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Adapting an optimization task for pre-service mathematics teachers in their secondary-tertiary transition in Chile

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Abstract
The purpose of this article is to identify the key elements for enriching mathematical activity during initial teacher education, in line with the skills acquired at secondary school. This is a qualitative study that presents an original methodological contribution: a three-step framework for adapting mathematical tasks, grounded in the activity theory in didactics of mathematics. The study focused on an optimization task given to 50 pre-service mathematics teachers. Data collection included work-in-progress on collaborative whiteboards (Jamboards) and final written assignments. Results demonstrate that overcoming the secondary-tertiary transition discontinuity depends on shifting the didactic contract toward autonomous exploration. GeoGebra emerged as a catalyst for heuristic reasoning and a mediator for conjecture validation, enabling mathematics pre-service teachers to manage complex subgoals and ensure procedural coherence (control activity). In conclusion, this methodological adaptation framework provides a replicable tool for enriching teacher education programs and shortening the gap between these two educational levels.

Keywords: optimization task, secondary-tertiary transition, calculus, teacher education, Klein's first discontinuity

INTRODUCTION

The upper secondary transition is a phenomenon experienced by students who leave secondary school to begin studies in higher education institutions such as universities (Gueudet & Thomas, 2020). In fact, there are ruptures and continuities in this extraordinarily complex change of educational institution. To analyze it, studies use different approaches, involving varied factors such as the individual, socio-cultural, institutional, and epistemological (Di Martino et al., 2023a; Gueudet, 2008; Thoma & Nardi, 2018).

A lot of research has been done to study the upper secondary transition for non-mathematics students (e.g., in engineering or in sciences), and there are numerous international initiatives to address this transition. We find, for example, the transition courses (Viirman, 2022) created to help level students and better prepare them for university courses, or also specialized support programs for students who encounter difficulties in mathematics (Biehler et al., 2018; Lawson et al., 2019). Whatever form these schemes take, they are attempts that consider the contexts of regional institutions, specific to each country. But few studies study this transition for future secondary mathematics teachers in countries where teacher education begins in the first year of university, which is known as the first Klein (2016) discontinuity (Kilpatrick, 2019). Klein (2016) describes the first discontinuity as: "The young university student found himself, at the outset, confronted with problems, which did not suggest, in any, the things with which he had been concerned at school (...)" (Klein, 2016, p. 1). Although research has shown evidence that this is a phenomenon shared in several countries and cultures (Di Martino et al., 2023a, 2023b), and that at the international level, this first discontinuity presents a variety of scenarios (Viirman, 2022), there are very few studies for South America, and the case of Chile has not been studied so far.

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Contribution to the literature

- This study proposes a three-step framework to adapt secondary curriculum tasks into university-level tasks, fostering secondary-tertiary continuity through rich and autonomous mathematical activities.
- It identifies specific epistemological and cognitive ruptures in optimization learning within the Chilean context, providing the study of a new topic at the beginning of initial teacher education, such as optimization, as well as original empirical evidence of institutional gaps in initial teacher education.
- It demonstrates that GeoGebra acts as a tool that mediates for the opportunity to develop autonomous control activity: helps to shape the reasoning of MPST, enabling them to argue and to verify the coherence of their mathematical models. It also advocates for a training model based on the articulation of heuristic mathematics using didactic mathematics, surpassing traditional remedial approaches that focus solely on technical re-teaching.

In Chile, the story of the teacher education program starting from the first year of university dates back to the 19th century and continues until now, where many reforms and curricular changes have been carried out. As is the case in other countries, Chile exhibits a high dropout rate at the outset of university studies among students pursuing mathematical careers (Acuña et al., 2010). Considering the socio-economic context of students, existing leveling strategies to facilitate the secondary-tertiary transition tend to prioritize students who have been historically marginalized (Miranda-Molina, 2022). In different contexts and universities, it has been creating strategies of support students through tutoring or remedial courses to accompany them in the transition to university (Gallardo et al., 2014). In this country, there is a notable absence of studies examining Klein's (2016) first discontinuity from the disciplinary didactic perspective. Moreover, most universities appear to lack strategies specifically designed for teacher education, which are essential for facing the transition of mathematics pre-service teachers (MPST) who have completed secondary education and are enrolled in university (Gueudet & Biza, 2021).

Internationally, there is a paucity of studies addressing this type of first discontinuity in teacher education. A recent study by Liang et al. (2023) to address this issue demonstrates the necessity of training future secondary school teachers to prepare their future students for the transition from secondary to higher education. Then we aim to incorporate this aspect of training into the curriculum at different points in teacher education, beginning in the first year. This would allow training to take place on a continuum, taking advantage of MPTS proximity to high school during their first years as college students.

In line with what has been presented, the present article explores Klein's (2016) first discontinuity in Chile. To this end, our investigation will concentrate on the field of calculus, with a particular emphasis on optimization using the activity theory in didactics of mathematics (ATDM) (Vandebrouck, 2013, 2018) as a theoretical framework. This topic is particularly appropriate for this study, since in the Chilean context calculus is a mathematical domain that is introduced at the end of high school and at the beginning of university. This allows us to analyze the transition from secondary to tertiary education in the initial training of secondary mathematics teachers.

Optimization problem solving at the beginning of university is a little-studied topic (LaRue & Engelke Infante, 2015, Mkhatshwa, 2019), and research on this subject in teacher education does not seem to have been tackled previously. The objective of this study is to undertake a critical analysis of the content offered in secondary education in the Chilean context and in the topic mentioned. In order to address this objective, we seek to answer the following inquiries: What are the defining characteristics of a task that has been adapted from the secondary curriculum to the university curriculum? Does this type of task facilitate continuity in the transition from secondary to higher education? What mathematical activities do prospective teachers develop in response to such a task in the field of optimization?

FRAMEWORK

We use the ATDM (Abboud et al., 2018; Vandebrouck, 2013, 2018), which considers the complexity of mathematical activity in classrooms from an individual and social point of view. Initially introduced by Leontiev (1978) and used in occupational psychology and cognitive ergonomics, this theory has been adapted to Didactics of mathematics with a multidimensional posture focusing on the cognitive and mediative dimensions. The theory distinguishes between task and activity: On the one hand, a *task* is understood as *a goal to be attained under certain circumstances* (Leplat, 1997), which means that we approach the statement to be solved with specific expectations. On the other hand, *activity* is *what a subject engages in during the completion of the task* (Rogalski, 2013), meaning that we consider not only externalized acts, but also internalized acts such as hypotheses, decisions, inferences, etc. Thus, activity is related to the *subject*, and task is related to the *objects of action*. We focus on:

- The study of the *relief* of the mathematical notion at issue, with an analysis of the context (curricula, mathematical notions involved, particularities of the school, the class and the students). This includes an epistemological and curricular analysis, and an analysis of the students' difficulties with the notion of being studied and designing the task(s).
- Analysis of pre-service teacher's *mathematical activity*. Here, it is important to analyze the *tasks* proposed to characterize the intended mathematical activity. For this analysis, it is also important to study how the MPST previous knowledge enables them (or not) to solve the task through the *possible activities*.

To study mathematical activity, Robert (1998) defines *simple or isolated tasks* and *complex tasks*. Simple or isolated tasks do not require any *adaptation of knowledge*, while complex tasks require several adaptations. Among the *adaptations of knowledge*, we find *mixtures of knowledge; the use of intermediaries; a change of mathematical domain; the introduction of steps; the introduction of results from previous questions, etc.* (Horoks & Robert, 2007). In addition, we focus on three mathematical activities that are essential for solving complex tasks (Flores González et al., 2022; Vandebrouck, 2018)[1]:

Recognizing activities mainly occur when students must recognize the mathematical knowledge that can be used to solve the task. They may also be asked to recognize how they can apply or adapt this knowledge. Students can also recognize a method and that various steps in their reasoning can be connected.

Treatment activities refer to all the mathematical activities associated with execution on mathematical objects. Students may be asked to draw a figure, compute, substitute, transform expressions (with or without giving the steps), change semiotic registers (Duval, 2017), change mathematical domains, etc.

Control activities are found when students must highlight the coherence of their mathematical reasoning, by introducing several check points. They also ensure that the answer produced corresponds to the intended goal of the activity.

Finally, we use the notion of didactic contract as used by Pepin (2014) to study the secondary-tertiary transition in mathematics as follows: "a system of rules, mostly implicit, associating the students and the teacher, for a given piece of knowledge" (Brousseau, 1997, p. 15). In our study, the didactic contract occurs between the teacher trainer of the university course, and the pre-service teachers.

AIM AND RESEARCH QUESTIONS

We aim to study the transition from secondary to tertiary education in the initial training of high school mathematics teachers in Chile from a curricular, cognitive and epistemological point of view, and identifying mathematical activities that allow a more continuous transition between both educational levels. Considering the theoretical framework, the research questions are as follows:

1. What are the curricular, cognitive and epistemological characteristics of Klein's (2016) first discontinuity in mathematical optimization within the Chilean context from the perspective of ATDM?
2. What mathematical activities are promoted by a task that seeks to facilitate continuity between secondary and higher education? What mathematical activities do future teachers perform when faced with such a task?

METHODS

This study employs a qualitative three-step framework based on theoretical tools from the ATDM to analyze the secondary-tertiary transition in a bridge course of calculus. We focus on the interpretive analysis of mathematical activities performed by MPST, contrasting potential and effective mathematical activities during the adaptation and implementation of an optimization task.

Context

This work is framed by a reform of a training plan of MPST, which is mainly considered

(1) definition of an explicit graduation profile, considering teaching practices, didactic-disciplinary and pedagogical ambits,

(2) integration of all disciplines in the plan,

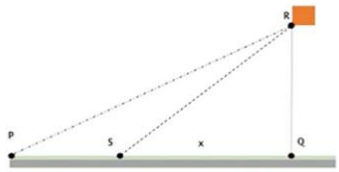
(3) early practices, and

(4) reform of the first year of university, designing bridge courses In coherence with the university educational model, the national curriculum and the standards for the mathematics teacher profession in Chile.

This reform created introductory mathematics courses for the first year of initial teacher education, particularly in algebra, geometry, and calculus. The central idea behind their creation is to address the gaps between secondary school and university that exist in this area. To this end, we considered global research on the subject and Klein's (2016) idea of the first discontinuity. In particular, the work presented here is

[1] In the section titled "Analysis of step 2: Analyzing the original task and an adaptation of this task," readers will find examples of each of these activities as applied to this study.

A water pipeline construction company won the bid for a project to connect a rural house to an existing water and sewer service, which is far away from the house. The schematic drawing below shows the location of the house, whose lot extends into a neighborhood street outlined by the gray strip.



The existing water and sewage service reaches to point P, next to a street that continues in direction Q. House R has the shortest distance QR from the street of 300m. Segment PQ has a length of 500m. Due to the geological situation of the soil, the estimated costs per running meter of the new connection directly next to the street are 3 UF and for the land towards the house are 6 UF.
As the company wants to maximize profits, the connection is modeled by determining the point S on the street side, where the bifurcation of the pipeline to the rural house should start.
a. Determine the costs that would be incurred in construction by direct routing PR.
b. Determine the costs that would be incurred in construction using the composite route PQ and QR.
c. Determine the costs that would be incurred in construction using the composite layout PS or SR with x = 450m.
d. Determine the costs that would be incurred in construction using the composite layout PS or SR with x = 350m.
e. Work out the equation of the function C, which models the costs of the connection to the house in dependence on x.
f. Determine the derivative C' of the function C, applying the composite function rule.
g. Determine the values for which costs reach a minimum.
h. Conjecture about the condition under which the direct duct connection PR is the cheapest. Argue and communicate the conjecture.
i. If the costs in constructing the conduit through the land increase more and more compared to the costs at the street side, in which direction does the location in the schematic drawing that determines the point S of the bifurcation move? Explain and argue without making calculations.

Figure 1. Original task (Ministry of Education of Chile, 2020) (UF, Unidad de Fomento, is a Chilean currency unit)

carried out as part of the Introduction to calculus course, created as part of this reform. In this course, we follow the ideas of Klein (2016) for calculus (Winsløw & Grønbæk, 2014), aiming to revisit the school contents, abilities and tasks from a university standpoint. The course corresponds to the didactic-disciplinary ambit, and it has the secondary school as requisite. It consists of three classes, one exercise class and two workshop sessions in a math laboratory, per week. During COVID-19 pandemics, this was made online by zoom. The adapted task is given to MPST to be solved in one of the workshop sessions. In the room there is the professor and 2-3 teaching assistants, with the mission to mediate the activity of the groups: they move between the groups (during pandemics, using breakout rooms of the zoom platform).

Setting and Participants

Participants included 50 MPST in their first semester, organized into 13 working groups. Prior to the study, MPST covered foundational calculus topics (derivatives, extrema, and optimization criteria). Ethical standards were strictly followed: participants provided informed consent, and all data were anonymized to ensure confidentiality and voluntary participation.

Data Collection

Data collection process uses the following timeline and sources:

1. Documentary analysis (2020-2023): Review of national secondary school curricula (science stream) and university calculus programs to compare optimization learning objectives.
2. Experimental phase (September 2021):
   (a) synchronous: 75-minute Zoom sessions (video/audio) capturing group interactions,
   (b) process monitoring: Real-time access to Google Jamboard collaborative whiteboards to observe MPST draft constructions, and
   (c) asynchronous: a *formal delivery* of the final solution after 8 hours of independent group work.

Data Analysis

A three-step qualitative methodology for task design is proposed due to the nature of the research questions of the study. These are described below.

Step 1. Relief study–Epistemological, cognitive, and curricular analysis

Due to the curriculum to which secondary students and MPST are exposed, we explain differences between institutions and substantial intercultural practices (Di Martino et al., 2023a), and, also, offer rich perspectives on the study of this phenomenon (Gueudet, 2023). We conduct an epistemological, curricular and cognitive analysis to identify the gap between secondary and university optimization tasks. We focused on identifying common difficulties documented in the literature to anticipate potential mathematical activities.

Step 2. Analyzing the original task and an adaptation of this task, considering the possible mathematical activity in both educational levels

The original task appears originally in the program of the elective course for secondary students in the last 2 years of high school, called "limits, derivatives, and integrals" (**Figure 1**).

A water pipeline construction company won the bid for a project to connect a rural house to an existing water and sewer service. The lot where the house is located is rectangular and abuts a neighborhood street. The existing water and sewer service runs to the corner of the lot that intersects with the street. The distance from the house to the street is 300 meters. From the corner of the land that intersects with the neighborhood street to the height of the house going along the same street is 500 meters. Due to the geological situation of the soil, the estimated costs per meter of the new connection going along the street side are 3 UF and along the land towards the house are 6 UF. How many meters of water conduits should go along the street and how many along the land so that the cost is the least possible?

Figure 2. Adapted task for first year of MPST (Source: Authors' own elaboration)

This one transforms into an adapted task for the university bridge course that aims to provide learners with a transit to university calculus based on their prior skills and contributes to developing new university knowledge (**Figure 2**). The criterion for adaptation is:

- Increasing complexity: transitioning from a set of simple tasks to a complex task requiring the coordination of multiple sub-goals (Robert, 1998).
- Variation of representation registers: motivating the interaction between algebraic, graphic, and functional registers (Duval, 2017).
- Change levels of autonomy: moving from a guided school procedure to an autonomous university-level modeling process (model of education of the Chilean university).

We analyze the potential activity by predicting the goals and actions required to solve the adapted task.

Step 3. Implementing the task in class and analyzing the MPST results

We analyzed the effective activity (MPST actual performance) by examining written productions and group interactions: that is, not only external actions to which we have access by means of written productions; but also, internal actions that can give rise to "inferences, hypotheses, thoughts and actions that they decide to undertake or not" (Vandebrouck, 2018). Finally, a comparative analysis was performed between the potential activity (step 2) and the effective activity (step 3) to evaluate the success of the adaptation in fostering university-level mathematical activity.

ANALYSIS AND SYSTEMATIZATION OF THE RESULTS

This section presents the findings of the three steps of our methodology. Using a qualitative and interpretive approach, we analyze the context in which the notion of optimization appears (step 1), then the tasks in question (step 2), and finally the actual mathematical activity of the MPST by comparing the planned activity with the data collected through Jamboard, Zoom, and the final reports (step 3). The analyses are structured according to the mathematical activities of recognition, treatment and control.

Analysis of Step 1: The Relief–Epistemological, Cognitive, and Curricular Analysis

From the curricular point of view, we seek to make explicit and compare what is actually presented in the curricula of both institutions. For this purpose, we focus on the topic of derivatives of real functions, appearing in both institutions, specifically, on optimization using a real context problem. In secondary school, it is seen in the course called "Limits, derivatives and integrals", whereas in the first year of university, this topic is commonly seen in the first semester, in the first course of calculus. **Table 1** displays the declarations made by each institution concerning the areas in which optimization will be introduced.

We observe continuity that infinitesimal calculus is dealt with in both institutions (such as limit and derivative, for example). However, although the secondary curriculum declares to care about the transition from one institution to the other and states *it prepares secondary students for the Calculus courses that are usually taught in higher education*, and the description of the first-year university course declares *MPST will be able to bridge the gap between high school and university,* nothing is said about what will be taken into account in this preparation, or how they will be capable of bridging the gap between both institutions. More specifically, we focus on the abilities that appear to be developed both in secondary and first year of university mathematics in these courses in the topic we are working on here with the task (**Table 2**).

According to the learning goals-results, they are coherent between both institutions, specifically in modeling, where the ability appears to be developed using real context. However, we observe a change of paradigm between secondary and university in terms of visualization. At university we see the importance of developing the ability to visualize a diversity of functions and their graphs, but in school, it seems to be only important to represent certain kinds of functions. The university level is promoted to explore and experiment with notions and objects rather than just solving problems as it is at the secondary level. We do not see an explicit learning goal at secondary level to bridge with university in terms of abilities, it is rather about covering a certain content brought from university level.

Table 1. Curricular statements from upper secondary school[2] and first term in university

Secondary curriculum (limits, derivatives, and integrals)	First year university course of calculus
This course provides the opportunity to understand and use fundamental concepts of infinitesimal calculus. The study is done from an approach that is based both on the abundant use of examples and on the accessible problem solving, as well as on the necessary formalization of the notions used. In this way, it provides opportunities to visualize concepts and situations concepts and situations, to raise and validate conjectures, and to experiment or propose and to experiment or propose solutions, using digital technologies. The course deals with concepts and results that are useful for high school students who wish to pursue higher, technical or university studies. Mathematics is a central tool; in particular, it prepares students for the calculus courses that are usually taught in higher education.	This course aims to provide a friendly, motivating and experiential introduction to the problem of infinity, contextualized in daily, cultural and scientific life, with a minimum of prerequisites. It motivates key notions of differential calculus, such as limit and derivative, through the exploration of processes accessible to MPST. It is intended that, by approaching problematic situations with their own resources, MPST will be able to reach a mathematical language coherent with the usual one and to bridge the gap between high school and university. MPST are expected to develop different strategies for the development of continuous and infinitesimal mathematical thinking. The achievement of this purpose is evidenced by the performance of MPST in several complementary assessments that consider the diversity of the MPST.

Table 2. Comparison of learning goals and results from secondary and university programs

Secondary curriculum knowledge learning goals (limits, derivatives and integrals)	First year university course of calculus learning results
(S1) Use various forms of representation when arguing about the result of the composition of functions and the existence of the inverse function of a given function.	(U1) Uses representations, metaphors and models in order to develop geometric visualization skills, using the graph of real variable functions as a reference to develop their intuition.
(S2) Model situations or phenomena involving instantaneous velocity of change and assess the eventual need to adjust the model obtained and evaluate the eventual need to adjust the model obtained.	(U2) Develops the ability to argue and communicate orally and in writing, results, facts and basic demonstrations of calculus, through the development of continuous and infinitesimal thinking, and an inductive attitude to improve their critical and self-critical capacity.
(S3) Solve problems involving increasing or decreasing, concavity, maximum, minimum or inflection points of a function, from the calculation of the first and second derivative, in handwritten form and using digital technological tools.	(U3) Mathematically model situations in context, using the basic tools of differential calculus, in order to conjecture and induce the future of the situation.

Meanwhile, there is not much research on student activity in solving optimization problems at the beginning of university (LaRue & Engelke Infante, 2015; Mkhatshwa, 2019), and we have not found any research on this topic in the context of teacher education, particularly on the first Klein (2016) discontinuity. Nevertheless, the findings of the studies allow us to posit that the difficulties of students with optimization problems require mastery of several mathematical notions, such as the notion of function with the different types of graphical and algebraic representations and introduction of geometric figures with knowledge of their properties. To complete the relief of the notion, we analyze these difficulties in terms of mathematical activity:

- *Recognizing activity* frequently causes difficulties in solving optimization problems. This involves the definition of variables that represent the quantities present in the problem and that allow the problem to be equated; and the notation of the optimization function, including the use of function composition (often necessary in this type of problem) (Klymchuk et al., 2010; LaRue & Engelke Infante, 2015; Mkhatshwa, 2019). Here, when the problem does not provide the function that we want to optimize, students are less successful at optimization tasks due to difficulties with the definition of variables in the problems and their use (White & Mitchelmore, 1996).
- The *treatment* is also a source of difficulties, in particular the properties of geometric figures such as the rectangle and the key words of the problem (with a view to modelling it) (LaRue & Engelke Infante, 2015), the differentiation of the optimization function, and the graphical representation of the optimization function. Regarding the latter, although it was possible to solve the problem without having to graph the function, the students found it difficult to give

[2] https://www.curriculumnacional.cl/614/articles-91414_bases.pdf

Table 3. Comparison of possible mathematical activities for solving the two tasks

Activity	Original task	Adapted task
Recognizing	The methods to be used and the definitions of variables are explicit in the statements. The function to be optimized is not given, but a geometric diagram is part of the statement. This means that the recognition activities do not need to come from the student.	Methods, variable definitions, geometric diagrams, and small tasks in a chain have been removed. In this way, MPST must develop an activity of recognizing the mathematical concepts to be used to solve the problem and the geometric diagram that represents the situation.
Treatment	Essentially students are asked to use the given geometric diagram (statement) as a reference, calculate the cost of certain paths to pass the water pipe, calculate the derivative, the minimum and argue in the last questions.	The construction of a geometric diagram (starting from diagrams that models the situation) is encouraged. The properties of the derivative, the chain rule, and the use of the first or second derivative criterion are also considered as objectives.
Control	There are no opportunities for explicit control activities, except in the final questions, where students are asked to argue and explain their answers. Moreover, as the statements are written, control is not the focus of the task, as the most common verb is "*to determine*", with no need for students to verify or ensure their answers.	This task is open-ended, with some parameters left free to make decisions: defining variables, the function and validating the minimum, requiring further control activities. The didactic contract established in the classroom and the context allow control occasions: MPST can use graphic calculators to test the coherency of answers to a question.

meaning to the answer found in the graph of the optimization function (LaRue & Engelke Infante, 2015).

- *Control* has not yet been sufficiently studied. Literature argue that the verification of the maximum (or minimum) value is a problem for students; and point out that although some students manage to solve the problem well with the correct procedures, they are not able to argue why these procedures work to answer (Brijlall & Ndlovu, 2013; LaRue & Engelke Infante, 2015; Mkhatshwa, 2019).

Analysis of Step 2: Analyzing the Original Task and an Adaptation of This Task, Considering the Possible Mathematical Activity in Both Schools

The original task is categorized as one that works the ability of modeling in a real context, specifically, it is seen as an optimization problem with a given context, as an application of the derivative topic (Ministry of Education of Chile, 2020). The task at this level of schooling is presented to apply the chain rule to derive a composition of functions. The original task (**Figure 1**) works on the learning goal S3) in **Table 2**. The adapted task (**Figure 2**) promotes an activity on the learning results U1, U2, and U3 in **Table 2**.

From a theoretical perspective, the original task is presented in a way that the secondary students just need to follow steps, blocking any possibility of developing a recognizing activity during the problem solving. The task is presented as a succession of simple tasks and the verb chosen to present is "determine," to order them to calculate the cost of following a certain given path of the pipe. In the letter e) the order is to find the equation of the function using the given variable x (in the figure). Thus, exploration and discussion to recognize the mathematics that help them to solve the task is not present. To find a solution to the problem they should make mathematical treatments such as derivation of a composition of functions, find the cost, use the figure to write the function, determine the minimum value of the function, etc. We observe only a few opportunities to develop control activities in the last two statements in the original task, because the MPST are invited to write the arguments and explanations of those answers, enabling them to highlight coherence in their reasoning and check that the results found correspond to what has been requested (see **Table 3**).

To identify the cognitive processes that the MPST must follow, we differentiate

(1) *Diagrams: figural representations that interpret and organize the information of a real situation.*

(2) *Geometric diagram: figural representation that interprets and organizes the information of a real situation, using geometric properties and theorems.*

The task was adapted in the way it is presented and in the type of activity it proposes from a theoretical point of view, following the methodology exposed. Both tasks offer different recognition, treatment, and control activities (**Table 3**). *Change levels of autonomy:* the adapted task is written as an open-ended statement and a question to be discussed among MPST. *Increasing complexity* given the requirements of the university curriculum, as well as the authors' teaching experience in the institution where the task is to be proposed, the original problem does not represent a valid task to propose as it is at university: it does not encourage significant discursive or deductive work. In addition, the fact of removing the geometric diagram that represents the situation (**Figure 1**), provokes the need for a complex activity of *variation of representation*

Table 4. Sub-goals to achieve to solve the adapted task by the three mathematical activities

Recognizing activities	Treatment activities	Control activities
Sub-goal 1. *Show a diagram to represent the given situation*		
Identify the data in the problem statement to draw a final geometric diagram that represents the situation.	Construct a geometric diagram of the situation described in the problem. Link the identified data to the drawing. This involves a shift in semiotic registers from natural language to figural representations.	Test different positions of the selected data to ensure coherence in the model and check that the relationship is correct. Verify that the correct geometric diagram representing the situation is a right-angled triangle.
Sub-goal 2. *Use the Pythagorean theorem to connect given distances*		
Get to know that the Pythagorean theorem will help to find a solution to the problem.	Effectively use Pythagoras' theorem to relate the given distances and calculate one in terms of the others (leads to a shift in the semiotic register from geometric diagrams to algebraic relationships).	Using Pythagoras's theorem, check the coherence of the data used and verify the possible paths where the pipe could be built and the costs of these paths.
Sub-goal 3. *Define a variable to find the optimization function*		
Distinguish that they need to choose a variable that allows them to define the optimization function.	A variable in meters is shown in the text or diagram. It is defined in writing, namely, "*x is the number of meters of duct that goes by the street*".	The variable chosen is coherent with the problem data and with the diagram constructed.
Sub-goal 4. *Define an optimization function to represent the cost per meter*		
Identify the parameters used to define the equation that will be used to identify an optimization function that represents the cost in UF.	Write an optimization function algebraically using the variable chosen in the previous step. Use the addition and composition rules to write $f(x) = 6\sqrt{(500-x)^2 + 300^2} + 3x$. Define a domain and codomain for this function.	The optimization function defined gives the exact cost of building a pipe as a function of x, and the domain defined of this function considers the problem data.
Sub-goal 5. *Apply a criterion to find critical points*		
Identify the derivative of the function as the first step to find critical points: properties of derivation as the chain rule and the definition of a critical point.	Calculate the derivative of the function. Calculate the value of x such that f' is equal to zero.	Check that the calculation and the properties of derivatives has been carried out correctly. Check the definition of critical points.
The algebraic expression found above can be set to zero to obtain the candidate's minimum value.	Equalize the expression of the derivative to zero and, using known methods for solving the equation (such as the Cardano method), find the value of x when this happens.	Check that the methods used to calculate the solution to the equation have been carried out correctly.
Sub-goal 6. *Find the minimum value and conclude the problem*		
The methods to follow to find that the solution found really corresponds to the minimum sought.	Choose one of the possible methods to find that the value found corresponds to the minimum. Depending on the method chosen, this may lead to changes in the representation register.	Check the coherence of the answer found, find where the error lies and fix it to find a correct minimum value. Write a conclusion about the problem to ensure that the goal of the problem is reached.

registers, in which the MPST must arrive with their own diagrams (and at least one geometric) that represents the situation correctly and coherently in a heuristic way, and also articulates it with algebraic and graphics registers.

Part of the didactic contract presented throughout the first year is to consider control activities as part of all task solutions, even if they are not explicitly requested in the statements. MPST are allowed to use mathematical argumentation in natural language if they do not know how to be more formal about writing an answer, but there must be an explanation of each step, of the decisions made in terms of mobilized mathematical properties, and the conclusions. They can also use a graphing calculator such as GeoGebra; the introduction of digital tools that could help with visualization is part of the course, and it can be used for conjecturing, verifying, and proving.

In the adapted task, we consider the advice in the findings of LaRue and Engelke Infante (2015), by creating opportunities for MPST to explore different examples (even non-optimal ones) of graphs that might correspond to the situation. This will allow MPST to explore the different components of the optimization function in detail. It is then up to the MPST to examine and consider the defined variables, the axes of the optimizing function graph, and all the elements that have an impact on solving the problem.

Following the previous analysis in **Table 4**, we identify the sub-goals associated with each sub-task to be performed that MPST needs to achieve to find a solution to the complex task.

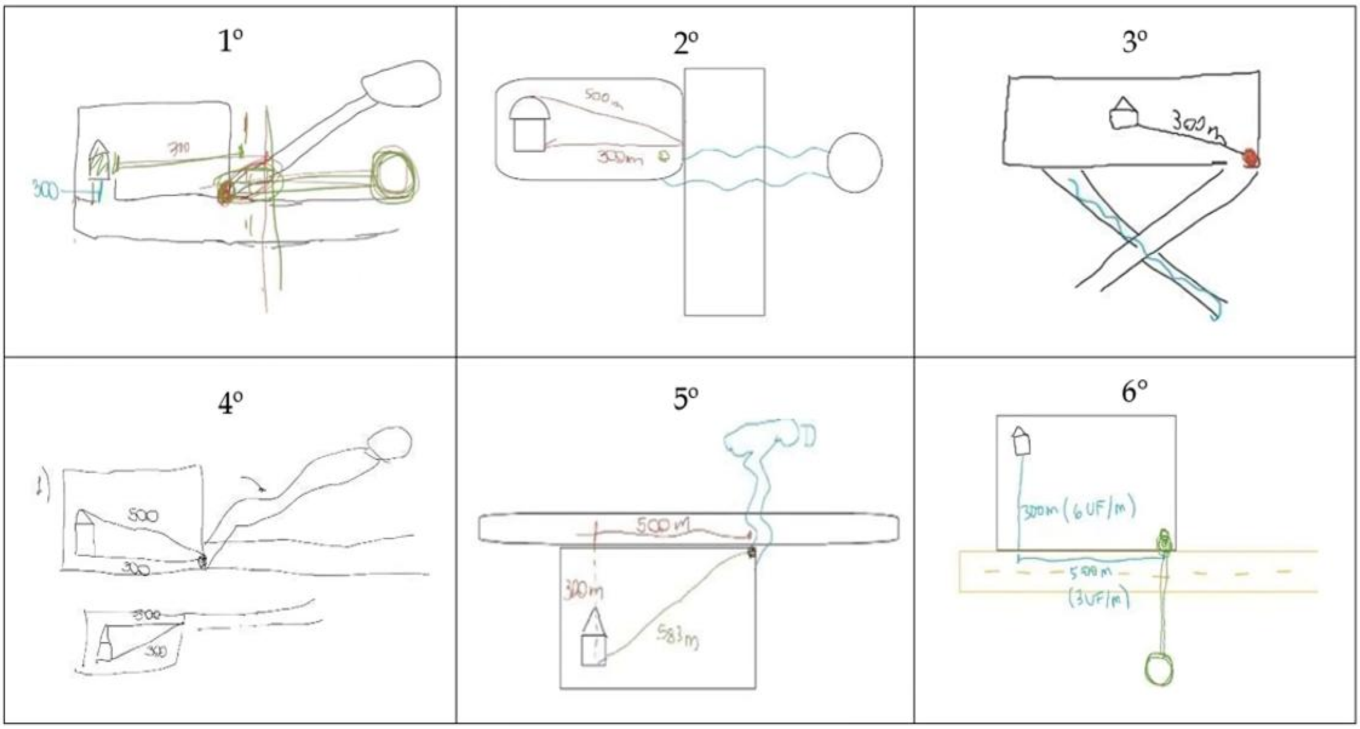


Figure 3. Group 7 trajectory to create a geometric diagram (sub-goal 1) (Source: Student's coursework)

Analysis of Step 3: Implementing the Task in Class and Analyzing the MPST Results

We analyze the responses of 13 groups of MPST obtained during the experiment. First, we present an analysis of the MPST' responses when they used the Jamboard to share their ideas (virtual workshop). Then we present their "formal delivery" after completing the synchronous activity 8 hours later.

Virtual workshop

MPST began solving the problem on the Jamboard drafts. MPST realizes that they need to find a good diagram representing the given situation. To do this, they were told that all the hypotheses of the problem should be included in the final agreed diagram.

In general, the groups in this stage are not very careful writing explanations or a full treatment, because at some moment in the session, the teacher said that those writings will be considered as drafts just to get an idea of their resolution's paths, due to several of them using plenty of time to just reach the sub-goal 1.

All the groups achieve the sub-goal 1 and sub-goal 2 (according to the treatment), as minimum. For sub-goal 1, in the process of finding a good diagram to represent the situation given by the problem, all the groups check the given hypotheses. One by one, they must be checked to confirm that the drawing is coherent.

In **Figure 3**, it is possible to observe the trajectory of one of the groups (group 7) to create the right model, given by the geometric diagram that they define, that represents the situation described in the problem. From left to right, the first and second rows, we see initial diagrams that represent attempts to obtain such a model, which satisfy all the assumptions given in the task. We can see that the first diagram is far from a good one, i.e., recognition activity using the right problem data and treatment activity.

For diagram 2, the recognition and treatment of the data in the statement and their representation are progressing. The right-angled triangle appears, but the data used are not yet correctly placed on the diagram. The MPST needs to mobilize control activity at this point.

We can see that it's in diagram 4 that the MPST begins to mobilize a control activity. Indeed, this is the first time that the correct right-angled triangle appears. Finally, it is from the fifth diagram onwards that the correct right-angled triangle becomes more formalized. Similarly, between the end of diagram 5 and diagram 6, there is not much difference from a mathematical point of view. What changes in 6 is that they add the UF, and this seems to control that they are drawing a correct diagram: the geometric diagram searched.

For sub-goal 2, all groups use the Pythagorean theorem, but some groups are unable to develop control activities. This means that only some of them test different positions of the data on the drawn diagrams (places where the pipe could be built and the costs of these paths) using the Pythagorean theorem. Specifically, groups 1, 2, 3, 5 and 12 do not test the possibilities, meaning they do not control possibilities (**Figure 4**).

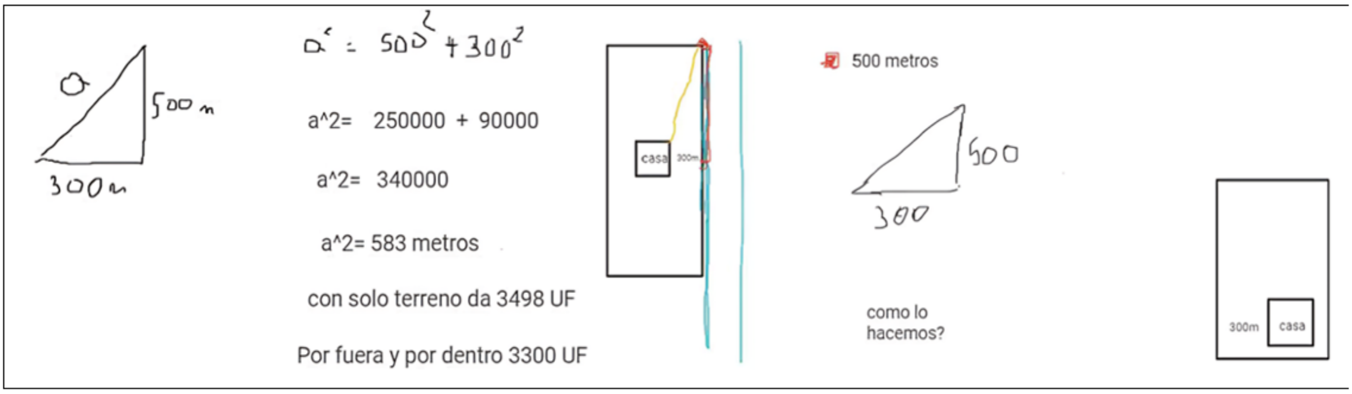


Figure 4. Group 5 checks only one price by the cathetus but does not check other possibilities (Source: Student's coursework)

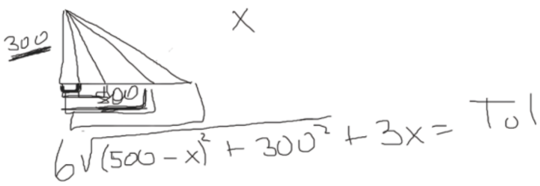


Figure 5. Group 13 attempts to model a function without defining variable x (Source: Student's coursework)

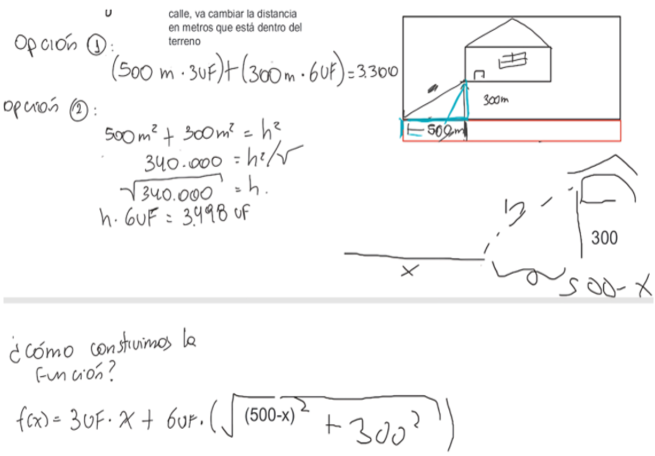


Figure 6. Group 9 attempts to model a function without defining a domain of the variable x (Source: Student's coursework)

For sub-goal 3, groups 1, 7 y 13 did not do the recognizing activity in this stage, because they use plenty of time in sub-goal 1, which is possible to see in the Jamboards (**Figure 5**). The rest of the groups show the variable to find the optimization function in the diagram they found. None of them define it explicitly in the text, but it still appears in the diagram, then the activity of treatment is achieved and control is not achieved.

For sub- goal 4, the groups that recognize the defined optimization function (groups 2, 3, 4, 5, 6, 9, 10, 11, 12) did not do the treatment completely because they forgot to define the domain of the function. Then none has the treatment and control activity (**Figure 6**).

For sub-goal 5 on the application of a criterion to find the critical points, it is interesting that the groups 5 and 6 reach the sub- goal 6 without the 5: they do not use any criteria to minimize, they use the graphic of the function found in GeoGebra and notice graphically the existence and value of a minimum. Finally, groups 4, 10 and 11 develop the activity of recognizing the sub-goal 5 (**Figure 7**).

For sub-goal 6, only the groups 4, 5, 6 and 11 find the minimum, but among them, groups 5, 6 and 11 do activities of treatment and control, which means finding the minimum, checking that it is actually a minimum and concluding the problem (**Figure 8**).

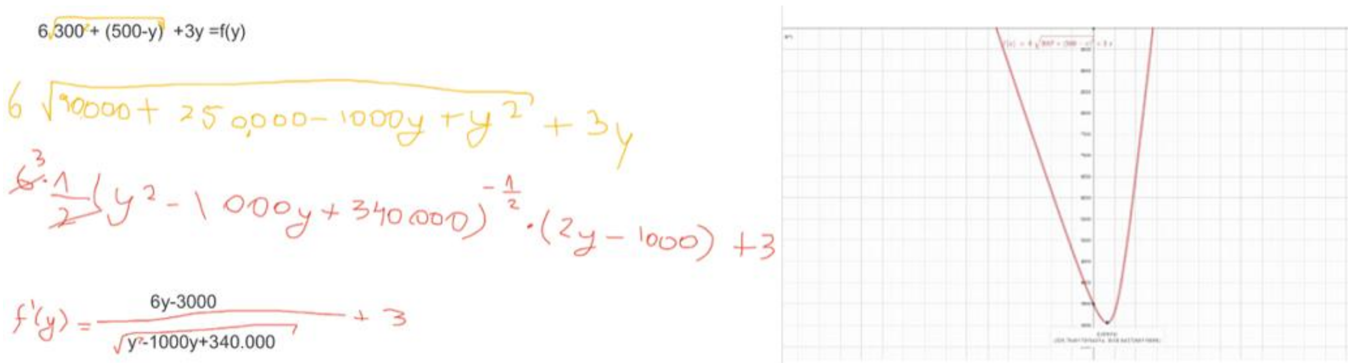


Figure 7. Group 5 checking minimum value with GeoGebra without algebraic criteria (Source: Student's coursework)

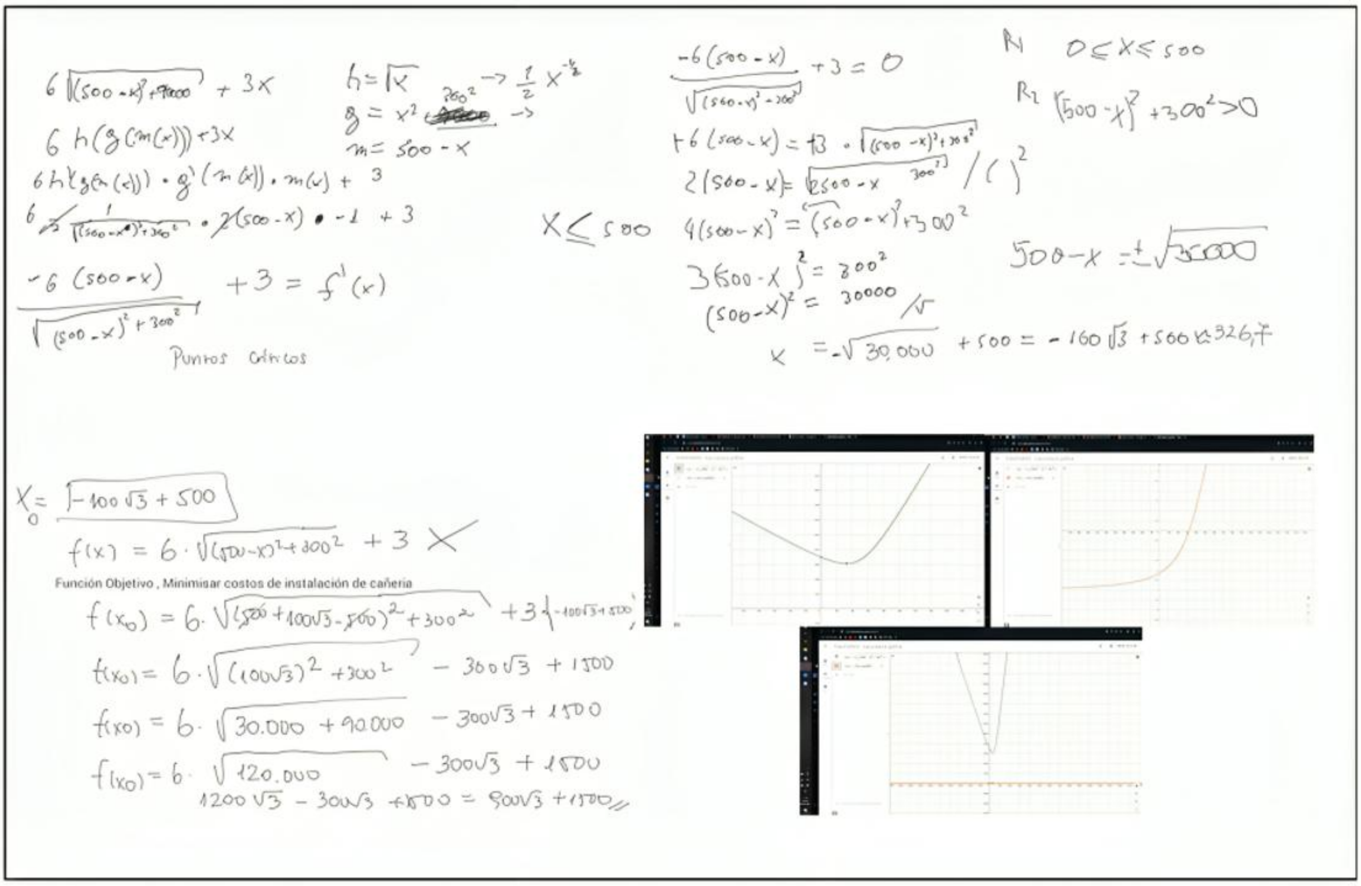


Figure 8. Group 11 doing treatment and controlling sub-goal 6 (Source: Student's coursework)

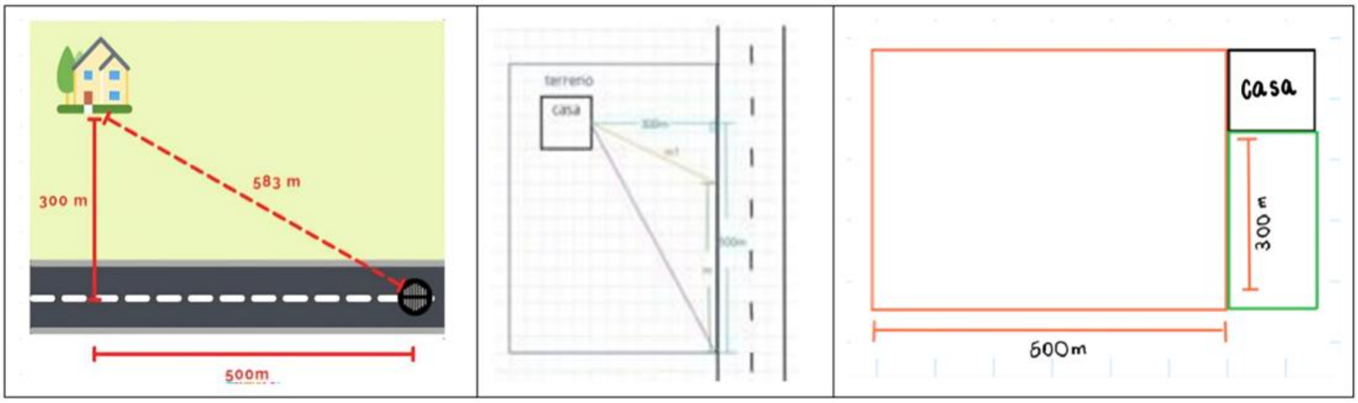


Figure 9. Geometric diagram of groups 7, 2 and 1 ("casa" means house and "terreno" means land) (Source: Student's coursework)

Formal delivery

Sub-goal 1: Show a diagram to represent the given situation: Once they have a geometric diagram defined, it is possible to observe that models are different regarding the level of abstraction. Some of them are more concrete (considering drawing the street, the house, the land, realistically), pictorial (considering some elements of reality but abstracting some elements as the house like a square for example) and others more symbolic (representing the house as a rectangle, square or dot, the street as a line, etc.), going from left to right in **Figure 9**.

Recognizing activities occur where they adapt their knowledge to code the information given in natural language to create a model for the situation and the steps to connect the ideas. Treatment activities occur when MPST chooses to draw a geometric diagram to have a representation of the situation. They all achieved this sub-goal 1, including the three activities expected. Also, they enrich the nature of the diagrams by changing the register from natural language into a figural register: concrete (groups 7, 8), pictorial (groups 2, 5, 6, 9, 12) or more symbolic (groups 1, 3, 4, 10, 11, 13) elements to do it.

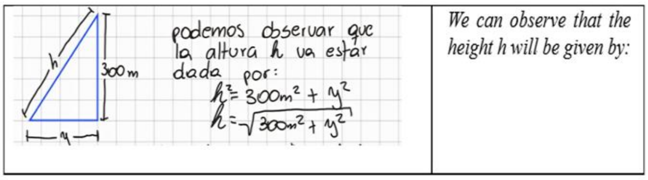


Figure 10. Use of Pythagorean theorem by group 8 (Source: Student's coursework)

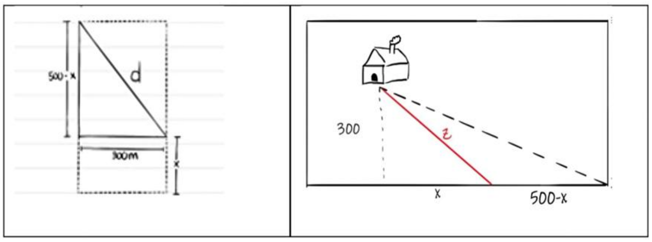


Figure 11. Production of group 12 and 9 for sub-goal 3 (Source: Student's coursework)

Sub-goal 2: Use the Pythagorean theorem to connect given distances: All the groups recognize the use of Pythagorean theorem to start writing a function to calculate the costs of building the water pipe as shown in **Figure 10**. There are no substantive differences, most of them draw a separate triangle to observe the possibilities. In terms of the treatment activities, MPST relate the distances and find an expression (groups 1, 2, 4, 5, 6, 7, 10, 11, 12, 13) for a function, and some of them also calculate the cost using the edges of triangle (groups 3, 8, 9), to use the known distances.

Sub-goal 3: Define a variable to find the optimization function: In this part, we observe that in the recognizing activities all the groups notice that they must choose a variable; they use different letters. In the treatment and control activities, everyone writes the variable in the diagram or in the text, choosing to place it in different places (**Figure 11**).

But only some of them define the variable in the text, as shown in **Figure 12** (groups 4, 5, 7, 10, 11). Group 3 ranges the variable, which is used later to define the domain.

Sub-goal 4: Define the optimization function to represent the cost per meter: In this part, we observe that in the recognition activities, all groups recognize that they have to define an optimization function using the variable given beforehand. For the treatment activities, only some of them defined the function completely correctly, considering the domain of definition of the optimization function, as shown in the image on the right of **Figure 13** (groups 6, 7, 11). Groups 9 and 10 never noticed the domain of the function, as shown in the left-hand image of **Figure 7**. Other MPST noticed the domain when they had to discard a value for a possible minimum, after sub-goal 5 (groups 1, 2, 3, 4, 5, 8, 12, 13).

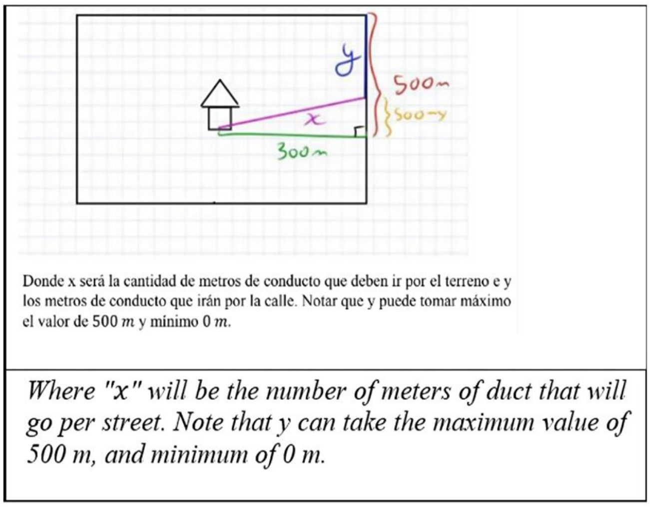


Figure 12. Production of group 5 for sub-goal 3 (Source: Student's coursework)

Figure 13. Production of groups 9 (left) and 11 (right) for sub-goal 4 (Source: Student's coursework)

Sub-goal 5: Apply a criterion to find critical points: All the groups apply the same procedure to find critical points, and they recognize the usage of it. They calculate the derivative of the function they define and calculate the value of the variable when the derivative is equal to zero. Until that stage, the treatment is algebraic for all of them, but after this calculation we observe different methods to pick the candidate of values in the x-axis where the minimum of the function is reached. Most of them notice the domain of the function and then discard the value that does not belong to the domain (groups 1, 2, 3, 4, 5, 8, 11, 12, 13), then there is only one candidate to analyze; this is part of the control activity (**Figure 14**). A couple of groups (groups 9, 10) also make a mistake calculating the square root and forgetting one of the solutions, getting exactly the root that works. There are no critical points because the derivative does not exist, but they do not write anything to control that fact.

Sub-goal 6: Find the minimum value and conclude the problem: All the groups find the minimum and check that the value they found is in fact the minimum of the function that they defined. They reproduce a coherent mathematical activity of recognition, treatment, and control. Most of the groups use GeoGebra to plot the function (groups 1, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13), another uses *only* algebraic methods (group 2), and another one a mixture of both to argue and validate the minimum (group 5) (**Figure 15**).

Group 11 shows a very powerful treatment and control activity, using the graph of the derivative of the function to further confirm that the value found is actually the minimum for f, graphically represents the criteria of first derivative noticing that at the value

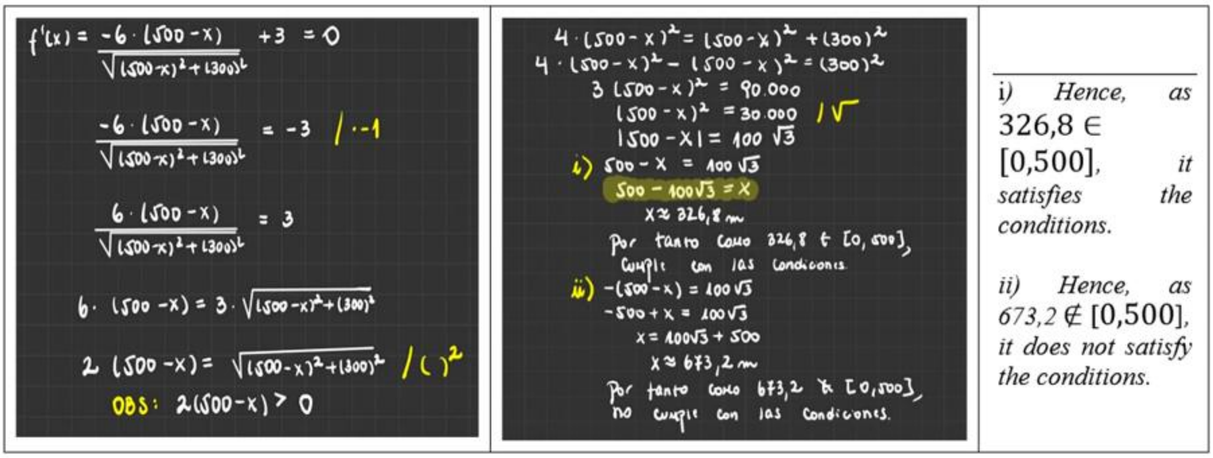


Figure 14. Treatment activity to reach sub-goal 5 by group 11 (Source: Student's coursework)

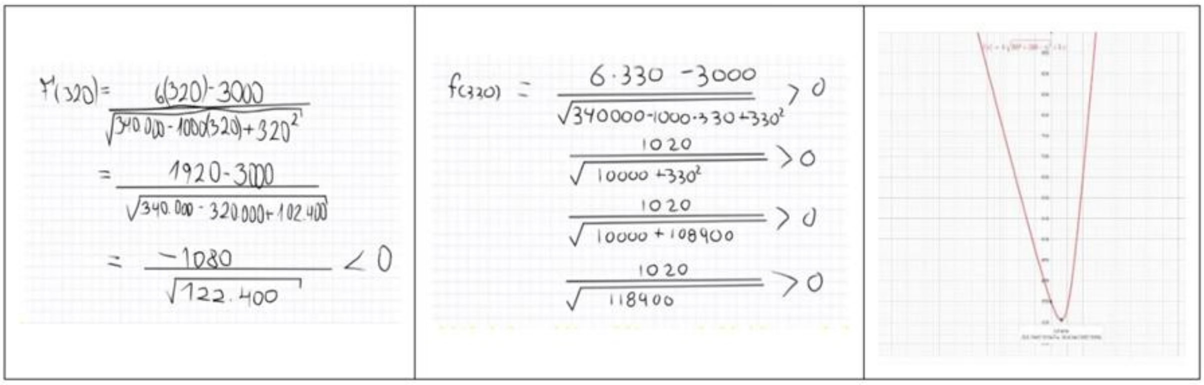


Figure 15. Group 5 algebraic and graphic methods (Source: Student's coursework)

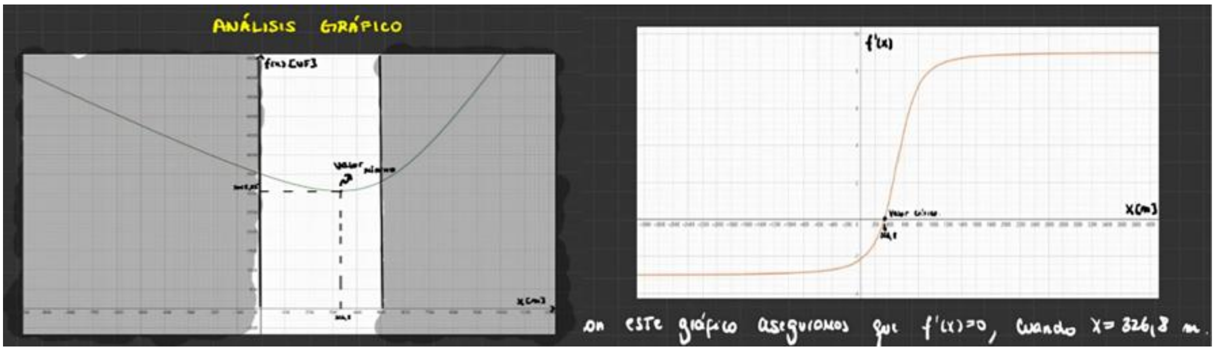


Figure 16. Checking the minimum by group 11, using the graph of f and f' (Source: Student's coursework)

found, f' passes by zero (**Figure 16**), and group 10 checks by comparing the values of f at the critical points and at the extreme values 0 and 500. They also write explanations in natural language. Also, all of them conclude the task, using the known symbol of three dots to summarize or just in Spanish writing "we conclude that …"

A Systematization of the Results

Table 5 summarizes the analysis and considers the sub-goals and quantifies the development of recognition, treatment and control activities as defined in **Table 4** for each group. We assign the number of each sub-goal when complete treatment in the productions of the MPST in each phase is shown. Then, we classify the groups in categories.

Category 1: Complete treatments up to sub-goal 2 in Jamboard

During the Jamboard, groups 1, 7 and 13 manage to represent a diagram (going through the three activities), and correctly use the Pythagorean theorem to connect distances. Group 7 and group 13 prove some costs using different paths with the Pythagorean theorem. They do not define any variables to find a function yet in this phase. In contrast, group 1 recognizes that they need to define a variable, but they do not perform the processing activity and only try some expressions.

Table 5. Groups progress activities per sub-goal in the Jamboard and the final delivery

Groups	Activities per sub-goal in Jamboard			Activities per sub-goal in Formal delivery		
	R	T	C	R	T	C
G. 1	1-2-3	1-2	1	1-2-3-4-5-6	1-2-4-5-6	1-4-5-6
G. 2	1-2-3-4	1-2-3	1	1-2-3-4-5-6	1-2-3-4-5-6	1-4-5-6
G. 3	1-2-3-4	1-2-3	1	1-2-3-4-5-6	1-2-3-4-5-6	1-2-4-5-6
G. 4	1-2-3-4-5-6	1-2-3	1-2	1-2-3-4-5-6	1-2-3-4-5-6	1-2-3-4-5-6
G. 5	1-2-3-4-6	1-2-3-6	1-6	1-2-3-4-5-6	1-2-3-4-5-6	1-3-4-5-6
G. 6	1-2-3-4-6	1-2-3-6	1-2-6	1-2-3-4-5-6	1-2-3-4-5-6	1-2-4-6
G. 7	1-2	1-2	1-2	1-2-3-4-5-6	1-2-4-5-6	1-2-3-4-6
G. 8	1-2-3	1-2-3	1-2	1-2-3-4-5-6	1-2-3-4-5-6	1-2-3-4-5-6
G. 9	1-2-3-4	1-2-3	1-2	1-2-3-4-5-6	1-2-3-4-6	1-2-3-6
G. 10	1-2-3-4-5	1-2-3	1-2	1-2-3-4-5-6	1-2-3-4-6	1-2-3-6
G. 11	1-2-3-4-5-6	1-2-3-5-6	1-2-6	1-2-3-4-5-6	1-2-3-4-5-6	1-2-3-4-5-6
G. 12	1-2-3-4	1-2-3	1	1-2-3-4-5-6	1-2-3-4-5-6	1-4-5-6
G. 13	1-2	1-2	1-2	1-2-3-4-5-6	1-2-4-5-6	1-2-4-5-6

In the final delivery, the three groups manage to reach sub-goal 6 by finding the minimum value going through the activities of recognizing, treatment and control in each sub-goal, except for the number 3, where they did not do the treatment to define the variable, less the control. Groups 1, 7 and 13 use GeoGebra to plot the function.

Category 2: complete treatments up to sub-goal 3 in Jamboard

In Jamboard, groups 2, 3, 4, 8, 9, 10 y 12 give an answer up to sub-goal 3 showing the definition of the variable in the diagram (recognition and treatment). Group 3 and 9 do some price tests to build the pipeline. Then, the three groups propose a function to optimize but only groups 2 and 9 calculate the derivative of the function they propose but without optimizing it. This means that they do not develop the recognition and treatment activities for the sub-goal 4. The control activities are only present in sub-goal 1 for groups 2, 3, 12 and until sub-goal 2 for groups 4, 8, 9, 10.

In the final delivery, all the groups manage to find the minimum value (sub-goal 6) by going through the three activities, and all groups except 9 and 10 achieve to define the domain of the function (control, sub-goal 4). The groups 9 and 10 do not complete correctly the application of the criterion to find the critical points (sub-goal 5). All the groups except number 2 use GeoGebra to plot the function (sub-goal 6), while group 2 only uses algebraic resolution.

Category 3: complete treatments up to sub-goal 6 in Jamboard

Groups 5, 6 and 11 belong here. Group 5 and group 6 check the possible cost by trying different paths. They do not complete the treatment of sub-goal 4 but they recognize the form of a function that could work. Group 5 makes a mixture of methods to argue and validate the minimum, using GeoGebra. Group 6 uses this technology to plot the function. Then they skip sub-goal 5 in terms of applying a criterion and try to calculate the derivative of the function found. Group 11 works on all sub-goals according to the 3 activities.

In the final delivery all groups manage to find the minimum value by going through the three mathematical activities. All groups use GeoGebra to control their results, but it is particularly more developed in group 11.

CONCLUSIONS AND DISCUSSIONS

This study contributes to research on Klein's (2016) first discontinuity from Chile. The institutional analysis (step 1) revealed a paradoxical tension: while both secondary and tertiary institutions claim to prioritize curricular continuity, a significant gap remains in the operationalization of infinitesimal calculus reasoning. This lack of explicit articulation often leaves MPST unsupported during the secondary-tertiary transition, particularly in optimization tasks where research remains sparse.

In addition, the findings (step 3) indicate that MPST initially struggled with recognition activities, specifically in variable identification, and treatment activities involving geometric properties. This is consistent with the work of LaRue and Engelke Infante (2015) and Mkhatshwa (2019), as well as that of White and Mitchelmore (1996). However, the most critical finding lies in the control activities, where MPST must verify the validity of their mathematical models (Jamboard). In this way, the control activity analysis remains in the optimization study at the beginning of university remains one of the major contributions of this study, and our results are also consistent with those of Brijlall and Ndlovu (2013), LaRue and Engelke Infante (2015) and Mkhatshwa (2019), since although several participants were able to argue and control their treatment activities, others were not.

In the bridge course, the didactic contract is redefined both explicitly in the curriculum and implicitly through the methodology of classes, workshops and exercise sessions. Besides, regarding the mathematical activities elicited by the adapted task (step 2 and step 3), the results demonstrate that shifting the didactic contract from a traditional prescriptive instructional design to one of autonomous exploration is pivotal, supporting the meta-synthesis by Hortelano and Prudente (2024). This shift allows MPST to move beyond procedural mimicry, assuming responsibility for their mathematical activity and fostering the complex reasoning required for a successful secondary-tertiary transition. By moving away from the step-by-step instructions typical of secondary school tasks, the adapted task compelled MPST to take ownership of the validation process. Notably, once MPST successfully constructed a geometric diagram, acting as a semiotic bridge, they demonstrated a robust command of algebraic treatments. This suggests that the transition gap is less about a lack of procedural knowledge but more about the cognitive load of autonomous recognition and the renegotiation of the didactic contract.

However, the epistemological-cognitive gap between high school and the university surfaced prominently in several of the participants' struggle to define the domain of functions. Multiple groups treated the function as an abstract algebraic entity rather than a model constrained by physical reality. This omission is not a mere procedural oversight but a manifestation of Klein's (2016) first discontinuity; MPST remains accustomed to a secondary school culture where functions are presented as decontextualized objects.

Finally, we observed that GeoGebra served as more than a verification tool; it acted as an instrumental mediator (Vandebrouck, 2018) that upped the threshold for control activity. The ability to visualize results and control mathematical coherence in real-time allowed MPST to argue their findings even when not explicitly prompted. This meant that the use of GeoGebra contributed to the process of Instrumental Genesis of the MPST (Rabardel, 1995; Trouche, 2004): the GeoGebra artifact led to the development of instrumentation by helping to shape the reasoning of the MPST. This primarily impacted on the development of control activity, transforming the artifact into a control instrument and promoting and sustaining coherence in the processing activity in subtasks 5 and 6.

Recommendations for Teaching and Teacher Education

Traditional curricula emphasize the derivative as the focal point of optimization, but the results here demonstrate that the activity of control and the definition of the domain are where the transition to university-level thinking truly occurs. For MPST, our suggestion is that navigating this complexity is essential; it transforms optimization from a set of steps into a rich site for mathematical reflection and mathematical activities. In this regard, we suggest that middle school assignments take this new way of thinking by incorporating more complex tasks into math courses, rather than limiting themselves to the execution of simple tasks. The relevance of these recommendations based on our findings lies in the shift from a procedural view of optimization to a structural modeling perspective.

The instrumental mediation of GeoGebra evidenced in this study suggests that technology must transcend its role as a mere computational shortcut. Instead, it should be embraced as a vital cognitive mediator that reshapes the very nature of mathematical reasoning and visualization. Rather than being relegated to discrete interventions, technology should form a seamless continuum within mathematical activity. In this context, GeoGebra emerges as a powerful catalyst for autonomous validation, a cornerstone competency for prospective mathematics educators. Consequently, we advocate for the pervasive use of dynamic tools to drive the instrumentation in complex tasks. Such tools should not be confined to standalone courses but must be established as structural pillars integrated throughout the entire mathematics teacher education curriculum, both in Chile and internationally, and also be present with this strength in the secondary schools.

Study Limitations and Projections

The limitations of this study stem from the fact that it is a pioneering effort in the Chilean context and, as such, the results raise new lines of research and open theoretical issues. Indeed, further research is needed to characterize the first Klein (2016) discontinuity in greater depth and to gain additional insight into the mathematical achievements MPST are capable of at this level. Moreover, additional studies are needed to determine the extent to which our findings align with trends in other countries; however, we maintain that our methodology provides a three-step framework for studying and addressing the secondary-to-tertiary transition in contexts beyond Chile. This would allow us to extrapolate and apply the findings of this study to the international arena.

Furthermore, we believe that the process of instrumentalization and the use of GeoGebra warrants closer examinations of further study. For example, we highlight the importance of mathematical autonomy in the first year of teacher education and raise questions for future research, such as: how can we foster the control activity through the use of technological tools that enhance visualization in other calculus or algebra courses?

On the other hand, the concept of the didactic contract could be studied in greater depth in the context of Klein's (2016) first discontinuity. For example, one could examine changes in the didactic contract specifically in the initial training of mathematics teachers in collaborative settings. Building on Presutti's (2025) work, one could analyze at the micro level the different responsibilities assumed and the participation of MPST in each group in the construction of knowledge, by analyzing micro-didactic contracts.

In conclusion, this research argues that addressing the secondary-tertiary transition requires moving beyond traditional remedial models that focus on intensive re-teaching of secondary basics (Viirman, 2022). Instead, we propose a bridge-building approach that articulates mathematics and didactics. This study underscores the need for learning situations that explicitly target control activities early in teacher education. By fostering mathematics for teaching rather than mathematics for its own sake, we empower future educators and teacher trainers to recognize these discontinuities in their own classrooms. As Gueudet (2023) suggests, incorporating these reflective practices into a curricular continuum is essential for preparing teachers to facilitate a smoother transition for the next generation of students

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Ethical statement: The authors stated that this research adhered to a rigorous process as certified by Ethic Committee of Science Faculty of University of Chile, following the Law No. 20.120 on Scientific Research on Human Beings, their genome, and prohibits human cloning, and its Regulation approved by D.S. No. 114, 2011, of the Ministry of Health. As this article does not utilize sensitive data pertaining to individuals, approval from the ethics committee is not a prerequisite. However, in order to comply with the ethics of the study, it should be noted that written consent was obtained from the participants who took part in the study and all data obtained has been anonymized.

AI statement: The authors used the generative AI tool Gemini to improve the English language clarity and check the structure of the manuscript. No original content was generated by the AI.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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