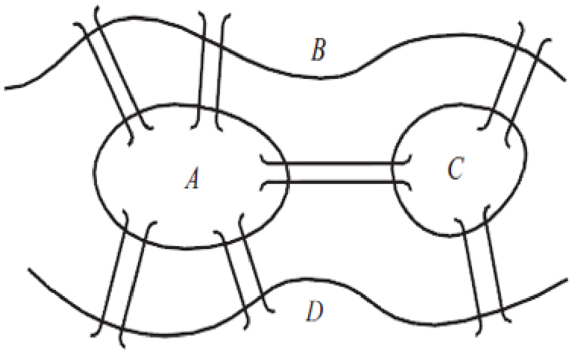

Figure 3. The problem situation of the Königsberg (Burton, 2011, p. 534)

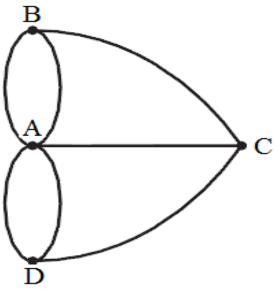

Figure 4. Graphical expression of Königsberg problem

and roughly using them by utilizing metaphors or analogies is an initial process involved in generating abductions. As such, creative abduction creates new rules when it is difficult to explain a given situation using preexisting rules. Thus, the use of undercoded abductions that retrace the use of existing rules is needed to generate creative abductions that enable the creation of new mathematical rules. Therefore, it is important to consider undercoded abduction a foundation of the exogenous expansion of discourse and the creation of new mathematical objects and procedures.

Modeling with Creative Abduction

Euler invented the mathematical object, the graph, while resolving the Königsberg problem (Burton, 2011). The Königsberg problem involves finding a walk through the city that would cross each of seven bridges once and only once (see **Figure 3**).

To resolve this problem, Euler mathematized the problem situation. To be more specific, he mathematized the regions of Königsberg (A, B, C, D) into vertices and bridges into edges, and modeled the situation into a graph as shown in **Figure 4**.

Euler defined a graph as a mathematical object with a set of vertices and edges, as shown in **Figure 4**. Euler then proved that there is no solution to the Königsberg problem using this graph. The problem solving process of the Königsberg problem can be interpreted using the modeling perspective.

Observing the problem situation (the Königsberg problem)
⇓
Searching for mathematical objects and procedures related to problem situation
⇓
There is no mathematical rule to explain the problem situation
⇓
Creating new mathematical objects (Graph)
⇓
Interpreting the given situation as a case in which a graph is applicable

Euler first attempted to resolve the Königsberg problem mathematically, but he could not find appropriate mathematical objects or procedures. He then created and defined a new mathematical object, the graph, and determined the mathematical theorems valid in graph theory to resolve the problem. We can summarize Euler's modeling and abductive reasoning as follows:

Abductive reasoning	Modeling
Observation results (Result)	Königsberg problem
⇓	⇓
Rule is created based on observation results (Rule)	Graph
⇓	⇓
Consider observation results to be a case in which the rule is applicable (Case)	Interpret problem situation as graph

As such, Euler's use of abduction was creative, since he created new mathematical rules to interpret and explain the given problem situation.

Modeling and abductive reasoning are not restricted to real-world problems. We can consider the creation of the logarithm by Napier to be a case of creative abduction using modeling. Napier invented the logarithm to replace the task of multiplying two big numbers with the simpler task of adding two other small numbers. The following number table shows the relationship between arithmetic progression and geometric progression.

Arithmetic Progression	0	1	2	3	4	5	6	7	8	…
Geometric Progression	1	2	4	8	16	32	64	128	258	…

Napier tried to map the multiplication of geometric progression in the second row as the addition of arithmetic progression in the first row.

$$2^1 \times 2^2 = 2^3 :: 1 + 2 = 3$$

To represent this relationship, Napier invented a mathematical object, the logarithm, to describe the multiplication of a geometric progression as the addition of an arithmetic progression (Burton, 2011). Napier's achievement can be interpreted as a result of modeling multiplicative reasoning on big numbers by additive reasoning on small numbers.

Observing multiplication of big numbers
⇓
Searching for mathematical objects and procedures related to the problem situation
⇓
Devising a way to replace multiplication of big numbers by addition of small numbers
⇓
Creating a new mathematical object (Logarithm)
⇓
Interpreting multiplication of big numbers as a case in which the logarithm is applicable

Abductive reasoning	Modeling
Observation results (Result)	Multiplication of big numbers
⇓	⇓
Rule is created based on observation results (Rule)	Logarithm
⇓	⇓
Consider observation result to be a case in which the rule is applicable (Case)	Replace multiplication of big numbers with addition of small numbers using the logarithm

Euler and Napier invented new mathematical objects to explain unfamiliar observation results. These two mathematicians' ways of reasoning using modeling can be interpreted as cases of creative abduction since they created new mathematical objects and procedures to explain a given problem situation.

Also, we can identify exogenous expansion of mathematical discourse in the above abductive reasoning. In other words, Euler and Napier created new mathematical objects to explain unfamiliar situations that are difficult to describe using existing mathematical objects. In these cases, abductive reasoning facilitated by mathematical modeling was closely related to exogenous expansion of mathematical discourse since it supported the creation of new mathematical rules.

DISCUSSION AND CONCLUSION

In this study, we aimed to investigate the nature of mathematical modeling and identify characteristics of mathematical inquiries triggered by mathematical modeling. As a result, we identified the abductive nature of mathematical modeling. Mathematical modeling involves establishment of an explanatory hypothesis for a given problem situation, so mathematical modeling entailed abductive reasoning. We also found that the abductive nature of modeling was a main trigger of mathematical inquiries.

The following characteristics of mathematical inquiries triggered by mathematical modeling can be identified in the results of this study. First, mathematical inquiries triggered by modeling have an abductive aspect. Mathematical inquires resulting from modeling were progressed mostly in abduction-driven ways rather than inductive or deductive ways. Mathematical inquiries resulting from modeling expanded the usage of mathematical language or created new objects by utilizing abductions to build mathematical modeling of an observed problem situation.

Second, mathematical inquiries triggered by mathematical modeling have a recursive aspect. As we have shown, mathematical modeling progresses cyclically. Given that, mathematical inquiries using modeling recursively progress by revising the usage of the mathematical objects that were roughly utilized in building the initial models of the given situation.

Third, mathematical inquiries triggered by mathematical modeling have an analogical aspect. Mathematical inquiries using modeling entail endogenous expansion of mathematical discourse while attempting to transplant the existing usage of mathematical language to unfamiliar contexts. On the other hand, mathematical inquiries using modeling also entail exogenous expansion of mathematical discourse while creating new mathematical objects and transplanting them to other similar situations in order to verify the validity of their usage.

Fourth, mathematical inquiries triggered by mathematical modeling have a *context-dependent* aspect. The validity of the mathematical language used while doing mathematical modeling is confirmed by the problem situation as well as its existing usage. That is, the problem situation and context are the key criteria for the validity of the usage of mathematical language in mathematical inquiries using modeling.

We can raise the following issues from the results of this study. First, we didactically and historically showed that attempts to model a problem situation can facilitate endogenous/exogenous expansion of mathematical discourse. This was closely related to the abductive nature of mathematical modeling, and undercoded abduction and creative abduction can entail endogenous and exogenous expansion of mathematical discourse, respectively.

Second, it is noteworthy to consider that mathematical modeling can be supported by several overlapped abductions. The first phase of mathematical modeling is understanding a problem situation (A->B). As we reviewed, model building involves simplification of a given problem situation so that we can deal with it, and this phase involves interpreting a given situation using already known rules. Given that, Kehle and Lester (2003) pointed out that the first phase of modeling involves abductive reasoning while representing a problem situation different ways. In this study, we mainly focused on the model-building phase (B->C) to identify the abductive nature of modeling and the key characteristics of mathematical inquiries triggered by mathematical modeling. However, abductive reasoning can be utilized in other phases of mathematical modeling, so further studies investigating overlapped abductions in mathematical modeling are encouraged.

Third, we focused in particular on learning mathematical objects or procedures while doing mathematical modeling, but investigation of modeling competency from a Peircean perspective is also encouraged. Peirce emphasized that the method of discovering methods is pure rhetoric. He noted that the pure rhetoric is the highest and most living branch of logic (C.P. 2.332), and it is a method of discovering methods (C.P. 2.108). Modeling competency is related to the ability to perform all of the processes of mathematical modeling. Thus, investigation of modeling competency from the perspective of pure rhetoric is encouraged, because modeling competency requires the ability to devise mathematical modeling methods.

In this study, we investigated the nature of mathematical modeling and identified characteristics of mathematical inquiries triggered by mathematical modeling. Further studies identifying various aspects of mathematical inquiries triggered by modeling are encouraged in order to verify the possibility of making connections between teaching and learning mathematics and modeling various situations.

REFERENCES

Ärlebäck, J. B., & Doerr, H. M. (2015). At the core of modelling: Connecting, coordinating and integrating models. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the 9th Congress of European Research in Mathematics Education.* Prague: Charles University.

Bailer-Jones, D. M. (1999), Tracing the Development of Models in the Philosophy of Science. In L. Magnani, N. J. Nersessian, & P. Thagard (Eds), *Model-based reasoning in scientific discovery* (pp. 23–40). New York: Springer. https://doi.org/10.1007/978-1-4615-4813-3_2

Burton, D. M. (2011). *The history of mathematics*. New York: McGraw-Hill.

Eco, U. (1983). Horns, hooves, insteps: Some hypotheses on three types of abduction. In U. Eco & T. Sebeok (Eds.), *The sign of three: Dupin, Holmes, Peirce* (pp. 198–220). Bloomington, IN: Indiana University Press.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*, Dordecht: Reidel.

Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modelling process, *ZDM, 38*(2), 143-162. https://doi.org/10.1007/BF02655886

Kehle, P. E., & Lester, F. K. (2003). A semiotic look at modeling behavior. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: a models and modelling perspective* (pp. 97–122). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Lee, K.-H. (2009). The role of analogical reasoning in mathematical knowledge construction. *Journal of Educational Research in Mathematics, 19*(3), 355-369.

Lenhard, J. (2005). Deduction, perception and modeling: The two Peirces on the essence of mathematics, In M. H. G. Hoffinann, J. Lenhard & F. Seeger (Eds.) *Activity and Sign - Grounding Mathematics Education* (pp. 313-324). New York: Springer. https://doi.org/10.1007/0-387-24270-8_27

Lesh, R., & Doerr, H. (2003). In what ways does a models and modeling perspective move beyond constructivism? In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 519-556). Mahwah, NJ: Lawrence Erlbaum Associates.

McClain, K., & Cobb, P. (2001). Supporting students' ability to reason about data, *Educational studies in mathematics, 45*, 103-129. https://doi.org/10.1023/A:1013874514650

Ottesen, J. T. (2001) Do not ask mathematics can do for modelling. In D. Holton (Ed) *The Teaching and Learning of Mathematics at University Level: An ICMI Study (pp. 335–346.).* London: Kluwer Academic Publishers.

Park, J., & Lee, K.-H. (2013). A semiotic analysis on mathematization in mathematical modeling process. *Journal of Educational Research in Mathematics, 23*(2). 95-116.

Park, J., & Lee, K.-H. (2016). How can students generalize the chain rule? The roles of abduction in mathematical modeling. *Eurasia Journal of Mathematics, Science & Technology Education, 12*(9), 2257-2278. https://doi.org/10.12973/eurasia.2016.1289a

Park, J., Park, M., Park, M.-S., Cho, J., & Lee, K.-H. (2013). Mathematical modelling as a facilitator to conceptualization of the derivative and the integral in a spreadsheet environment, *Teaching Mathematics and Its Applications, 32*, 123-139. https://doi.org/10.1093/teamat/hrt012

Peirce, C. S. (C.P.) (1931–1935, 1958) *Collected Papers of Charles Sanders Peirce*. Cambridge, MA: Harvard UP.

Peng, Y., & Reggia, J. A. (1990). *Abductive inference models for diagnostic problem-solving*. New York: Springer. https://doi.org/10.1007/978-1-4419-8682-5

Prawat, R. S. (1999). Dewey, Peirce, and the learning paradox, *American Educational Research Journal*, *36*(1), 47-76. https://doi.org/10.3102/00028312036001047

Rothenberg, J. (1989). The Nature of Modeling. In L.E. William, K.A. Loparo, N.R. Nelson (Eds.). *Artificial Intelligence, Simulation, and Modeling* (pp. 75-92). New York: John Wiley and Sons, Inc.

Sfard, A. (2008). *Thinking as communicating, human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press.

http://www.ejmste.com