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A study on the representations and generalization strategies of numerical sequences among secondary school students

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Abstract
Algebraic thinking in secondary education has been widely studied through the exploration of patterns. This study aimed to identify the representations developed by secondary school students to express a numerical sequence and to describe the strategies they employed for its generalization. To this end, a questionnaire was administered to 25 students aged 14 to 15, who were asked to represent four different numerical sequences. The results revealed four levels of representation: pre-structural, emerging, partially structured, and fully structured. Similarly, it was found that the strategies varied according to the type of task. In immediate and near generalization, students primarily relied on counting and additive operations, whereas in distant generalization, the strategies diversified, with the correspondence strategy being the most frequently used. Finally, it was evident that the representations produced by the students themselves served as an essential resource that facilitated the construction of their generalizations.

Keywords: figurative patterns, numerical sequences, generalization, strategies

INTRODUCTION

In recent years, research has focused on introducing algebra through patterns, which constitute a key foundation for algebraic thinking and represent an important topic in secondary school mathematics. When addressing patterns and sequences, it is important to distinguish between the two concepts, as many authors closely associate them. Mulligan et al. (2020) note that a lack of awareness of patterns can lead to difficulties in both research and teaching.

A pattern refers to the perception of a recognized regularity within an ordered collection of elements (numerical or figurative) and may exist at non-symbolic levels of generality, such as gestures, everyday language, or drawings (Radford, 2008). In contrast, a sequence involves the mathematical formalization of this regularity through an explicit rule that allows any term to be generated, rather than merely recognizing repetition (Kaput, 2008). In this sense, a pattern may serve as a perceptual starting point, but it becomes a sequence when the student formulates the general relationship–that is, when generalization is expressed through algebraic language (Radford, 2008).

The recognition and analysis of patterns constitute an essential component in the development of mathematical thinking, as they enable the identification of regularities in numbers, shapes, and measures. Learning is understood to begin with concrete situations and gradually progress toward higher levels of abstraction, thereby fostering processes of generalization and representation. This highlights the need for research that effectively integrates the pattern construction with spatial abilities and structural thinking in mathematics education (Acosta & Alsina, 2021, 2022; Mulligan et al., 2020; Vanluydt et al., 2021).

Likewise, the role of representations in the construction of mathematical knowledge has been widely recognized. In this regard, Freudenthal (1991) argues that the progressive development of

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Contribution to the literature

- This research expands the understanding of early algebraic thinking by analyzing how secondary school students construct their own figural representations to generalize numerical sequences.
- The results provide evidence on the relationship between representation and generalization.
- The results offer an integrative perspective that complements the existing literature on the development of algebraic thinking in secondary school students.

mathematical ideas and procedures moves from concrete to abstract and can be expressed through physical objects, natural language, drawings, or conventional symbols. For this reason, Duval (1995) asserts that "no knowledge can be used by an individual without engaging in representational activity."

However, Reed (2001) argues that despite this broad conception of representations, little attention has been given to the "mathematical tool of drawing." The act of drawing involves the creation and manipulation of symbols, which is fundamental to logical-mathematical development.

A recent study reported a strong correlation between mathematical ability and creativity in the invention of figurative patterns (Assmus & Fritzlar, 2022). Another study suggested that presenting growing patterns in a figurative form may enhance academic performance in this area (Mielicki et al., 2021).

Furthermore, representation is closely linked to generalization and algebraic thinking (Kaput, 2008; Radford, 2018; Ureña et al., 2022, 2023; Wilkie, 2016). Generalization plays a fundamental role in the development of knowledge and constitutes a key resource from both mathematical and didactic perspectives (Abramovich & Connell, 2021; Mason et al., 2005). Figurative pattern tasks typically require students to extract the relationship between a term and its position and to use this relationship to determine terms in other positions (Rivera, 2010). As students progress in their education, their generalizations are expected to be expressed through conventional symbolic systems (Kaput, 2008).

The way patterns are presented to students directly influences the strategies they use to solve them and reach generalizations (Mielicki et al., 2021). Therefore, analyzing these strategies helps identify when a particular procedure is most appropriate, providing insights to improve or redirect various forms of reasoning and problem solving (Ureña et al., 2023). The strategies employed also offer valuable information about students' learning, experiences, and practices. Understanding these strategies can serve as an important resource for decision-making in both research and educational contexts (El Mouhayar & Jurdak, 2015; Ureña et al., 2023).

Within this context, the present study focused on analyzing the representations (drawings or figures) produced by secondary school students (aged 14-15) based on a numerical sequence, in contrast to previous studies in which figurative patterns were explicitly provided (Amit & Neria, 2008; El Mouhayar & Jurdak, 2015; Stacey, 1989). Additionally, the study aimed to examine the strategies students used to generalize numerical sequences based on these representations.

This focus arises from the observation that, despite the importance of the topic, few studies have specifically addressed this aspect. Accordingly, we posed the following research questions (RQs):

RQ1. What types of representations do third-grade secondary students use to express a numerical sequence?

RQ2. What strategies do students employ to generalize a given numerical sequence based on their own representations?

Based on these questions, the following study objectives arise:

1. To identify the representations that secondary school students use to express a numerical sequence.
2. To describe the strategies students employ to generalize a given numerical sequence based on their own representations.

NUMERICAL SEQUENCE

A numerical sequence is an ordered list of numbers in which each number follows a specific pattern or rule. The numbers in the sequence are called terms, and each term is related to the previous one according to a defined rule.

This rule can be expressed in several ways: through a general formula defining the nth term, through a set of instructions indicating how to obtain a term from the previous ones, or by providing a series of consecutive, ordered terms, one after another, as in an infinite list of numbers (Bajo-Benito et al., 2021; Nuñez-Gutiérrez & Cabañas-Sánchez, 2023).

In the educational context, Kaput (2008) and Radford (2008) extend this conception by considering the numerical sequence not only as a formal structure but also to fostering generalization and algebraic thinking, in which students progress from recognizing perceptual regularities to expressing them through symbolic rules or algebraic language.

REPRESENTATION

Through the process of representation, students simulate and anticipate reality with the purpose of organizing and directing their activity, while simultaneously producing and structuring action. Within this process, mathematical concepts and principles are identified, interpretations are made, actions are generated, and predictions are formulated (Medrano et al., 2022; Vergnaud & Récopé, 2000).

According to Goldin (2014), the concept of representation refers to visible or tangible elements (external representations), mental or cognitive constructions of individuals (internal representations), or even the act of creating or generating such representations. In mathematics, external representations serve as a means to understand elements of the student's internal representational process; that is, they allow us to identify which knowledge the student associates with a problem, the conflicts they encounter, how they resolve them, the meanings they attribute to mathematical symbols and operations, and the relationships they are able to recognize (Medrano et al., 2022). In this regard, Karmiloff-Smith (1990) found that internal representations reflect the processes of conceptual change during the construction of new knowledge.

In this study, only external representations are considered, understood as visible productions generated by the students. Specifically, the drawings or figures they created to express numerical sequences are analyzed, as these representations allow observation of how students convey their ideas and provide insights into their generalization processes.

In the current literature, the study of representations used by students when working with numerical sequences has primarily focused on symbolic or verbal processes, with an emphasis on how students express generalization through algebraic language or oral explanations. However, research examining external representations in the form of drawings or figures remains scarce, even though these productions offer valuable information about how students visualize and structure numerical relationships when attempting to generalize a pattern.

Recent studies have shown that the use of graphical and figurative representations improves students' understanding of patterns and mathematical relationships across different ages. For example, Acosta and Alsina (2022) reported a 23.8% reduction in incorrect representations when patterns were presented in a graphical format. Similarly, Becker and Rivera (2005) found that students who more frequently employed figurative rather than numerical reasoning were better able to explain and justify closed formulas through correspondence analysis. In line with this, Medrano et al. (2022) observed that third-grade students (8-9 years old) produced more complex representations–such as drawings, diagrams, or self-created symbols–and demonstrated better performance on functional thinking problems, particularly those requiring the identification of patterns or regularities in relationships between variables. These findings highlight the importance of fostering sophisticated representations to enhance understanding and effective problem-solving in mathematics.

PATTERN

A pattern is any predictable regularity or situation that repeats consistently (Mulligan & Mitchelmore, 2009; Stacey, 1989; Steen, 1988). It arises from a generative core that can be repeated or extended in an orderly manner (Castro, 2013) and comprises two components: a cognitive component, related to knowledge of structure, and a metacognitive component, associated with the ability to search for and analyze patterns (Mulligan & Mitchelmore, 2009). Its structure may follow repetitive rules, known as sequencing patterns (MacKay & De Smedt, 2019), or exhibit growing relationships, known as growth patterns, which are particularly relevant for introducing algebraic thinking (Junker et al., 2024; Mielicki et al., 2021; Wijns et al., 2021).

The creation of patterns plays a central role in learning mathematics, as it is considered the core of mathematical ideas and processes (Steen, 1988; Wijns et al., 2019). Developing figurative patterns offers broad creative potential, as they allow for multiple forms of representation and structuring–numerical, geometric, or figurative. This variety of approaches demonstrates how the same sequence can be expressed through different figures, or how visually similar representations can correspond to different numerical sequences (Assmus & Fritzlar, 2022).

Several studies have examined different perspectives on the creation of figurative patterns. Rivera and Becker (2016) analyzed generalization ability in seventh- and eighth-grade students by asking them to provide multiple continuations of patterns. Similarly, Assmus and Fritzlar (2022) compared the creativity of gifted and non-gifted elementary students in inventing patterns using manipulatives, finding slight advantages for the gifted students. Other studies have focused on equations derived from linear or quadratic functions. In this regard, Wilkie (2019) asked secondary students to invent figurative patterns, while Wilkie (2021) explored how prospective teachers created figurative patterns based on quadratic functions.

GENERALIZATION

The concept of generalization is fundamental in mathematical learning, as it enables knowledge to be transferred from previous experiences to new situations and fosters both the understanding of particular cases and the rigorous development of theory (Abramovich & Connell, 2021; Barahmand & Attari, 2024). This process involves moving from the particular to the general in the pursuit of universality (Zhang, 2023).

Kaput (1999) defines generalization as the intentional extension of reasoning beyond specific cases through the identification of common aspects and a focus on patterns, procedures, structures, and their relationships. Expressing a generalization requires translating it into some form of language, which may be formal or, in the case of children, expressed through gestures or intonation.

The generalization of patterns represents a key process in mathematical learning, as it involves identifying regularities and formulating general rules that allow any term of a sequence to be determined (Merino et al., 2013; Radford, 2008; Stacey, 1989). Such tasks foster the development of algebraic thinking and can be classified as near, when they can be solved step by step by verifying all terms, or far, when they require extending the rule beyond practical limits, such as in the case of very advanced terms (Mouhayar & Jurdak, 2013; Stacey, 1989).

Research on generalization among elementary and secondary students, as well as adults, shows that participants employ diverse strategies when solving tasks, particularly those involving figurative patterns, with their choice influenced by multiple factors (Amit & Neria, 2008; El Mouhayar & Jurdak, 2015; Stacey, 1989; Ureña et al., 2023; Wilkie, 2016; Wilkie & Clarke, 2016; Zapatera Llinares, 2018). It has also been found that, while numerical representations predominate in elementary education, secondary students tend to advance toward generalization through algebraic symbolism (Akkan, 2013; Amit & Neria, 2008; Wilkie, 2016).

STRATEGIES

Studies on the generalization of figurative patterns show that students attend to different properties depending on their knowledge and experience, which determines the use of various strategies to approach the same pattern (Rivera & Becker, 2008, 2011). These strategies–understood as procedures applied to representations of concepts and their relationships within a conceptual structure (Rico, 1997)–include the identification of patterns as a fundamental resource for solving generalization tasks (Merino et al., 2013).

Although the literature describes various approaches to strategies for achieving generalization, this study adopts the proposal of Ureña et al. (2022), who offered a detailed classification of the strategies used by students, which includes:

1. **Counting:** The result was obtained by counting certain elements in a pictorial representation.
2. **Additive operationality:** The answer was found through explicit or implicit isolated additions not related to operations performed in previous or subsequent responses to the task.
3. **Multiplicative operationality:** The answer was found through explicit or implicit isolated multiplication or division not related to operations performed in previous responses to the task.
4. **Proportionality:** Proportional reasoning was used to obtain one term as a product of others. This strategy is separated from the previous one to emphasize the specific reasoning and procedure involved.
5. **Correspondence:** A functional correspondence between the associated variables was established and used to describe the situation.
6. **Direct answer:** Answers were obtained without specifying the procedure followed.
7. **Other:** The procedure used could not be classified in any of the above categories.

METHOD

This study employed a qualitative approach with an empirical, descriptive design, as the research aimed to identify the representations students use to express a numerical sequence and to describe the strategies they employ to generalize such a sequence (Cohen & Manion, 2002).

Participants

The study was carried out in a public school located in the metropolitan area, with the participation of 25 third-grade secondary students (14 girls and 11 boys) aged between 14 and 15 years. The selection of this population was justified because, according to the Ministry of Public Education (2024), at this educational level students possess prior knowledge about numerical sequences and patterns, which includes the algebraic representation of sequences and the recognition of arithmetic and geometric progressions in figures and numbers.

Instrument

For this study, a data collection instrument was used and later complemented by an interview. The instrument, designed by El Mouhayar and Jurdak (2015), included four numerical sequences with an increasing number pattern. Of these four numerical sequences, three were linear and one was quadratic. The sequences were as follows: (1, 3, 5, 7, …), (6, 10, 14, 18, …), (10, 17, 24, 31, …), and (2, 5, 10, 17, …). An example of the proposed instrument is shown in **Figure 1**.

1, 3, 5, 7, ….

In the box below, make a series of drawings or figures that represent this sequence of numbers for you.	Explain in detail the reasons why you chose the drawings or figures to represent the sequence.

Figure 1. First task of the questionnaire (Source: Authors' own elaboration)

For each numerical sequence, students were asked to create drawings or figures to represent the pattern within that sequence. Afterwards, they were asked to explain the reasoning behind their chosen representations.

Interview

In this phase of the research, individual semi-structured, task-based interviews were conducted, allowing participants to interact not only with the interviewer but also within a working environment (Goldin, 1997). The interviews were organized into two parts. In the first part, one student per representation level was selected with the purpose of exploring their reasoning in greater depth and obtaining more detailed explanations about the drawings or figures they had created in the questionnaire. The students were intentionally selected to analyze the various forms of thinking characteristic of each level of representation, without aiming for statistical representativeness.

In the second part, the goal was for students to achieve generalization by solving the four tasks included in the instrument. For this phase, four students were selected who, based on the results of the first instrument, demonstrated a developed structural representation level. This selection of students allowed for a deeper understanding of the generalization processes and the quality of mathematical reasoning, giving greater relevance to the detailed analysis of arguments than to the number of students. Each student was assigned one of the previously proposed sequences.

The interview consisted of two main phases:

1. **Introduction phase:** In this phase, students were informed of the purpose of the activity. Instructions were provided both visually (using the drawings or figures they had created in the questionnaire) and verbally (through oral explanations).
2. **Work phase:** This phase lasted 25 minutes, during which students were asked to explain in detail the drawings or figures they had produced in the questionnaire. They were then given the first four terms of the sequence, along with their own figurative patterns. For each numerical sequence, students were asked to predict the subsequent figures. Specifically, they were required to determine the fifth figure (as an *immediate generalization* task), the ninth figure (as a *near generalization*), and the hundredth figure (as a *far generalization* task).

For each instruction, students were asked to explain how they arrived at their answers and to record their procedures in writing.

The selection criteria for the tasks presented to students were based on the different levels of complexity proposed by El Mouhayar and Jurdak (2015), as well as on the types of pattern generalization known as immediate, near, and far.

Data Collection

Prior to its administration, the data collection instrument was validated by a group of experts. In addition, informed consent forms were signed by school authorities and parents, and oral consent was obtained from the students. Data collection was carried out in the classroom for two 50-minute sessions, in the presence of the subject teacher. In the first session, the instrument was administered, and in the second session, the interviews were conducted. Students completed the questionnaires individually.

Data Analysis

Questionnaire analysis

For the instrument, the analysis procedure was carried out based on the studies by Assmus and Fritzlar (2022) and Cruz-Hernández et al. (2024), who established a five-step process:

1. **Identification of a valid figurative pattern:** A pattern was considered valid if it presented regularities across the four terms of the sequence, either in the number of figures or in the corresponding numerical terms.
2. **Determination of characteristics for the differentiated description of valid patterns:** The objective was to describe the characteristics of valid patterns in a differentiated way through a deductive-inductive analysis. Attention was given to the modality of the underlying regularity (number, shape, or both), the continuity of the pattern, written explanations, and the orientation of the construction process.
3. **Evaluation of students' flexibility in pattern creation:** To evaluate flexibility, both the variety of patterns generated and, to a lesser extent, their descriptions were considered. A descriptive system was developed based on relevant characteristics to explain variety in a differentiated and practical manner. This system included five aspects: the type of mathematical relationships, referring to the mathematical relation underlying the figurative pattern's regularity. The shape, considering what figure emerges in the pattern (such as a cross, bar, L-shape, or pyramid) and what elements change regularly. The construction principle, which examines how the drawings or figures are built and aligned. The number of extension directions, which observes how many directions the patterns develop step by step. The focus of the students' written descriptions, which reveals their pattern construction process.
4. **Formation of representation types in drawings or figures:** The goal was to identify the types of representations created by students, defining three types. Type A, drawings or geometric figures represented through points arranged in certain geometric configurations. Type B, realistic drawings or figures, meaning representations of common objects from the everyday environment and type C representations that do not correspond to either of the previous two types.
5. **Levels of representation:** According to the study by Mulligan et al. (2009), responses were classified according to the structural development of a pattern. The pre-structural level indicates that the student has not yet understood the topic; the representations lack structural evidence in the drawings or figures and display idiosyncratic characteristics. The emergent level (inventive-semiotic) shows that the representations include some relevant elements of the given structure, but the figurative structure itself is not represented there is no order or relationship among the drawings or figures. The partial structure level shows most relevant aspects of the figurative structure, but the representation remains incomplete. Finally, the developed structure level integrates the figurative structural characteristics correctly and coherently.

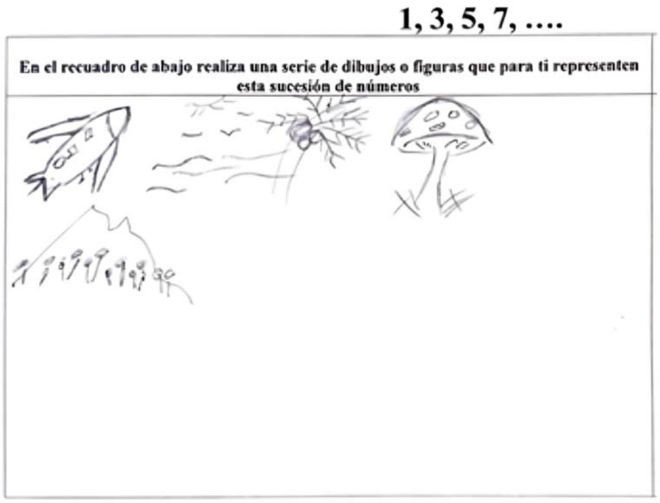

Figure 2. Representation at the pre-structural level (Source: Authors' own elaboration)

RESULTS

To address **RQ1**, this section presents the levels of representation of the identified figural patterns. It then describes their main characteristics and analyzes specific cases in greater depth.

Pre-Structural Level

At this level, 15 representations were identified, all classified as type B. These productions do not show an organization that represents a numerical sequence, nor do they allow the identification of an extensible rule or regularity. The representations are individual and disconnected from each other, responding mainly to the students' imagination rather than to a mathematical relationship (**Figure 2**).

During the interview, the responses of student (E5) showed that their representation was focused on a personal idea of visual growth and revealed a lack of relationship between their drawings and the numerical values of the sequence.

> Interviewer: When you were given the first four terms of the sequence, what did you notice?

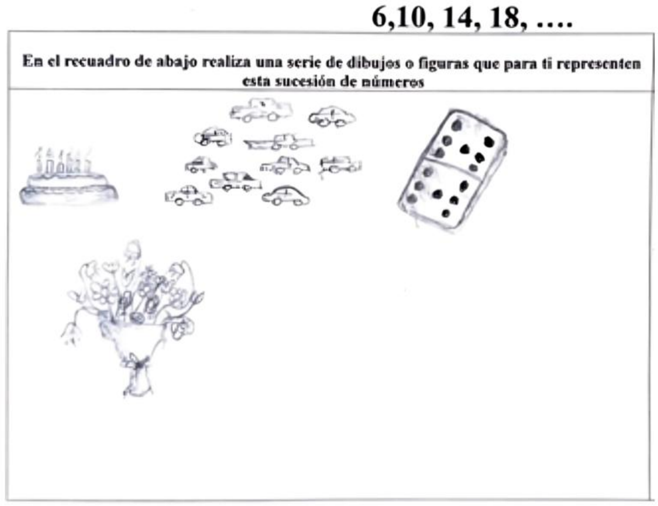

Figure 3. Representation at the emergent level (Source: Authors' own elaboration)

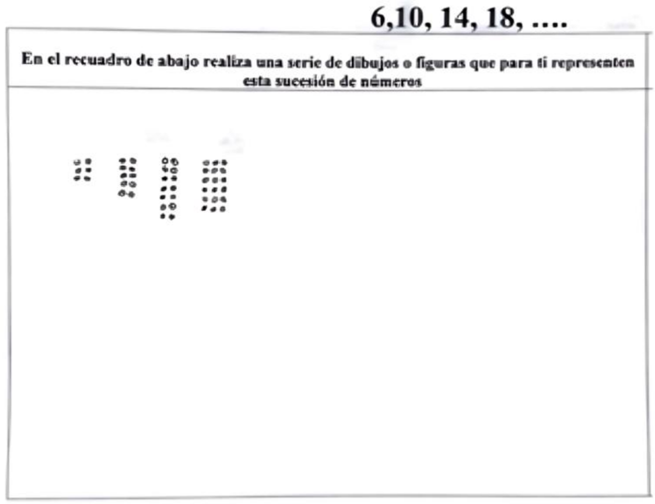

Figure 4. Representation at the partial structure level, type A (Source: Authors' own elaboration)

> E5: When I saw the numbers … the first thing I noticed was that there were numbers, and then I read that we had to make drawings according to the numbers.
>
> Interviewer: Could you describe in detail how you constructed each of the figures you drew?
>
> E5: I assigned a value to each drawing, representing each event I wanted to create–1 being the first priority and 7 the last option.

Emergent Level (Inventive-Semiotic)

Eleven representations were found. These correspond mainly to type B. While some productions show a certain visual or thematic coherence, they do not present a figurative organization that suggests a recognizable pattern or progression. Furthermore, there is no clear connection between the figures, as each one seems to function independently and does not contribute to the development of a sequence or generalization (**Figure 3**).

In the interview, student (E8) demonstrated an initial understanding of the numerical sequence that focused solely on the quantity of elements, rather than on a structure or growth pattern. When describing their drawing, the student explained:

> "I drew a cake with eight candles, then ten cars … because those were the numbers that came next."

When asked whether they noticed any pattern, the student responded:

> "No, just that they were increasing. I mean, there were more each time."

These responses indicate that student's representations were limited to the quantities specified in the numerical sequence, without establishing a structure or recognizing a figurative regularity.

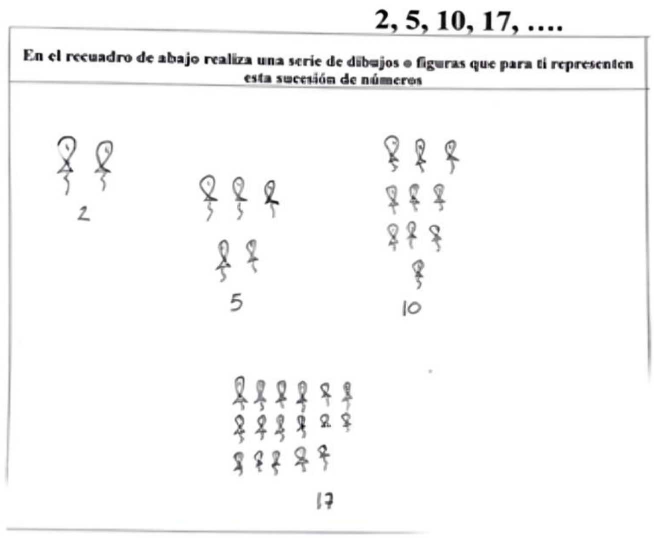

Figure 5. Representation at the partial structure level, type B (Source: Authors' own elaboration)

Partial Structure Level

At this level, 32 representations were identified. Both type A and type B. These representations show relevant aspects of a pattern, such as repetition or growth. Although there is a clear intention to organize the drawings or figures, the sequence is not always maintained consistently or completely. Nonetheless, progress toward structural thinking is evident, as students begin to establish relationships among their drawings or figures and move toward a possible generalization (**Figure 4** and **Figure 5**).

In **Figure 4**, student (E2) represented each term of the sequence through groupings of circles arranged in a rectangular shape, attempting to maintain the structural consistency of the figures.

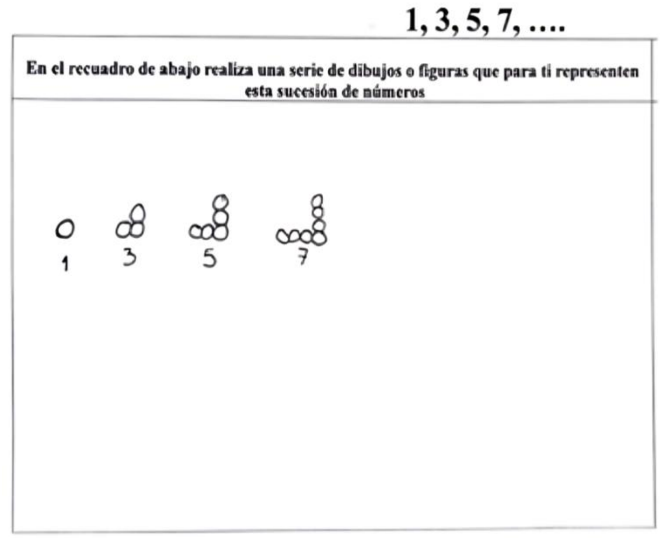

Figure 6. Representation at the developed structure level, type A (Source: Authors' own elaboration)

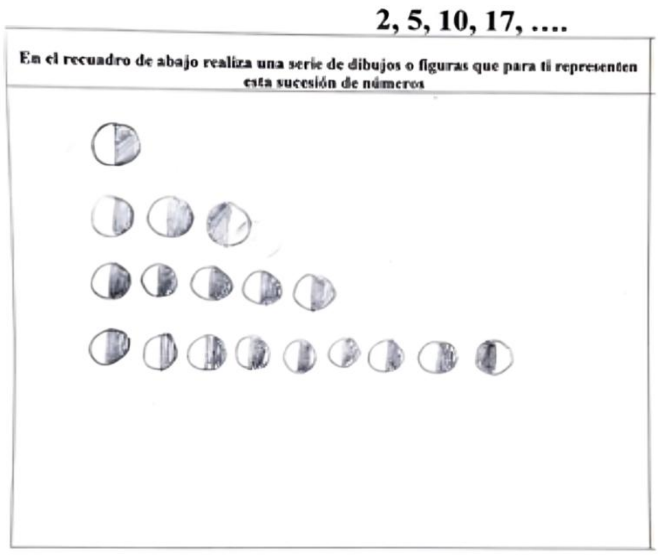

Figure 7. Representation at the developed structure level, type B (Source: Authors' own elaboration)

During the interview, when asked about how each figure was constructed, the student explained:

> "I kept adding more circles in each drawing, two at a time, because that's how it was increasing."

However, when asked about the last drawing which did not follow the same structure as the previous three terms the student argued:

> "I felt that the columns were getting too long, so I preferred to add them in groups of three instead."

These responses show that the student was able to partially recognize the regularity in the sequence's growth but did not fully grasp the figurative structure.

Developed Structure Level

At this level, 42 representations were identified, corresponding to both type A and type B. These representations clearly reflect the expected figurative structure, incorporating features such as repetition, growth, and symmetry. The drawings or figures are organized in a way that makes the sequence easily recognizable. Moreover, the students' work demonstrates a solid understanding of the concept, as they were able to construct diverse forms that convey a coherent mathematical structure (Figure 6 and Figure 7).

In Figure 7, it can be observed that student (E11) used shaded semicircles to represent numerical values, establishing a correspondence between the drawings and the terms of the numerical sequence. During the interview, the student explained:

> "Each figure represents a quantity; the circle shaded on the left side is worth one, and the one shaded on the right is worth two. In each row, I increase the amount just like in the numbers."

This excerpt shows that the student was able to recognize the numerical pattern and successfully create a figurative representation of growth.

Generalization Strategies Used by Students

RQ2 is addressed through the analysis of students' generalization strategies in the interview tasks, which allowed us to identify that students used a variety of strategies, the use of which depended on both the task and the assigned sequence. In total, twelve tasks corresponding to four students were examined, each of whom was presented with a different sequence. Despite this variation, similarities emerged in the strategies used, particularly regarding the level of difficulty associated with the generalization tasks.

In the immediate generalization task, where students were asked to determine the next figure in the sequence, two students were found to have used the counting strategy, while the other two relied on additive operationality (Figure 8). These strategies were influenced by the representations previously created by the students. Those who employed the additive operationality strategy applied addition both implicitly and explicitly.

Similarly, in the near-generalization task–which involved representing the figure in the ninth position of the sequence–all four students again used the counting and additive operationality strategies (Figure 9). This indicates that the students who applied counting based their responses on their previously created representations. However, in the third sequence, one participant chose not to complete the full drawing, explaining that the increasing size of the figures made construction difficult. Instead, they decided to directly calculate the number corresponding to the requested position (Figure 10).

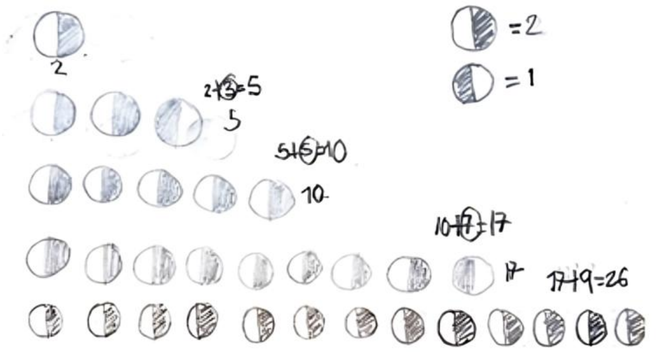

Figure 8. Response to the fourth numerical sequence of the questionnaire: Immediate generalization case (Source: Authors' own elaboration)

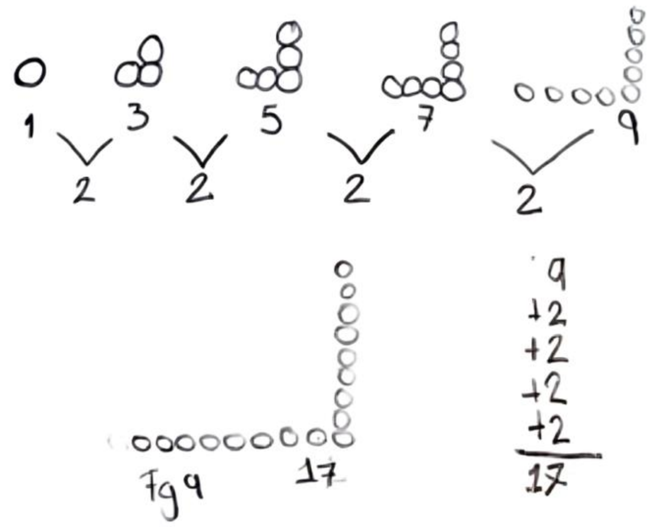

Figure 9. Response to the first numerical sequence of the questionnaire: Near generalization case (Source: Authors' own elaboration)

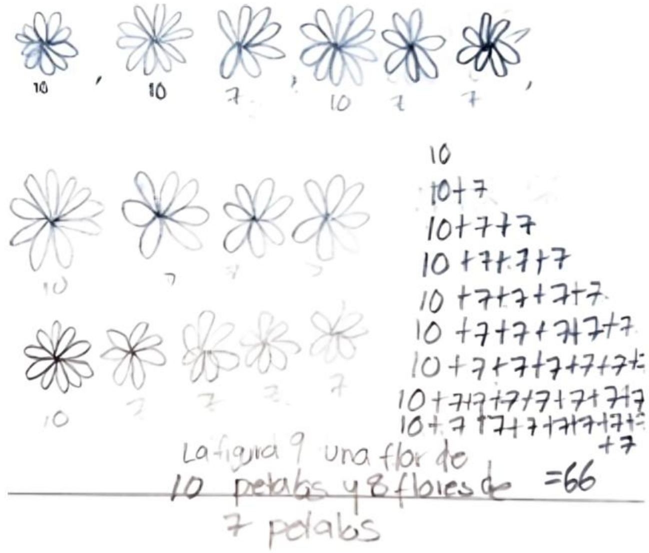

Figure 10. Response to the third numerical sequence of the questionnaire: Near generalization case (Source: Authors' own elaboration)

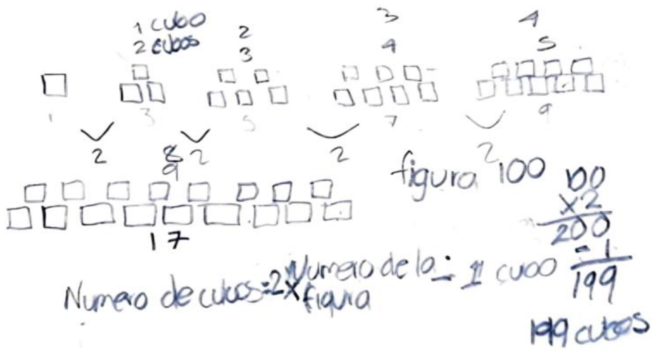

Figure 11. Response to the first numerical sequence of the questionnaire: Far generalization case (Source: Authors' own elaboration)

In **Figure 10**, it can be observed that the student began to show signs of identifying a rule for the numerical sequence. In this case, their own representation served as a resource that allowed them to develop a clearer and more organized idea to progress in the generalization.

Finally, in the far-generalization task, three students employed different strategies–such as multiplicative operationality, additive operationality, and correspondence–to formulate a rule that would allow them to determine the requested term (**Figure 11**).

For this task, the students moved from counting to the correspondence strategy, through which they were able to establish generalization by relating the position of each term in the numerical sequence to the constant difference between them. In this case, the students did not draw the figure corresponding to the 100th term but instead relied on their previous representations to describe it.

In contrast, in the fourth proposed sequence, one student was unable to establish a rule or represent the required figure.

DISCUSSION

The study presented here consisted of a questionnaire administered to secondary school students with prior knowledge of patterns and numerical sequences. It involved 25 third-grade students (aged 14-15) and six individual interviews, each lasting approximately 25 minutes. Although the sample size was relatively small, cautious conclusions could be drawn from the results. This is particularly important for analyzing the four levels of representation identified in this study and the strategies used to achieve generalization.

For the section concerning representations, we analyzed the diversity of drawings created by students in a differentiated manner. Based on this analysis, we assessed the types and levels of representation individually in the creation of figurative patterns. Four levels of representation were identified: pre-structural, emergent, partial structure, and developed structure. These levels, therefore, range from an absence of structure to a correct structural understanding of a pattern.

This progression is consistent with the findings of Mulligan et al. (2009), who also identify levels ranging from a lack of structure to a fully developed structural organization in students' mathematical representations. In this sense, the results of the present study not only confirm the validity of this sequence of levels in a different context but also expand upon it by showing how these levels emerge from representations freely generated by the students themselves in generalization tasks.

As previously described, clear differences were observed among students regarding the types of representations and the occurrence of originality. At the pre-structural and emergent levels, students used only type B drawings or figures. At the partial structure and developed structure levels, students demonstrated the use of both type A and type B representations. Notably, four students exhibited three representation tasks at the developed structure level and one task at the partial structure level. This discrepancy may be attributed either to the influence of the higher numerical position in the sequence or to the fact that students did not recognize that one of their representations failed to meet certain previously established criteria.

Furthermore, it is important to highlight that most of the drawings created by students at any level of representation were original productions. Assmus and Fritzlar (2022) point out that, in the case of figurative patterns, prior experience and knowledge in handling patterns and structures foster the development of original ideas oriented toward making rule-based modifications.

The generalization strategies used by third-grade secondary students represent one of the main contributions of this research, as it was found that students relied on their own representations a fundamental resource that facilitated the discovery of generalizations in numerical sequences.

In most cases, students used two different strategies depending on the task. The counting strategy was the most frequently employed in immediate and near-generalization tasks, whereas the additive operationality strategy appeared more often in near-generalization tasks. The use of both strategies was linked to the students' visual representations of the task (El Mouhayar & Jurdak, 2015; Stacey, 1989), as well as to how they interpreted and approached the given situation. In most cases, the application of these strategies aligns with results reported in other studies (Merino et al., 2013; Ureña et al., 2022; Zapatera Llinares, 2018).

Similarly, students generally did not go beyond the specific cases proposed, nor did they create additional drawings beyond what was requested, limiting themselves to representing only the terms indicated in the generalization task. When asked to describe the 100th figure, students' strategies became more diverse. Because this term was neither small nor close to the previous ones, they could no longer rely on counting or additive operationality strategies. Consequently, they sought alternative procedures, such as formulating a rule; among those who responded, the correspondence strategy was the most prevalent.

One of the main contributions of this study lies in identifying the relationship between the representations created by students and the strategies employed to address generalization tasks. It was observed that both dimensions are closely connected: the representations constructed by students not only reflect their way of conceptualizing the numerical sequence but also serve as a resource that guides and facilitates their choice of strategies.

According to Duval (1995), the visual organization of a representation can foster specific cognitive transformations; in this study, the visual grouping of elements facilitated the transition to additive reasoning by making partial quantities and their increments explicit. In this sense, the representations act as a key support for achieving generalization in numerical sequences.

Moreover, when tackling far generalization, students based their reasoning primarily on the numerical answers obtained from previous questions and the expressions used in their calculations. This indicates an emphasis on numerical rather than visual elements of the tasks, possibly as a result of their learning experiences. This suggests that solving increasingly complex growing pattern problems–those oriented toward finding rules tends to be associated more with convergent thinking than with pattern creation (Amit & Neria, 2008; Ureña et al., 2022). Likewise, the exclusive use of increasing sequences may have limited the range of observable strategies.

CONCLUSION

This study aimed to identify the representations used by secondary school students to express a numerical sequence, as well as to describe the strategies employed to generalize a given numerical sequence based on their own representations.

Based on the analysis, four levels were identified: pre-structural, emergent, partial structure, and developed structure, which demonstrate a gradual evolution from limited and poorly structured representations to constructions showing a clear and coherent organization of the pattern. The finding that most of the productions were original suggests that, even with structural limitations, students tend to develop proposals grounded in their prior knowledge or related to their environment.

Regarding the strategies, students most frequently used counting and additive operationality in immediate and near generalization tasks, while in far generalization tasks, more complex strategies emerged, such as correspondence and multiplicative operationality. This confirms that task complexity directly influences both the choice of procedures and the level of reasoning demonstrated.

In conclusion, one of the main contributions of this study lies in the fact that students created original representations, which in turn significantly facilitated the construction of their generalizations.

Unlike previous studies on pattern generalization that use pre-designed figurative sequences, this research focuses on representations freely generated by the students themselves. This approach allows researchers to analyze not only whether students succeed in generalizing, but also how they construct, transform, and justify their representations throughout the process. In this way, the study provides a deeper understanding of the cognitive and semiotic mechanisms involved in generalization, making visible decisions, difficulties, and conceptual transitions that often remain hidden when working with pre-structured patterns.

We believe that future research could be useful in comparing two groups of students–one provided with pre-defined representations of the sequence and another required to generate their own–in order to analyze how this difference affects their ability to generalize.

They could also consider decreasing numerical sequences to broaden students' understanding of structural reasoning in the face of different types of patterns.

Furthermore, the study may have implications for teaching practice, as the findings suggest that the invention of figurative patterns holds strong potential for fostering creative mathematical activity among all secondary school students, and even at lower educational levels.

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