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A hypothetical learning trajectory for vector subspace: An example with engineering students

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Abstract

The concept of vector subspace presents significant challenges in linear algebra courses due to its abstract nature and the required mathematical formalism. This research characterizes how first-year engineering students construct the concept of a vector subspace through a hypothetical learning trajectory (HLT) grounded in the emergent models design heuristic and inclusive mathematics education. Two cycles of teaching experiments were implemented with 26 and 30 students, respectively, using design-based research. The results show conceptual progression through four milestones: identifying properties in solution sets of homogeneous systems of linear equations, recognizing these as vector subspaces, establishing verification procedures, and autonomously applying the test theorem in various contexts. The integrated inclusive strategies (activation of prior knowledge, collaborative learning, comparative tables, and differentiation) supported students with diverse mathematical backgrounds in moving from situational models to formal reasoning. This study provides empirical evidence on the design of inclusive HLTs for abstract algebraic concepts in university mathematics education.

Keywords: hypothetical learning trajectory, emergent models heuristic, inclusive mathematics education, subspace, homogeneous systems of linear equations

INTRODUCTION

The concept of vector subspace is part of the curriculum in linear algebra courses in many university programs, including engineering. However, learning and teaching vector subspaces represents a significant challenge, mainly due to their high level of abstraction (Dorier, 1995). This challenge has motivated various studies focused on university students' understanding of vector subspaces in $\mathbb{R}^n$ in linear algebra (Stewart & Thomas, 2010; Wawro et al., 2011) and the difficulties they face with this concept (Britton & Henderson, 2009; Mutambara & Bansilal, 2019). These difficulties limit the construction of knowledge and hinder progress in other topics in linear algebra, such as bases and linear transformations.

The problem generated by these difficulties is intensified in today's university context, characterized by increasing student heterogeneity. Greater access to higher education has increased the diversity of educational backgrounds and prior experiences, which demands new teaching strategies (Kosunen, 2018; UNESCO-IESALC, 2020). In particular, mathematics courses in engineering often bring together students with different levels of mathematical preparation, posing an additional challenge for instructors, since this variation affects both student retention and academic performance (Alting & Walser, 2006; Lockhart et al., 2023). In response, mathematics education has emphasized the need to promote inclusive approaches that recognize this diversity and guarantee equitable learning opportunities (Abtahi & Planas, 2024; Jackson-Summers et al., 2024).

Within this framework, hypothetical learning trajectories (HLTs) have emerged as a promising tool for the teaching and learning of linear algebra (Andrews-Larson et al., 2017; Cárcamo et al., 2021). An HLT proposes a path for learning a new mathematical concept, providing valuable information for researchers, curriculum developers, and teachers (Andrews-Larson et al., 2017). An HLT has three components: a learning goal, learning activities, and a hypothetical learning

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Contribution to the literature

- This study addresses a critical gap in university mathematics education by integrating the emergent models design heuristic with inclusive pedagogical strategies to teach the abstract concept of vector subspaces.
- Through a design-based research (DBR) methodology, the article provides empirical evidence of a four-milestone conceptual progression, detailing how students transition from situational models to formal mathematical reasoning.
- The research offers a replicable and inclusive HLT that illustrates how mathematical rigor and equitable participation can be jointly pursued with diverse student profiles.

process (HLP) (Simon, 1995). Although previous studies have documented the difficulties associated with vector subspaces and have proposed specific strategies to address them, such as constructing non-trivial examples, using counterexamples, or exploring subspaces in technology-assisted environments (Caglayan, 2019; Chandler & Taylor, 2008; Mutambara & Bansilal, 2018), these contributions remain isolated suggestions rather than a complete instructional sequence for this concept. In addition, to our knowledge, no HLT has been reported for vector subspace, and no proposal has integrated a heuristic design for progressive formalization with inclusive teaching strategies that respond to the growing heterogeneity of university classrooms. This gap constitutes the primary rationale for the present study. To address it, the objective of this study is to characterize how students construct the concept of vector subspace through an HLT grounded in the emergent models design heuristic and inclusive mathematics education. This study contributes to the field of university mathematics education by providing empirical evidence on the design and implementation of inclusive HLTs for abstract algebraic concepts and by characterizing the conceptual progression of engineering students through the activity levels of emergent models. This information is relevant both for research and for the design of future pedagogical interventions in linear algebra.

THEORETICAL FRAMEWORK

This research is based on two theoretical pillars: the emergent models instructional design heuristic and inclusive mathematics education. These approaches guide the design of learning tasks for the concept of vector subspace and the description of students' progress in understanding this concept. Additionally, a review of previous studies on vector subspace is presented, which contextualizes the specific difficulties of the concept and supports the need for the proposed approach. This section begins by specifying the conceptual and operational definition of vector subspace adopted throughout the study.

The Concept of Vector Subspace

Conceptually, a vector subspace is defined as a subset W of a vector space V that is itself a vector space under the operations of vector addition and scalar multiplication defined on V (Larson, 2014; Poole, 2011). Operationally, this study adopts the formulation of the subspace test used in the learning tasks, according to which a subset W of V is a vector subspace of V if the zero vector of V belongs to W and W is closed under vector addition and under scalar multiplication, where the first condition guarantees that W is nonempty (Larson, 2014; Poole, 2011). This operational definition is used consistently throughout the study: the learning tasks ask students to examine these three conditions, the HLP anticipates how students verify them, and the retrospective analysis characterizes students' progression in terms of how they identify, apply, and generalize these conditions in different vector spaces.

Emergent Models Instructional Design Heuristic

The emergent models instructional design heuristic, proposed by Gravemeijer (1999), aims to support a gradual learning process in which models and mathematical conceptions evolve together. The term "emergent" refers to the process by which models arise and to the process by which these models support the emergence of formal mathematics (Gravemeijer, 2004). According to this heuristic, the model initially emerges as a *model-of* the students' informal strategies. Over time, the model gains independence and becomes a *model-for* more formal and meaningful mathematical reasoning (Gravemeijer, 2004). This process develops through four levels of activity:

(1) situational level (or within the context of the task) involves interpretations and solutions that depend on understanding how to act in the context of the problem,

(2) referential level includes *model-of* that refer to the environment described in the instructional activities,

(3) general level includes *model-for* that facilitate a focus on interpretations and solutions independent of specific images of the situation, and

(4) formal level involves reasoning with conventional symbolism, without relying on the support of *model-for* for mathematical activity (Gravemeijer, 1999).

This heuristic has its roots in realistic mathematics education, in which learning is conceived as a process of progressive mathematization through which students reorganize their informal activity into increasingly formal mathematical reasoning (Gravemeijer & Doorman, 1999). Its relevance for university mathematics has been documented in linear algebra, where it has guided the design of learning trajectories for concepts such as matrices as linear transformations (Andrews-Larson et al., 2017) and spanning set and span (Cárcamo et al., 2021). These studies show that students can construct formal concepts when instruction supports the transition from models-of two models-for, a progression that Zandieh and Rasmussen (2010) characterize as a movement from informal to more formal ways of reasoning. In relation to the topic of this research, the heuristic is particularly appropriate for vector subspace because this concept requires precisely the type of transition that the heuristic promotes, from operating with familiar objects, such as solution sets of systems of linear equations (SLEs), to reasoning with the formal definition in arbitrary vector spaces. In this study, the emergent models heuristic is operationally defined through its four levels of activity, which fulfill a double function: they are design criteria for sequencing the learning tasks of the HLT and analytical categories for characterizing students' conceptual progression.

Inclusive Mathematics Education

Inclusive mathematics education must be accessible and understandable for a diverse range of students (Espinoza, 2023), seeking to provide equitable and meaningful learning opportunities regardless of their abilities, backgrounds, or learning needs (Scherer et al., 2016). From this perspective, there is an emphasis on shifting from an instruction-based approach (Boghossian, 2006) to one focused on student learning, with secondary support from the teacher (Alsina & Franco, 2020). On the other hand, implementing inclusive mathematics education requires finding organizational responses to manage the diversity of all students within the classroom. However, the main challenge is to combine general classroom education with differentiated teaching strategies to address the diversity of students (Laubenstein et al., 2019).

Rather than a set of prescriptions, inclusion in mathematics education has been characterized as both an ideology and a way of teaching, which implies that its implementation must be specified through concrete decisions in the design of instruction (Roos, 2019). In line with this view, inclusive pedagogy proposes extending what is available to everybody, instead of offering different provisions only for some students (Florian & Black-Hawkins, 2011). At the university level, this perspective is especially pertinent in mathematics courses for engineering, where the heterogeneity of students' previous preparation may turn abstract concepts into barriers to participation.

Conceptually, inclusive mathematics education is understood in this study as an approach that seeks to guarantee equitable and meaningful learning opportunities for all students, regardless of their abilities, backgrounds, or learning needs (Espinoza, 2023; Scherer et al., 2016). Operationally, it is defined by four inclusive teaching strategies embedded in the learning tasks of the HLT: activation of prior knowledge, collaborative learning (Slavin, 1995), use of comparative tables, and differentiated instruction (Tomlinson, 2014). In this HLT, differentiation is implemented as a range of task demands available to all students, rather than as separate provision for some of them (Florian & Black-Hawkins, 2011). These strategies are complemented by a scaffolded progression that is realized through the level-by-level sequencing of the four tasks, so it permeates the sequence as a whole rather than constituting a separate strategy.

In the design of the HLT, the emergent models heuristic and inclusive mathematics education fulfill complementary and clearly differentiated roles. The emergent models heuristic constitutes the progression dimension of the design because it determines the sequence of the four learning tasks according to the four levels of activity and provides the analytical categories with which students' conceptual progression is characterized. Inclusive mathematics education constitutes the access dimension of the design because it defines the pedagogical format with which each task is implemented so that the intended progression becomes accessible to students with diverse mathematical backgrounds. Consequently, activation of prior knowledge, collaborative learning, comparative tables, and differentiated instruction are neither steps of the HLT nor levels of the emergent models heuristic. They are inclusive strategies embedded in the learning tasks, whereas the levels of activity describe the expected evolution of students' mathematical reasoning as they move through those tasks. Both dimensions converge in one design because they share the principle of gradual progression: the heuristic organizes the route from situated experiences toward formal reasoning and the inclusive strategies ensure that all students can travel that route.

Previous Studies on Teaching and Learning of Vector Subspace

The concept of vector subspace has been the subject of various studies in university mathematics education due to the difficulties students face in understanding it. Previous research has identified challenges associated with the formalism and abstraction of the concept,

Table 1. Main results of previous studies on the learning and teaching of vector subspace

Study	Main finding
Dorier (1995)	• The concept of vector spaces presents a high level of abstraction, which makes initial understanding difficult.
Stewart and Thomas (2010)	• Lack of construction of visual and embodied ideas about span and subspaces. Teaching usually focuses on symbolic calculations rather than conceptual understanding. • Promote frameworks that combine the visual, symbolic, and formal.
Wawro et al. (2011)	• Students generate varied conceptual images of the subspace: as a geometric object (planes, lines), as a “part of a whole” (contained within a larger space), and as an algebraic object (sets of vectors or algebraic conditions). These images are not always coordinated with the formal definition of subspace (closed under addition and scalar multiplication). • Several students expressed the misconception that $\mathbb{R}^k$ is a subspace of $\mathbb{R}^n$ for $k < n$ (“nested subspaces”).
Britton and Henderson (2009)	• Difficulties with mathematical formalism and the use of logic and set theory, the transition between abstract and algebraic modes, and understanding the very definition of a vector subspace. • Previous concepts for studying vector subspace. • Didactic proposal: Provide students with a table that serves as a template to verify the closure property with respect to addition.
Mutambara and Bansilal (2018)	• Use counterexamples to verify the properties of a vector subspace. • Mathematical concepts associated with vector subspace.
Mutambara and Bansilal (2019)	• Difficulties with the formalism required for proofs, the use of logic and set theory, the transition between abstract and algebraic modes, and understanding the very definition of a vector subspace. • Use counterexamples to verify the properties of a vector subspace.
Chandler and Taylor (2008)	• The construction of non-trivial subspaces facilitates conceptual understanding.
Caglayan (2019)	• The importance of understanding the meaning of the null vector in a subspace and the process of demonstrating the closure properties of the operations. • Mathematical concepts associated with vector subspace.
Cárcamo et al. (2023)	• Specific errors such as incorrect procedures for determining whether a set is a vector subspace, incoherent relationships between concepts (for example, incoherently relating vector subspace with linear independence), and erroneous representations of the elements of a vector subspace. • Previous concepts for studying vector subspace.

including problems with logic and set theory, the transition between abstract and algebraic modes, and specific errors in verification procedures. To address these difficulties, several teaching strategies have been proposed, such as constructing and demonstrating non-trivial subspaces (Chandler & Taylor, 2008), using counterexamples to verify properties (Mutambara & Bansilal, 2018), and emphasizing the understanding of the zero vector and closure properties (Caglayan, 2019). **Table 1** summarizes the main contributions of the literature on vector subspaces and the gaps that justify the need for new teaching proposals.

Taken together, the studies presented in **Table 1** have made it possible to identify the main difficulties and some instructional strategies related to learning about vector subspaces. This information serves as a valuable input for the design of the HLT proposed in this study. It is important to note that none of these studies have explicitly complemented the heuristic of emergent models with inclusive mathematics education in the design of an HLT. Therefore, the novelty of our research lies in articulating both approaches to promote an accessible and equitable conceptual progression for engineering students, thus making an original contribution to the field of university mathematics education.

METHODOLOGY

This study used DBR, which has the general objective of investigating possibilities for educational improvement through the creation and analysis of new forms of learning (Gravemeijer & van Eerde, 2009). Specifically, this research seeks to improve the learning process of the concept of vector subspace by applying an HLT specially designed for this purpose. According to Cobb and Gravemeijer (2008), DBR is developed in three phases:

(1) preparation and design,

(2) teaching experiment, and

(3) retrospective analysis.

In the second phase, classroom interventions are carried out, along with successive iterations of three-step cycles: design, test (classroom intervention and data collection), and revision (analysis of the collected data with the purpose of reviewing and reformulating the hypothesis).

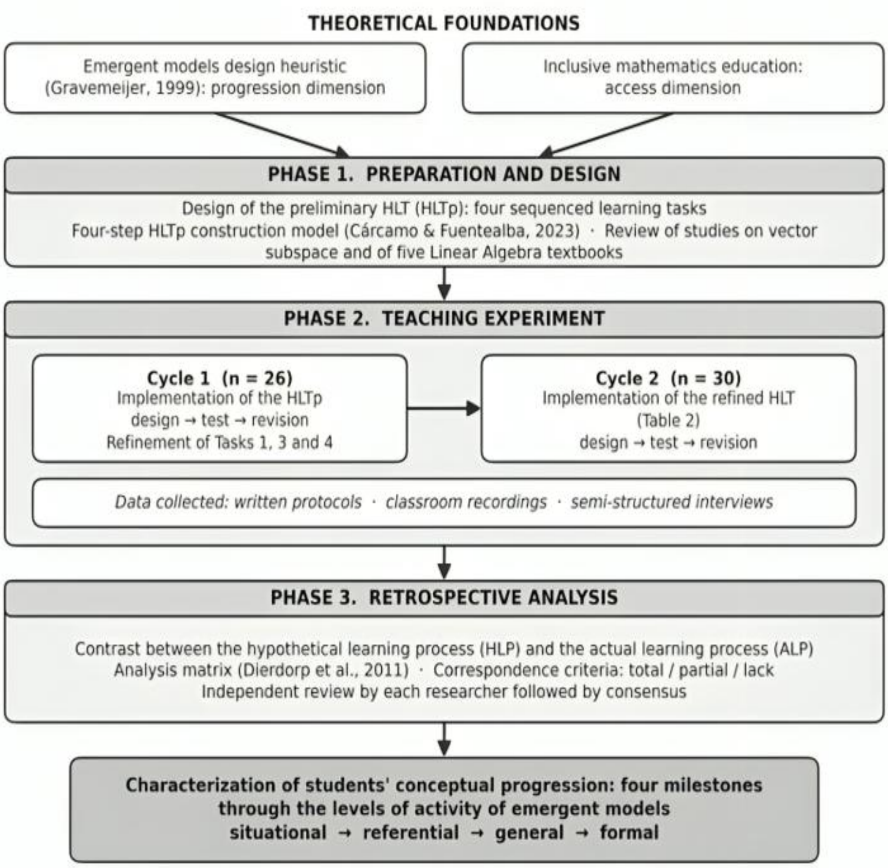

Figure 1. Schematic representation of the design-based research methodology of the study (the authors' own elaboration)

Figure 1 presents a schematic representation of the methodology of the study. It shows how the theoretical foundations informed the design of the HLT in the first phase, how the HLT was implemented and refined across two cycles in the second phase, and how the retrospective analysis contrasted the hypothetical and the actual learning processes (ALPs) to characterize students' conceptual progression.

Preparation and Design Phase: HLT Design

In the preparation and design phase, a preliminary HLT (HLTp) was formulated based on the four-step HLTp construction model (Cárcamo & Fuentealba, 2023):

(1) choose the theoretical approach that will guide the construction of the HLTp,

(2) select the learning objective and identify the central mathematical concept of the HLTp,

(3) review mathematics education studies related to the central mathematical concept of the HLTp, and

(4) examine curricular materials that include the mathematical concept of interest to the HLTp.

In our research, the approaches that guided the construction of the HLTp were the emergent models design heuristic (Gravemeijer, 1999, 2020), which has been used for designing HLTs at the university level (Andrews-Larson et al., 2017; Cárcamo et al., 2021), and inclusive mathematics education. In this design, the two theoretical approaches operated on different planes: the emergent models heuristic guided the sequencing of the learning tasks through its four levels of activity, while inclusive mathematics education guided the pedagogical format of each task through one specific inclusive strategy, as detailed below.

Thus, the learning objective of the HLTp is for students to identify and demonstrate whether any given set is or is not a vector subspace of a particular vector space. To achieve this, students are expected to use the subspace test theorem (Alayont & Schlicker, 2019; Larson, 2014). Therefore, the central concept of the HLTp is the vector subspace. The review of studies on the concept of vector subspace included the search for articles published both in journals and international conference proceedings on mathematics education and engineering education. From this search, we found the following information: prior conceptions necessary for constructing vector subspaces (Britton & Henderson, 2009; Cárcamo et al., 2023), mathematical concepts associated with vector subspaces (Caglayan, 2019; Mutambara & Bansilal, 2018), errors, difficulties, or obstacles related to vector subspaces (Britton & Henderson, 2009; Mutambara & Bansilal, 2019; Wawro et al., 2011; Cárcamo et al., 2023), and teaching suggestions for vector subspace instruction (Britton &

Henderson, 2009; Caglayan, 2019; Chandler & Taylor, 2008; Mutambara & Bansilal, 2018). Finally, we reviewed five linear algebra textbooks (Grossman & Flores, 2019; Larson, 2014; Lay, 2012; Poole, 2011; Strang, 2007) that include the concept of vector subspace in order to analyze the structure they use to explain it.

Based on the selected theoretical approaches and the information obtained from the literature review and curricular materials, an HLTp was designed composed of four sequenced learning tasks, each aimed at promoting students' progression through the levels of activity in emergent models by integrating specific inclusive strategies. Learning task 1 was designed to activate prior knowledge about solution sets of homogeneous and non-homogeneous SLEs, promoting the situational level through analysis of properties such as the presence of the zero vector, closure under addition, and scalar multiplication. Learning task 2 incorporated collaborative learning to facilitate the referential level, aiming for students to recognize vector subspaces using the subspace test theorem. Learning task 3 integrated comparative tables as a tool to promote the general level through the systematization of procedures for verifying vector subspaces in different spaces. Finally, learning task 4 implemented differentiation through tasks involving verification, refutation, and the creation of examples, with the goal of facilitating the formal level by working with vector subspaces in various vector spaces.

Teaching Experiment Phase

In the teaching experiment phase, two cycles of experimentation were carried out in the linear algebra course. In the first cycle, 26 students participated, and the results made it possible to adjust certain learning tasks. Specifically, the wording of the instructions in learning task 1 and learning task 3 was improved for greater clarity, and the level of complexity of some examples in task 4 was adjusted to better address the diversity of students. Consequently, in the second cycle, the refined version of the HLT was implemented (see **Table 2**) with 30 students from the same course but from a different academic year (identified as S_1 to S_{30}). In both cases, the sample corresponded to all students enrolled in the course, and participation was voluntary. All students agreed to take part in the study, so the 30 participants in the second cycle represented both the entire cohort and a group that consented to participate. Therefore, the sample was natural to the course context and aligned with ethical research procedures.

As part of the process, information was gathered from participants across three dimensions. First, their prior knowledge was explored, confirming that all had worked with matrices, determinants, and SLEs—content that served as the starting point for introducing the concept of vector subspaces. This exploration served as a diagnostic baseline of the group and, together with the demographic information and the students' perceptions of inclusion, informed the characterization of its heterogeneity. Moreover, learning task 1 was conceived as an entry point that made students' initial reasoning visible at the situational level and provided an empirical record of their starting points for the retrospective analysis. Second, personal and demographic data were collected, including gender and sex, along with other background information that students themselves considered relevant. The 30 students formed a diverse group in terms of geographic origin (mainly from two regions in southern Chile), with a presence of both men and women, and a significant percentage of first-generation university students. Finally, the perception of inclusion was explored, with students being asked to indicate which aspects facilitated their learning and with which teaching strategies they felt most included in the linear algebra course. This information was fundamental in selecting inclusive practices that matched the needs of this group of students (Marchesi et al., 2009).

During the implementation, the students completed the learning tasks guided by their teacher, integrating the disciplinary aspects of linear algebra with the inclusive practices defined in the HLT.

The qualitative data we collected consisted of the students' written protocols for each of the learning tasks, classroom recordings of the dialogues that took place while they worked on them, and semi-structured interviews with the students to further explore how they resolved the learning tasks.

Retrospective Analysis Phase

The retrospective analysis was based on a qualitative approach aimed at contrasting the HLP defined in the HLT with the ALP observed during the second implementation cycle. This article reports the results of the analysis of the data collected during that cycle. Special attention was paid to how the different levels of activity of the emergent models were manifested in relation to the concept of vector subspace and how inclusive practices were associated with the students' conceptual progression.

The analysis procedure followed three stages. First, each researcher independently reviewed the collected data, identifying relevant excerpts that evidenced the students' progress in reconstructing the ALP. Second, the researchers collectively discussed their selections, reached a consensus on differences, and generated a common characterization of the ALP. Finally, the ALP was compared with the HLP to evaluate the achievement of the learning objective, identify patterns of conceptual progression, and determine which inclusive strategies were associated with this progression.

Table 2. Description of the refined HLT on vector subspace applied in the second cycle of the teaching experiment

Learning objective	The students identify and verify whether a set is or is not a vector subspace of a given vector space.
Learning task	Hypothetical learning process
Task 1. Students are asked to analyze the solution set *S* of homogeneous and non-homogeneous SLEs with infinite solutions and answer the following questions: (a) Is the zero vector in *S*? (b) Write 2 vectors from *S*. Then, add them together and indicate whether the resulting vector is in the set *S*, (c) Write a vector from *S* and any real number. Then, multiply this vector by the real number and indicate whether the resulting vector is in the set *S*.	The students check whether the zero vector satisfies the conditions of each solution set *S* with infinite solutions of non-homogeneous SLEs and conclude that the zero vector is not in any of these sets. Then, the students come up with two particular vectors from each solution set *S* with infinite solutions of non-homogeneous SLEs. They then add these vectors together and conclude that the resulting vector is not in set *S*. Next, the students choose a particular vector from each solution set *S* with infinite solutions of non-homogeneous SLEs and a scalar belonging to the real numbers. They then multiply the vector by the scalar and conclude that the resulting vector is not in set *S*. Finally, the students identify that these sets *S* do not contain the zero vector nor do they satisfy the closure properties for vector addition or scalar multiplication.
Teacher's intervention	*Introduce the definition of a vector subspace and the test theorem for a subspace.*
Task 2. Students are asked to determine whether the sets *S* from task 1 are vector subspaces of $\mathbb{R}^n$.	The students recognize whether the sets *S* they worked with in task 1 satisfy the properties required to be vector subspaces. The students identify that the solution sets with infinite solutions for homogeneous SLE meet the three properties to be vector subspaces. Therefore, they conclude that these are vector subspaces of $\mathbb{R}^n$. In turn, the students identify that the solution sets with infinite solutions for non-homogeneous SLEs do not meet the properties to be vector subspaces of $\mathbb{R}^n$. Therefore, they conclude that these are not vector subspaces.
Teacher's intervention	*Provide feedback on what the students did in task 2.*
Task 3. Students are asked to complete two tables. In the first table, students must propose steps to be followed to verify that a set is a vector subspace of a given vector space and illustrate these steps with three given vector subspaces. In the second table, students must propose steps to be followed to show that a set is not a vector subspace of a given vector space and illustrate these steps with three given sets.	The students conjecture steps to verify that a set is a vector subspace of a certain vector space by using the *subspace test theorem* and what they did in task 2. The students include specific steps to prove the three properties required for a set to be a vector subspace. Consequently, the students exemplify these steps with three given vector subspaces. Afterwards, the students conjecture steps to verify that a set is not a vector subspace of a certain vector space. The students include giving a counterexample as one of these steps. Then, the students exemplify these steps with three given sets.
Teacher's intervention	*Establish, together with the students, the main steps both to verify that a set is a vector subspace of a certain vector space and to determine that it is not.*
Task 4. Determine whether sets from different vector spaces are vector subspaces. Additionally, provide examples of sets that are vector subspaces and sets that are not.	Students consider the *test theorem for a subspace* to identify whether a set is or is not a vector subspace. Then, the students verify that a set is a subspace by checking the three properties. Additionally, the students show that a set is not a vector subspace by indicating which property of the *test theorem for a subspace* is not satisfied and by providing a counterexample. Finally, the students propose examples of sets that are or are not vector subspaces based on the *test theorem for a subspace*.

To organize this process, an analysis matrix adapted from Dierdorp et al. (2011) was used, structured into four columns:

(1) HLP indicators formulated in the HLT,

(2) representative excerpts from students' written or oral responses,

(3) description of the reconstructed ALP, and

(4) assessment of the correspondence between HLP and ALP.

Specific criteria were established for assessing correspondences: total correspondence when the ALP fully met the expectations of the HLP; partial correspondence when some elements of the HLP were observed, but with limitations or minor errors; and lack of correspondence when the ALP differed substantially from the HLP or showed significant conceptual errors. **Table 3** presents an example of how this matrix was applied to analyze students' progression in learning task 2, where they had to recognize which solution sets from learning task 1 were vector subspaces.

This analysis made it possible to identify how students progressed from the empirical exploration of properties (learning task 1, situational level) to the application of mathematical criteria (learning task 2, referential level). In the example from **Table 3**, the

Table 3. Example of an analysis matrix applied to learning task 2

HLP indicator	Audio excerpt	Reconstruction of the ALP	HLP-ALP correspondence
The students identify that the solution sets with infinite solutions of homogeneous SLEs satisfy the three properties required to be a vector subspace of $\mathbb{R}^n$.	S_2: *"Well, in the previous assignment I had already verified that when the system is homogeneous with infinite solutions, the zero vector is always there, and when I add vectors from the set or multiply them by numbers, the result is still in the set. So when I learned the conditions that a set must satisfy to be a subspace, I realized that S is a subspace of $\mathbb{R}^3$ because it met the three conditions of the theorem".*	Student S_2 identifies that the solution set of a homogeneous system of linear equations with infinitely many solutions satisfies the three properties required by the subspace theorem (contains the zero vector, is closed under addition and scalar multiplication) and correctly establishes the connection between their empirical observations from Task 1 and the formal theorem, appropriately concluding that it is a vector subspace of $\mathbb{R}^3$.	Total correspondence: The ALP meets the expectations of the HLP, demonstrating the transition from empirical verification (situational level) to the formal application of the theorem (referential level).

Table 4. Synthesis of the conceptual progression of students through the HLT

M	T	Activity level	Main achievement	Key inclusive strategy
1	1	Situational	They identify properties of the solution set of a SLE with infinite solutions.	Activation of prior knowledge
2	2	Referential	They recognize that the solution set of a SLE with infinite solutions is a vector subspace of $\mathbb{R}^n$, whereas the solution set of a non-homogeneous SLE is not a vector subspace.	Collaborative learning
3	3	General	They establish verification procedures to show that a set is a vector subspace of a certain vector space and to justify why a set is not a vector subspace.	Comparative tables
4	4	Formal	They apply the test theorem for a subspace in different contexts.	Differentiation

Note. M: Milestone & T: Task

evidence suggests that student S_2 was able to connect their previous observations with formal mathematical properties, indicating conceptual progression.

RESULTS

The results show how the process of constructing the concept of vector subspace unfolds through four milestones in the students' conceptual progression (see **Table 4**). In this study, a milestone denotes an empirically reconstructed state of the ALP that condenses the main achievement of one learning task and its corresponding level of activity. This progression, interpreted through the levels of activity of the emergent models heuristic, provides evidence of the role of the inclusive strategies integrated into the HLT in the conceptual understanding of first-year engineering students.

Below, we describe each milestone, illustrating how the students progressed through the four learning tasks of the HLT.

Identification of Properties of the Solution Set of a SLE With Infinite Solutions

Learning task 1 activated the students' prior knowledge about SLE to implicitly introduce the properties that characterize a vector subspace. The students analyzed solution sets S with infinite solutions of homogeneous and non-homogeneous SLEs, verifying three specific properties:

(1) the zero vector is in S,

(2) S is closed under vector addition, and

(3) S is closed under scalar multiplication.

The majority of the students managed to identify that the solution sets of homogeneous SLEs with infinite solutions have distinctive characteristics. They recognized that these sets contain the zero vector and are closed under vector addition and scalar multiplication, whereas the solution sets of non-homogeneous SLEs lack these properties. Student S_3 (**Figure 2**) illustrates this conceptual progress. For a solution set of a homogeneous SLE, they verified that the zero vector belongs to the set: "*if alpha is zero, z is zero, x equals negative four times zero, which is zero, and y equals five times zero, which is also zero. Thus, the vector (0, 0, 0) belongs to S.*" Accordingly, S_3 demonstrated closure under addition by verifying that $v_1 + v_2 = (-4, 5, 1) + (-8, 10, 2) = (-12, 15, 3)$ belongs to set S, arguing that "*the z component is 3, the y component is 5 times 3, and x is negative 4 times 3.*" Finally, S_3 showed closure under scalar multiplication by observing that $2v_1 = (-8, 10, 2)$ also belongs to S.

The main difficulty was related to understanding the concept of set membership. Some students, such as S_7, considered vectors that did not satisfy the conditions of the set when verifying the properties of closure, revealing confusion between the structural form of the

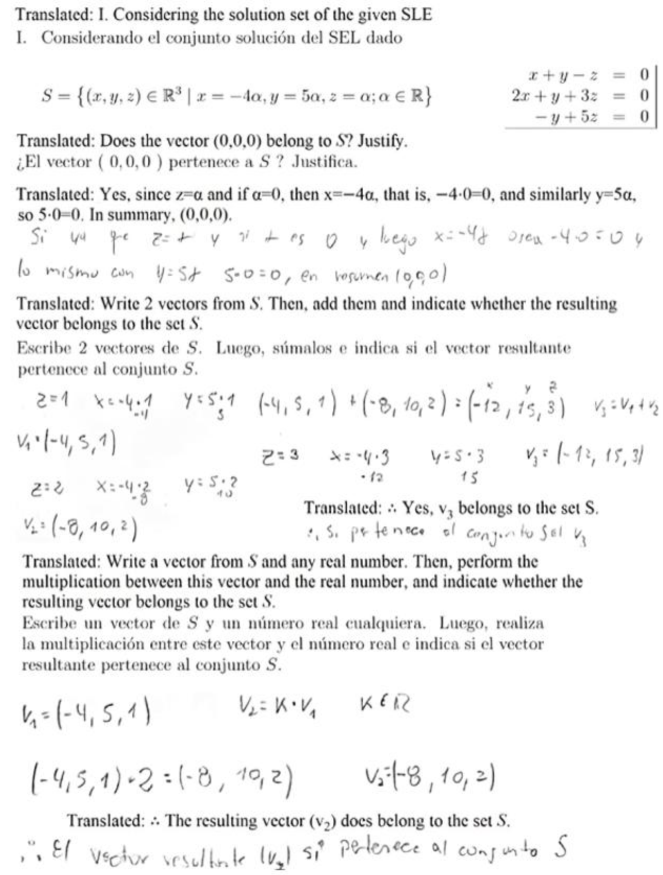

Figure 2. Written protocol of student S_3 for task 1: verification of properties in a solution set of a homogeneous SLE (study data collected by the authors)

vectors (belonging to $\mathbb{R}^n$) and meeting the specific conditions of the solution set S. This first milestone establishes the conceptual foundations for the further development of the concept of vector subspace. The students built intuitions about the three fundamental properties (presence of the zero vector, closure under addition, and scalar multiplication) without yet knowing the formal terminology, preparing them to progress towards the formalization that would occur in learning task 2. The inclusive strategy of activating prior knowledge about SLEs, vectors, and vector operations supported all students by providing a familiar and meaningful starting point, supporting the transition from the situational level to the referential level of the emergent models.

Recognition of Solution Sets of Homogeneous Systems of Linear Equations as Vector Subspaces

After the teacher's intervention introducing the definition of vector subspace and the subspace test theorem, the students tackled learning task 2. This task required them to identify which of the solution sets S analyzed in learning task 1 constituted vector subspaces, applying the newly presented mathematical theory. Most students were able to make the connection between their observations in learning task 1 and the subspace

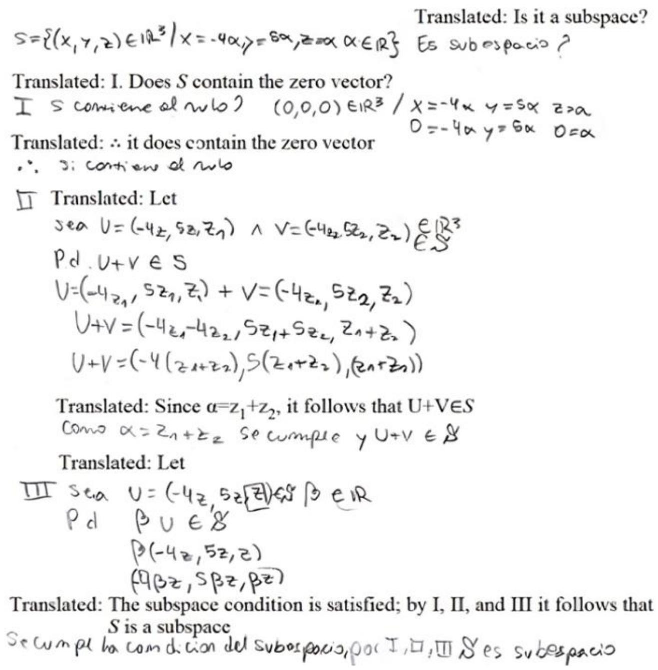

Figure 3. Written protocol of student S_2 for task 2: identification and verification of a solution set of a homogeneous SLE as a vector subspace of $\mathbb{R}^3$ (study data collected by the authors)

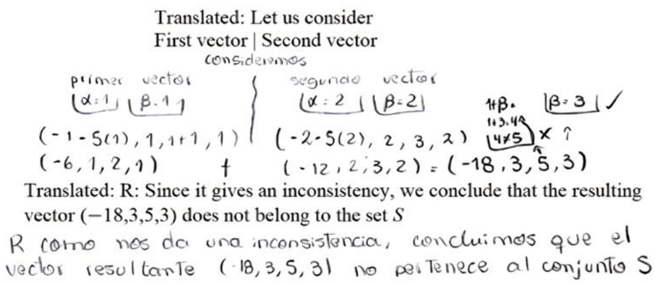

Figure 4. Written protocol of student S_{10} for task 2: verification of a solution set of a non-homogeneous SLE that is not a vector subspace of $\mathbb{R}^4$ (study data collected by the authors)

test theorem. They recognized that the solution sets of homogeneous SLEs satisfied the three properties required to be a vector subspace, while the solution sets of non-homogeneous systems did not. Student S_2 exemplifies this transition from the situational to the referential level (**Figure 3**). For the solution set $S=\{(x, y, z) \in \mathbb{R}^3 : x = -4a, y = 5a, z = a, a \in \mathbb{R}\}$ of a homogeneous SLE, S_2 correctly applied the test theorem: S_2 verified the presence of the zero vector, demonstrated closure under addition using generic vectors u and v, and confirmed closure under scalar multiplication with a generic vector and a scalar β. Finally, S_2 concluded that "it satisfies the subspace condition" and that "*S is a vector subspace of* $\mathbb{R}^3$," demonstrating understanding of the test theorem for a subspace.

In contrast, for solution sets of non-homogeneous SLE, student S_{10} appropriately used counterexamples (**Figure 4**). S_{10} showed that a specific set was not a vector

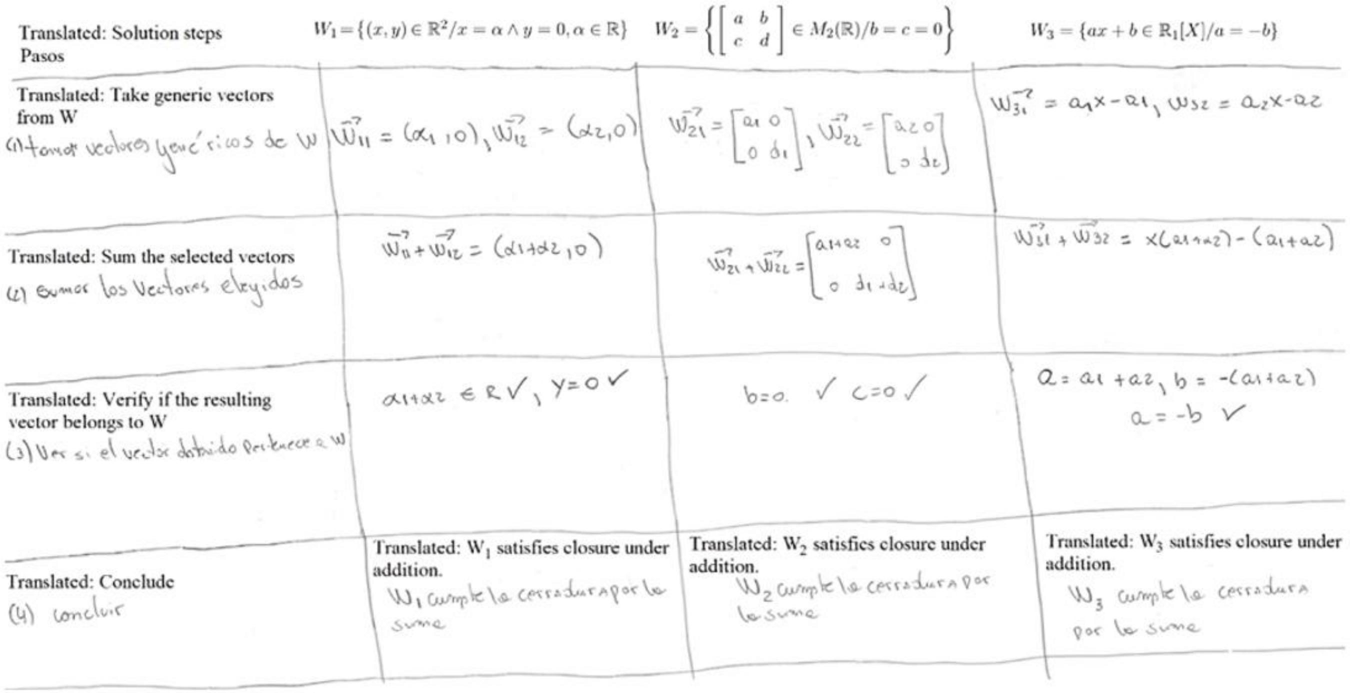

Figure 5. Written protocol of student S_{15} for task 3: comparative table of steps to verify closure under vector addition (study data collected by the authors)

subspace by demonstrating that the sum of two vectors from the set did not belong to it, concluding that "*S does not satisfy closure under vector addition. Therefore, S is not a vector subspace of $\mathbb{R}^4$.*"

Some students made mathematical notation errors when working with generic vectors, revealing difficulties in mastering the formalism. However, these mistakes did not hinder conceptual understanding. This second milestone initiates the transition from the model-of (solution set as a specific example) toward the model-for recognizing patterns of sets that are vector subspaces. The students began to develop a *model-for* identifying vector subspaces, laying the groundwork for establishing procedures that characterize milestone 3. The inclusive strategy of collaborative learning created conditions for students with different levels of mathematical formalization to participate in this conceptual transition.

Identifying Steps to Show That a Set Is or Is Not a Vector Subspace of a Certain Vector Space

Learning task 3 promoted the transition to the general level through the use of comparative tables as a systematization tool. The students completed two double-entry tables to organize the verification procedures: one to demonstrate that a set is a vector subspace and another to justify that it is not. This inclusive strategy was designed to support the visualization and structuring of the formal verification process.

The students progressed from working with solution sets of SLEs to broader contexts (vectors in $\mathbb{R}^n$, matrices, and polynomials), developing procedures applicable regardless of the specific vector space. Most proposed coherent sequences of steps to verify each property of the test theorem. Student S_{15} exemplifies this ability to generalize by proposing steps to verify closure under addition: "*(1) take generic vectors from W, (2) add the chosen vectors, (3) check if the resulting vector belongs to W and conclude.*" S_{15} consistently applied these steps to sets from different vector spaces ($\mathbb{R}^2$, $M_2(\mathbb{R})$, and $\mathbb{R}_1[x]$), working with generic representations in each case (**Figure 5**).

To demonstrate that a set is not a vector subspace, student S_{20} proposed a procedure that includes the strategic search for counterexamples: "*(1) identify the vector space being worked with (2) study the condition or conditions of that set (3) find a counterexample where one of the properties is not satisfied and (4) conclude that the set is not a vector subspace.*" The main difficulty arose with the set that belonged to the vector space of polynomials. Student S_{10}, although proposing correct steps, was unable to complete the verification for the set $W_3 \subset \mathbb{R}_1[X]$, stating: "*I don't know what I should look at in $\beta ax + \beta b$ [the resulting polynomial] to see if it's in the set. I'm confused by the fact that there are so many letters.*" This difficulty suggests that the greater notational complexity of polynomials requires additional support to consolidate understanding.

In this milestone, the students moved from the *model-of* (solution set as a specific example of a subspace) to the *model-for* (identifying subspaces in various contexts). The comparative tables functioned as cognitive organizers that accompanied this abstraction, in which students

Figure 6. Written protocol of student S_{25} from task 4: formal verification that W is a vector subspace using the test theorem (study data collected by the authors)

structured formal thinking independent of particular examples. This transition prepared the formal level where they applied these procedures autonomously and flexibly.

Applying the Test Theorem for a Vector Subspace

Learning task 4 assessed students' ability to apply the subspace test theorem autonomously and flexibly in different vector spaces. This task included both verifying given subspaces and creating their own examples, demonstrating a high level of conceptual understanding. Most students reached the formal level, providing evidence of being able to work with abstract mathematical representations and concepts without relying on specific situational contexts. The students applied the subspace test theorem both to verify and to refute statements about vector subspaces. Student S_{25} illustrates this by verifying that a set W is a vector subspace, rigorously applying the three properties of the subspace test theorem. Their protocol demonstrates structured and autonomous mathematical reasoning (**Figure 6**).

Students' ability to create their own examples demonstrates a higher level of understanding. Student S_{22} constructed a set $W \subset \mathbb{R}^2$ that is a vector subspace and justified their statement by verifying each property (**Figure 7**). In turn, S_{15} created a set that is not a vector subspace of $\mathbb{R}_1[X]$ (the set $W = \{ax + b \in \mathbb{R}_1[x] \mid b = 1\} \not\leq \mathbb{R}_1[x]$), arguing: *"if the vector $2x + 1$ belonging to W is multiplied by the scalar 5, the result is the vector $10x + 5$,*

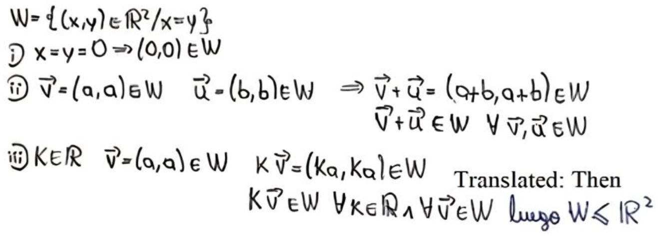

Figure 7. Written protocol of student S_{22} for task 4: creating an example of a vector subspace of $\mathbb{R}^2$ (study data collected by the authors)

which does not belong to W," providing evidence of their understanding of the strategic use of counterexamples.

Some students, although they correctly identified when a set was a vector subspace, had difficulties articulating complete justifications. This limitation was mainly evident in verifying that the resulting vector from operations met the conditions of the set, from which it can be inferred that demonstrative formalism requires additional practice beyond conceptual understanding. In this fourth milestone, students consolidated a *model-for* reasoning about vector subspaces in a general and abstract way. The progression from solution sets of SLEs (milestone 1) to the ability to create and analyze examples in various vector spaces (milestone 4) provides evidence of the potential of the designed HLT. The implemented differentiation strategy, which included verification, refutation, and creation tasks with varying levels of complexity, allowed students with different abilities to demonstrate their conceptual understanding and progress toward formal mathematical reasoning.

The Four Milestones as a Single Progressive Process

Although each milestone was illustrated with representative protocols of different students, the four learning tasks were implemented progressively, in the same order and with the same group of students, and they were anchored initially in a single authentic practical example: the solution set of a homogeneous SLE with infinite solutions. The same set $S = \{(x, y, z) \in \mathbb{R}^3 : x = -4\alpha, y = 5\alpha, z = \alpha, \alpha \in \mathbb{R}\}$ whose properties student S_3 verified empirically in learning task 1 (**Figure 2**) is the set that student S_2 formally proved to be a vector subspace of $\mathbb{R}^3$ in learning task 2 (**Figure 3**). The verification procedure constructed around this example was then generalized in learning task 3, when students systematized steps applicable to sets in other vector spaces, and was used autonomously in learning task 4, when students verified, refuted, and created their own examples. The continuity of this process is also observable among individual students. The excerpt in **Table 3** shows that S_2 connected learning task 2 with the properties verified in learning task 1, stating that "in the previous assignment I had already verified" the behavior of the homogeneous case. S_{10}, who refuted with a counterexample in learning task 2, proposed correct general verification steps in learning task 3, although the polynomial context revealed remaining notational difficulties. S_{15} progressed from proposing general verification steps in learning task 3 to creating a counterexample in the vector space of polynomials in learning task 4. Moreover, the four tasks were implemented with the entire cohort, and the milestone descriptions report tendencies observed across the group, with protocols selected as representative instances through the consensus procedure described in the methodology. Consequently, the milestones reported here are successive states of one and the same process of progressive mathematization anchored in a common example, rather than isolated cases.

DISCUSSION

The discussion addresses the main objective of this study, which was to characterize how the participating students construct the concept of vector subspace through a HLT grounded in the emergent models design heuristic and inclusive mathematics education. The analysis of students' work revealed a progressive construction of this concept across the four activity levels proposed by the emergent models framework, supported by inclusive strategies designed to promote access and participation for all learners.

Conceptual Progression Through the Heuristic of Emergent Models

This research sought to understand how the students develop the concept of vector subspace through an approach based on emergent models and inclusive mathematics education. The results show a progression through four milestones that illustrate the transition from a *model-of* (solution set of a homogeneous system of linear equations) to a *model-for* identifying sets that are vector subspaces in different contexts.

The solution set of homogeneous SLEs initially functioned as an exemplary *model-of* a vector subspace, allowing students to explore closure properties without knowing the formal terminology. This progression stands in positive contrast to the difficulties reported by Stewart and Thomas (2010), who found that traditional teaching focuses on symbolic calculations without gradual conceptual construction. Our HLT, by starting from a familiar context, made it easier for students to construct meaning before facing formal abstraction. Our results complement those of Wawro et al. (2011), who identified that students generate varied conceptual images of subspaces that do not always align with the formal definition. In our study, the systematic use of the subspace test theorem allowed students to progressively coordinate their intuitions with formal properties. For milestone 4, several students reached the formal level, applying the theorem and creating their own examples, suggesting a more robust conceptual construction than that reported in studies where mechanical application predominates (Britton & Henderson, 2009). The transition from the general to the formal level (milestone 3-milestone 4) was accompanied by comparative tables that acted as cognitive organizers, aligning with the recommendations of Britton and Henderson (2009) regarding structured templates.

Two design decisions help to explain why the progression emerged in the reported manner. In terms of the emergent models heuristic, the solution set of a homogeneous SLE functioned simultaneously as a familiar context and as a structurally rich example of the target concept, so each learning task invited students to reorganize their previous activity instead of confronting an unfamiliar formal definition. This mechanism of progressive mathematization (Gravemeijer & Doorman, 1999) explains why formalization appeared as a gradual reorganization of what students already knew rather than as an abrupt rupture, and it may also explain why the difficulties associated with formal definitions in traditional instruction (Britton & Henderson, 2009; Stewart & Thomas, 2010) did not prevent most students from progressing. In terms of inclusion, each strategy reduced a specific barrier at the moment in which heterogeneity could have interrupted the progression: activation of prior knowledge provided a common and familiar starting point, collaborative work created conditions for students with different levels of formalization to participate in the transition to the referential level, comparative tables made the verification procedure explicit for students with less experience in formal proofs, and differentiation allowed students to demonstrate the formal level through tasks of different demands. This combination helps to explain why students with diverse mathematical backgrounds could sustain the same conceptual progression. In turn, the persistent difficulties can be explained by two factors that the design addressed only partially, the syntactic load of polynomial notation and the specific demands of written mathematical argumentation, both of which require more practice time than was available within the sequence.

Contribution of the Inclusive Strategies to the Observed Progression

The integration of inclusive strategies in the HLT was associated with students' sustained participation in the conceptual progression. The activation of prior knowledge (milestone 1) provided a familiar context from which students, regardless of their previous educational background, could begin to build the concept of a vector subspace. Collaborative learning (milestone 2) created opportunities for students with different strengths to support each other in constructing the concept of vector subspace. Comparative tables (milestone 3) functioned as scaffolding that made verification processes explicit, which benefited students less familiar with formal proofs. Differentiation (milestone 4), through verification, refutation, and creation tasks, allowed students at different stages to demonstrate their abilities. These results align with the perspectives of Abtahi and Planas (2024) and Jackson-Summers et al. (2024), who emphasize that inclusion means recognizing and valuing the diversity of starting points and learning paces.

Persistent Difficulties and Limitations of the Study

Although the results show significant conceptual progress, challenges persist that require attention. Difficulties with mathematical formalism (Britton & Henderson, 2009; Mutambara & Bansilal, 2019) appeared as errors in notation or incomplete justifications. This persistence suggests that demonstrative formalism requires practice beyond conceptual understanding, indicating that the skills of "understanding what to verify" and "formally communicating a proof" are related but distinct abilities. The difficulty with polynomials (milestone 3) provides evidence that this mathematical notation requires more time than the transition from $\mathbb{R}^n$ to matrices. The difficulty with the membership relation (milestone 1) showed that some students did not distinguish between "being a vector in $\mathbb{R}^n$" and "satisfying the specific conditions of the set," highlighting the need to address this distinction explicitly at the start of the HLT.

This study presents some limitations that should be considered when interpreting the results. First, the participants were from a single institutional cohort with a small, non-random sample (26 and 30 students), which limits the generalizability of the findings. Second, the analysis focused on the second implementation cycle and was based mainly on written work and classroom records, which, while allowing reconstruction of conceptual progress, do not fully capture students' cognitive processes. Third, the study followed a DBR approach with a single group and did not include a comparison group or pre- and post-test measures. This decision is consistent with the purpose and the epistemology of DBR, whose aim is not to estimate the effect of an intervention against a control condition but to develop and refine a local instruction theory by contrasting an HLP with the ALP observed in a real classroom (Bakker & van Eerde, 2015; Cobb et al., 2003; Gravemeijer, 2004). For this reason, the claims of this study concern the characterization of how students constructed the concept and the plausibility of the designed trajectory, and no causal effectiveness claims are made. The diagnostic exploration of prior knowledge and the entry-level nature of learning task 1 allowed us to document the students' starting points, although prior achievement was not formally measured beyond this exploration, and we acknowledge that quantifying learning gains would require complementary designs. In this sense, and consistent with DBR, the study seeks transferability to similar instructional contexts rather than statistical generalization. To mitigate these limitations, measures were taken such as triangulating data sources (written protocols, recordings, and interviews), using an analysis matrix with explicit criteria (Dierdorp et al., 2011), and independent coding by two researchers followed by consensus. These actions do not completely eliminate the design's constraints, but they help increase the validity and reliability of the results.

The results of this research open new lines of inquiry into mathematics education at the university level. Future studies could replicate the HLT in different institutional contexts to assess its robustness and adaptability, as well as explore its effectiveness in programs not exclusively focused on engineering, in order to analyze how student characteristics influence the learning of abstract concepts. In particular, quasi-experimental studies with comparison groups and pre- and post-test measures could complement this line of research by estimating the learning gains attributable to the HLT. It is also relevant to conduct longitudinal studies that evaluate knowledge retention and its transfer to other topics in linear algebra, such as bases, linear transformations, and eigenvalues. Finally, it would be valuable to further explore the connection between emergent models and inclusive strategies in the design of HLTs for other advanced algebraic concepts.

CONCLUSIONS

The results provide evidence of conceptual progression through four milestones that illustrate the transition from situational models to formal reasoning. The research question regarding how this construction process develops is addressed by identifying consistent patterns of progression. Students built the concept starting from solution sets of homogeneous SLEs as familiar examples, gradually developing the ability to work autonomously with the subspace test theorem in various vector spaces. This progression was supported by the intentional integration of inclusive strategies at each stage of the HLT, which was designed to provide access to abstract reasoning for students with diverse mathematical backgrounds.

This study provides empirical evidence on the design and implementation of inclusive HLTs for abstract algebraic concepts in university mathematics education. In particular, it offers a detailed characterization of students' conceptual progression through the activity levels of emergent models, presents a concrete example of the articulation between design heuristics and inclusive education, and delivers situated knowledge about how the students construct a formal understanding of vector subspaces. In addition, this study offers a replicable model of an inclusive HLT for abstract algebraic concepts, which may guide future

DBR and instructional innovations in university mathematics education.

The HLT developed and refined across two cycles in this research can inform both curriculum design and teaching practice in linear algebra. Its structure (starting from familiar contexts, gradual progression through well-sequenced activities, integration of strategies that address diversity) is transferable to other concepts where mathematical abstraction and student heterogeneity converge.

This research suggests that it is possible to promote both equity and mathematical rigor simultaneously in university education. Inclusive HLTs represent a way for students with different starting points to access abstract concepts without compromising conceptual understanding or mathematical formalism.

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