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An analysis of errors for pre-service teachers in first order ordinary differential equations

Chipo Makamure [1*], Zingiswa M. Jojo [1]

[1] University of South Africa, Pretoria, SOUTH AFRICA

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Abstract
Literature has established that some learners encountered difficulties solving first order ordinary differential equations (ODEs). The use of error analysis in teaching ODEs is believed to make essential contribution towards calculus knowledge development. This paper therefore focuses on analyzing pre-service teachers' (PSTs) errors and misconceptions apropos of first order ODEs. The paper analyzed the nature of errors made in a test which was written by PSTs on the above topic. The test comprised various types of first order differential equations such as ODEs with separable variables, exact ODEs, ODEs that needed integrating factors, linear ODEs, and homogeneous ODEs. The purpose was to investigate the challenges faced by PSTs in various types of ODEs and the nature of misconceptions that they had in each particular type. This is a qualitative study that involved 63 PSTs who wrote a test on ODEs after being taught the topic for two weeks. The authors marked the work in order to ascertain the misconceptions and errors exhibited by the participants in the test. The PSTs' performance in the test was analyzed using the SOLO taxonomy and the Newman's theory mistake analysis. The study established that the topic was rather difficult for PSTs due to various reasons that included, among others, knowledge gaps in integration rules, algebraic computations and, in rare cases, differentiation, as well as misapplication of the rules of natural logarithms. This research therefore recommends that mathematics teacher educators ought to rather focus on the concept of integration and basic algebra before introducing the topic on ODEs to teachers on training.

Keywords: errors, integration, misconceptions, ordinary differential equations, pre-service teachers

INTRODUCTION

STEM education, which includes mathematics education, is meant to promote STEM skills and knowledge that include problem solving skills, critical thinking, creativity, and self-directed learning (Kusmin, 2019; White, 2014). This implies that mathematics curricula and the teaching of mathematics must be directly responsive to the needs and demands of the 21st century competencies such as innovation, which is key for boosting productivity.

Ordinary differential equations (ODEs) have been considered by many as essential in solving daily activities such as bacterial growth rates, heating and cooling, to mention a few (Farlina et al., 2018). It is therefore important to be able to solve ODEs in order to find the right solutions to these crucial problems in people's daily life experiences. However, despite ODEs being one of the most crucial courses which is calculus driven in higher education, many schools have registered challenges in dealing with the topic (Luneta & Makonye, 2010; Maat & Zakaria, 2011). Farlina et al. (2018) concur that there are still challenges among students in solving ODEs.

In this study, we therefore found it necessary to analyze these difficulties and errors in ODEs in order to improve pre-service teachers' (PSTs') understanding of the topic. By identifying learners' specific error types, we can provide suggestions to instruction targeted to PSTs' areas of need in ODEs. This study therefore explores the

✉ makamburec@gmail.com **(*Correspondence)** ✉ jojozmm@unisa.ac.za

Contribution to the literature

- Since ODEs are an important and pragmatic topic in people's daily life experiences, it is crucial that PSTs are knowledgeable about this topic in order to promote the needs and demands of the 21st century.
- The research unravels the types of errors and misconceptions PSTs make in ODEs, and it is essential to have errors analyzed at this level in order to correct and improve the teaching of calculus in high school since ODEs is a calculus driven topic.
- The use of error analysis in teaching ODEs is believed to have made contribution to calculus knowledge development.

errors and misconceptions exhibited by PSTs in ODEs. The study unravels the types of errors and misconceptions PSTs have in response to the ODEs test questions provided to the participants. It is essential to have errors analyzed and remediated at this level in order to correct and improve the teaching of calculus in high school since ODEs is a calculus driven topic.

Problem Statement

Literature documents the misconceptions and difficulties associated with the teaching and learning of mathematics (Larbi & Okyere, 2016), and it has been established that learners make various errors and mistakes when performing mathematics tasks (Brown et al., 2016), especially in ODEs (Yarman et al., 2020). According to Yarman et al. (2020), ODEs are an essential course in accordance with the demands of an applicable curriculum. However, students' mastery of ODEs is still relatively low and they have difficulties solving applied questions that relate to daily life (Yarman et al., 2020). Educators therefore need to be aware of the errors made in ODEs for effective error analysis. Error analysis assists educators to understand learners' thinking, which can actually enable the educators to adjust their teaching practices hence improving the performance of PSTs in mathematics. It is therefore fundamental to explore the errors made by PSTs in ODEs to assuage the common mistakes made. The study thus looked into performance of PSTs in finding right solution to ODEs.

Research Questions

The following research questions guided this study:

1. What common errors, if any, do mathematics PSTs make in first order differential equations?
2. What are the causes of the common errors and how can these be remediated?

LITERATURE REVIEW

Conceptual Anatomy of Error Analysis

What are errors and mistakes?

According to Brown et al. (2016), students make various errors and mistakes when performing mathematical tasks. For example, the misinterpretation of mathematical expressions and misapplication of mathematical properties may be due to some conceptual errors or mistakes (Brown et al., 2016). Teachers and educators need to be aware of these errors for effective error analysis.

According to Legutko (2008), a conceptual error reveals inadequacy of knowledge and is closely connected with limitations of imagination and creativity in new situations. The definition suggests that an error is caused by insufficient mastery of basic facts, concepts and skills. It is believed that an error takes place when a person chooses something false as the truth. Legutko (2008) also describes a mistake as one that is made when an individual, for example, incorrectly applies a formula or a theorem that she knows. Some misunderstanding of these terms may therefore impel educators to neglect conceptual understanding of learning mathematics in favor of procedural/factual corrections during error analysis (Russell & Masters, 2009). Lai (2012) identifies three types of errors. First, there is procedural errors where a learner fails to follow the correct steps to solve a problem. The second type is the factual error. These are types of mistakes that some learners make when they cannot recall a fact required to solve a problem.

According to Lai (2012), procedural and factual errors are also called 'slips.' These are normally not caused by inherent misunderstandings and are easier to identify. The third error, according to Lai (2012), is a conceptual error (bugs). These types of errors may look like procedural but are caused by the fact that the learner does not understand a specific mathematics content (Herholdt & Sapire, 2014). That means 'bugs' are more serious errors. A conceptual error is normally made when a learner cannot show and explain the steps used to solve the problem (Hudson & Miller, 2006). On the contrary, Luneta and Makonye (2010) and Riccomini (2005) define an error as a mistake or deviation from accuracy. Furthermore, Riccomini (2005) classifies errors into systematic and unsystematic errors. Unsystematic errors are defined as intended, non-recurring wrong answers which learners can readily correct by themselves. On the other hand, systematic errors are recurrent wrong responses methodically constructed and produced across space and time.

Researchers like Adu-Gyamfi and Bosse (2014) and Adu-Gyamfi et al. (2015) assert that all errors must be defined by three activities, namely syntactic elaboration, semantic elaboration and strong parallel activities. According to these researchers, syntactic thinking involves the learner incorrectly manipulating the problem but generally understands the concept. In semantic elaboration, the learners can work through all the steps of a mathematical problem although they do not understand the problem. Strong parallels are between syntactic and manipulation errors together with semantic thinking and conceptual errors.

However, despite different types of errors that exist, literature (Pomalato et al., 2020; Rushton, 2018) focuses on two types of error analysis. The first type includes identification and interpretation of learners' common error patterns as a result of misconceptions. In this type of analysis, the educator needs to possess strong mathematics content in addition to the ability to focus on students' aptitude level. McGuire (2013) refers to the ability to interpret students' level of understanding as a skill for good teaching. The second component of error analysis involves educators' best practices for instructional remediation. This implies that the educator should come up with the best strategies of teaching mathematics to address particular individual learners' ascertained errors. The authors in this study hence explored the errors made by the study participants as well as recommending some possible ways in which those errors could be remediated.

Justification of error analysis

Herholdt and Sapire (2014) and McGuire (2013) define error analysis as the ability to identify and interpret learners' errors in learners' work with a view to finding possible explanations about those errors. This view implies that error analysis is capable of empowering educators to assist students grow out of their incorrect understandings. This way, the PST teacher's knowledge of mathematical cognition and concept development are thus broadened (Herholdt & Sapire, 2014). Generally, the definitions alluded to above imply that error analysis assists teacher educators to understand PSTs' thinking. This can assist educators to adjust their teaching practices including the way they assess PSTs' work, hence improving PSTs' performance in mathematics. Moreover, according to Lai (2012), the first step of error analysis should include the identification of the error displayed in the learners' work. The second stage involves finding out why the learner made that error, for example, it could be due to a lack of knowledge or understanding (Hudson & Miller, 2006). Peng and Luo (2009) concur with Lai's (2012) suggestion and went further to develop a framework which identified, from the educators' perspectives, the four key *phrases of error* analysis. The four are identify (knowing the existence of mathematical error); interpret (interpreting the underlying rationality of mathematical error), evaluate (evaluating students' levels of performance according to mathematical error), and remediate (presenting teaching strategy to eliminate mathematical error). Using these concepts, we explored the errors made by mathematics PSTs in ODEs. ZIMSEC examiners reports (November 2010, November 2013, June 2017, June 2016), stated that, when solving first order differential equations, some students fail due to poor algebraic skills and others failed to correctly integrate functions, especially partial fractions. The reports further identified carelessness in implicit exponential and logarithm function notation. Students have observed that they perform poorly when given word problems involving differential equation which cannot be solved using procedures. Reports have also commented that some students fail to separate variables in solving first order differential equation and some learners end up with several variables in their different differential equations which could not be separated correctly.

Concept analysis of ODEs

According to Nykamp (2015), an ODE is an equation that involves some ordinary derivatives (as opposed to partial derivatives) of a function. Our goal, as authors of this article, was to establish PSTs' ability to solve ODEs, that is, to determine what function or functions satisfy the given differential equation. For example, given the function $\frac{dx}{dt}(t)=\cos t$, the function $x(t)$ is obtained by finding the antiderivative of $\cos t$, which is equivalent to $\sin t$. Hence, $x(t)=\sin t + A$, for some arbitrary constant A (Nykamp, 2015). However, by and large, solving an ODE, according to Nykamp (2015), is more complicated than simple integration and the difficult part is usually when students attempt to determine what integration is needed to solve an ODE. There are different types of ODEs such as separable ODEs, exact ODEs, linear ODEs, homogeneous ODEs and integrating factors.

Differential equations have applications in all areas of science and engineering. Mathematical formulation of most of the physical and engineering problems lead to differential equations. So, it is important for engineers, scientists and STEM learners to know how to set up differential equations and solve them. Farlina et al. (2018) define differential equations as those that have one or more variables. Differential equations that have one differential variable are called ODEs, whilst differential equations with two or more variables are called partial differential equations (PDEs) (Farlina et al., 2018). Differential equations are therefore of two types:

1. ODEs and
2. PDEs.

According to Farlina et al. (2018), ODEs are applicable in real life daily activities; hence, it is necessary that they are understood by mathematics PSTs. According to Chikwanha (2021), teachers and students have challenges in differentiation and

Table 1. Solo declarative and functioning learning verb (Adapted from Tarigan et al., 2019)

SOLO taxonomy levels	Verbs and skills
Uni-structural level	Define, identify, name, draw, find, label, match, and follow a simple procedure
Multi-structural level	Describe, list, outline, follow an algorithm, and combine
Relational level	Sequence, classify, compare and contrast, explain causes, explain effects, analyses (part-whole), form an analogy, organize, distinguish, interview, question, relate, and apply
Extended abstracts level	Generalize, predict, evaluate, reflect, hypothesize, theories, create, prove, plan, justify, argue, compose, prioritize, design, and perform

integration as these are the feeder topics of first order differential equations and the problems emanate from the errors and misconceptions found in topics like the power rule, chain rule and exponentials. Similarly, according to Yarman et al. (2020), ODEs are an essential course in accordance with the demands of an applicable curriculum. It is therefore essential for PSTs to have knowledge of ODEs in order to develop teaching knowledge of calculus at high school level. However, students' mastery of the course is still relatively low and they have difficulties solving applied questions that relate to daily life (Yarman et al., 2020). Jojo (2011) also observed difficulties in understanding and applying the concept of calculus using the chain rule among students. These difficulties might impact on PSTs' performance in ODEs since ODEs are a calculus driven topic (Makonye, 2016). It is therefore fundamental to explore the errors made by PSTs in ODEs in order to mitigate the common mistakes made by teachers in calculus since ODEs encompass calculus topics. The study thus looked into PSTs' performance in finding the right solution to ODEs.

Error analysis in ODEs

Farlina et al. (2018) found that there are several difficulties in solving ODEs among students. A study by Yarman et al. (2020) analyzed students' errors in ODEs using the structure of observable learning outcomes (SOLO) taxonomy. The SOLO taxonomy, according to Caniglia and Meadows (2018), focuses on observable outcomes; hence, provides a context for measuring, evaluating and analyzing how well a student understands a topic. The SOLO taxonomy describes five levels that classify students' abilities and understanding of a topic starting from simple to complex (Caniglia & Meadows, 2018). The five levels are pre-structural, uni-structural, multi-structural, relational, and extended abstracts level. Levels one to three show lower-level cognitive skills that involve deductive reasoning (quantitative), whilst the last two show complex inductive reasoning strategies (Biggs & Tang, 2007; Caniglia & Meadows, 2018). The verbs/skills associated with the SOLO taxonomy levels, according to Tarigan et al. (2019) are given in **Table 1**.

The level of thinking criteria of students, according to the SOLO taxonomy (Putri et al., 2017), are explained in the following.

Pre-structural level shows that the students do not understand anything about the topic. Biggs and Tang (2007) concur that, at this level, the students miss the point and there is little or no evidence of learning the topic because their performance or responses to the questions asked are full of inaccuracies.

Uni-structural level reflects the student's ability to identify a few ideas and follow simple procedures taught, but, missing some important parts of the topic. Some students' responses can be vague or too general.

In *multi-structural level*, the student may acquire some knowledge but cannot put the ideas together. This means the student at this level has superficial understanding of the topic and only relies on memorizing, remembering and parroting what they have learned (Caniglia & Meadows, 2018); hence, the concepts cannot be used in new or innovative contexts. Caniglia and Meadows (2018) likened students at this level to a builder who has all the pieces but without tools, and does not know how these pieces connect. This means the student has knowledge cluttered all over the mind but the concepts cannot be connected to solve problems.

On *relational level*, students are able to explain how ideas link together and can compare and contrast concepts to demonstrate a qualitative change in learning (Tarigan et al., 2019). According to Caniglia and Meadows (2018), the *extended abstracts level* involves application of knowledge in different contexts or real-life situations.

Yarman et al. (2020) used the SOLO taxonomy to determine the level of problems in ODEs, the quality of responses to ODEs test items and the analysis of test items given to the students. According to Yarman et al. (2020), the different SOLO taxonomy levels described above can be analyzed using different types of errors in solving ODEs. The errors that resonate with Yarman et al.'s (2020) analysis are, as follows:

1. **Conceptual error:** The indicators of the conceptual error in ODEs include (i) errors determining the formula in answering a problem and (ii) the use of formulas/theorems, which are not in accordance with the conditions of the pre-requisite for the enactment of the formula by the student, or the student does not write the theorem.

   Failure to describe and apply the formulas shows that the learner has superficial comprehension of the topic and cannot use it in different contexts.

This resonates with the multi-structural level of the SOLO taxonomy.

2. **Error using data:** The indicators include (i) not using the data that should be used (for example, instead of using an integrating factor to solve a particular equation, a learner tries to separate variables), (ii) errors entering the data into variables, and (iii) adding data that is not needed to answer a problem.

   The learner making such errors is still at the uni-structural level of the SOLO taxonomy because the learner can hardly identify the correct information to be used to solve a problem.

3. **Language interpretation error**: Indicators include (i) errors in expressing everyday language in mathematical language and (ii) errors in interpreting symbols, graphs, and tables into mathematical language.

   Inability to define and identify symbols and graphs reflects that the learner's level of understanding is at the uni-structural level.

4. **Technical errors:** The indicators are (i) miscalculations and (ii) errors in manipulating algebraic operations.

   Failure to follow algorithms to solve problems defines the multicultural level of the SOLO taxonomy because learners at this level depend on memorizing or parroting what they have learnt.

5. **Error making conclusions:** The indicators include (i) conducting conclusions without the right supporting reasons and (ii) concluding unauthorized statements with logical reasoning.

   Whilst at the SOLO taxonomy's relational and extended abstracts levels, learners are expected to generalize concepts, explain causes and justify arguments. Learners at this level can make mistakes or errors of failure to make proper conclusions.

These errors were explored in the PSTs responses to the ODEs test items given in the study using the SOLO taxonomy as defined by Tarigan et al. (2019).

THEORETICAL FRAMEWORK: LEARNING ODEs THROUGH ERROR ANALYSIS

It is pertinent that educators are privy to Newman's model in order to determine students' misconceptions in solving mathematics questions (Alhassora et al., 2017). Newman's error analysis (NEA) assists in diagnosing students' errors in solving higher-order thinking skills in mathematics (Abdullah et al., 2015; Maat & Zakaria, 2011).

In their study, Rohmah and Sutiarso (2018) present the various stages of Newman's theory mistake analysis. According to Rohma and Sutiarso (2018), there are five stages in recognizing students' errors in the NEA model:

1. Reading and decoding, in which the ability to read mathematical problems and identification of mathematical symbols is used;
2. Comprehension, which refers to the students' understanding in relation to the symbols and problems given in the questions;
3. Transformation, referring to the ability of students in choosing the appropriate formulae or method to solve the problems given;
4. Process skills, where students use rules to solve a problem but make some computation errors in the process; and
5. Encoding, which is the ability of the students in generating and justifying the answer they give.

In addition, the SOLO taxonomy model used to determine the level of students challenges in ODEs (Tarigan et al., 2019), was blended with the Newman's theory mistake analysis to explore the errors made by PSTs in ODEs. For each of the SOLO levels described, there were errors associated with the problems that included the concept error, error using data, language interpretation error, technical errors and error making conclusions. Using these models, namely Newman's theory mistake analysis stages and the SOLO taxonomy, this research presents an analysis of errors made by PSTs in first order ODEs.

After a thorough scrutiny, it emerged in the research that both the Newman's model and the SOLO taxonomy have common views in terms of errors committed by students. The two models were therefore integrated to analyze errors made by PSTs in ODEs.

METHOD

Sampling Procedures

This qualitative study involved 63 mathematics PSTs from a selected teacher training college in Zimbabwe. The participants were purposively selected to participate in the study. These were secondary school mathematics specialist trainees in their first year. The college admission requirements included 'O' level certificate with at least a 'B' pass in mathematics and at least a C pass in English language. After successfully completing this program (Diploma in Education), the graduates would be expected to teach secondary school mathematics up to ordinary level.

Procedure and Design

63 secondary school mathematics PSTs were asked to write a test with nine questions on first order ODEs after being taught the topic for two weeks. The nine questions embraced all the aspects of first order differential

Table 2. Analysis of errors made per particular level of understanding for each given question

Level of understanding		Pre-structural	Uni-structural	Multi-structural	Relational	Extended abstracts
Number of participants at a particular ability level	Question 1	57 (90.5%)	5 (7.9%)	1 (1.6%)	0 (0.0%)	-
	Question 2	26 (41.3%)	8 (12.7%)	16 (25.4%)	13 (20.6%)	-
	Question 3	35 (55.6%)	13 (20.6%)	6 (9.5%)	9 (14.3%)	-
	Question 4	16 (25.4%)	29 (46.0%)	9 (14.3%)	9 (14.3%)	-
	Question 5	23 (36.5%)	21 (33.3%)	9 (14.3%)	10 (15.9%)	-
	Question 6	43 (68.3%)	11 (17.4%)	8 (12.7%)	1 (1.6%)	0 (0.0%)
	Question 7	30 (47.6%)	5 (7.9%)	28 (44.5%)	-	-
	Question 8	21 (33.3%)	19 (30.2%)	11 (17.5%)	8 (12.7%)	4 (6.3%)
	Question 9	36 (57.1%)	3 (4.8%)	6 (9.5%)	18 (28.6%)	-

Note. *NB: Numbers represent the number of participants out of 63 who made a particular error; *NB: "-"means the question did not test the level of understanding up to a specific level of the SOLO taxonomy

equations that include exactness, separation of variables, linearity and homogeneity.

The purpose was to investigate the challenges, if any, faced by PSTs in various types of ODEs and the nature of misconceptions that they had in each particular type. The mathematics test items are given in the appendix section. To explore and identify the existence of errors in the PSTs work in ODEs, the researchers marked and analyzed the test scripts for each PST who took the test. A mark or score was allocated for each stage of the solution to the nine questions.

Analysis

On analyzing the results, the researchers analyzed the PSTs responses to each of the nine test items in the test. Making use of Newman's theory mistake analysis stages and the SOLO taxonomy blended, the responses to the test items on ODEs were analyzed to explore the errors made by the PSTs. The Newman's five-stage model recognizes students' errors based on five indicators, namely comprehension error or error understanding problem, transformation error, process skill error, and encoding error or error writing (Pomalato et al., 2020). Yarman et al.'s (2020) SOLO taxonomy was based on the concept error, error using data, language interpretation error, technical errors and the error making conclusions.

This research employed this model and heeded Lai's (2012) suggestion that the first step of error analysis should include the identification of the error displayed in the learners' work and that it should involve finding out why the learner made that error. In the research, we analyzed the performance of participants in response to the questions: What kind of errors were made? What are the possible causes of the errors (misconceptions that propagated the errors)? How can the errors be remediated (instructional strategies that can be used to remediate the learners' error patterns)? Responses to these questions were used as indicators to identify the PSTs' errors in ODEs. The study hence focused on the correct methodologies of the test takers, the questions not attempted, which could be an indication of area of weakness, hence full attention would be required. Furthermore, emphasis was also on identification of most commonly occurring errors per item. A prevalence of such errors could be a sign of general confusion which could reflect the need for further clarification of the concept (Herholdt & Sapire, 2014). The performances on each test item were compared to establish the areas in ODEs where the PSTs had difficulties. The errors and mistakes made by the participants on each test item were therefore identified, recorded and reported qualitatively.

RESULTS

The errors made by each PST were not written separately because some of them were repetitive. Each type of error made in each question was discussed using some representative excerpts from the participants. The data collected revealed the types of errors made and the possible causes. Some of the errors made were mostly conceptual/comprehension errors, process skill errors, technical errors, decoding/encoding errors as well as transformation errors. The type of errors made are presented first before they are discussed. **Table 2** shows an analysis of errors in ODEs made per particular level of understanding according to the SOLO taxonomy.

Generally, **Table 2** shows that PSTs are in the lower cognitive level of ability in ODEs as reflected by the high percentages below the relational level of the SOLO taxonomy. In all the questions, at least 85% of the PSTs never reached the relational level, whilst an average of at least 50% were at the pre-structural level. The most common errors shown in the table were conceptual and/or comprehension errors.

Description & Possible Causes of Errors Made in Test

Below is a description of the types and causes of errors made by participants in some of the questions asked in the ODEs test.

The reflections of each participant were not written separately. Only those responses that were rather representative of the participants' general views and ideas about a particular question were used to discuss the solution and causes of errors to each question item.

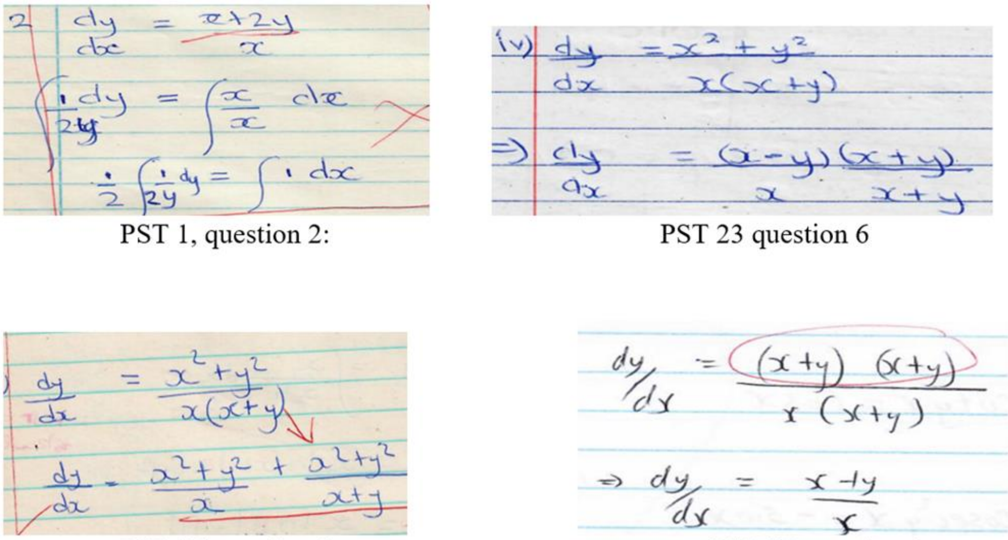
PST 1, question 2: PST 23 question 6
PST 56 question 6 PST 57 question 6

Figure 1. The rampant conceptual and/or comprehension errors made by PSTs

The descriptions below were extracted from participants' work as evidence of the errors made.

Failure to simplify simple algebraic expressions was a consistent problem with most PSTs. A significant number exhibited miscalculations errors in manipulating algebraic operations. These weak algebraic skills were exhibited and obtrusive when the PSTs were separating variables, hence separation of variables in solving ODEs was a mammoth task for the PSTs. **Figure 1** exhibits the rampant conceptual and/or comprehension errors made by PSTs, which were facilitated by weak algebraic skills.

PST 1 is dividing both sides by $2y$, to get a daunting result, $\frac{1dy}{2y}=\frac{x+0}{x}\,dx$ which simplifies to $\int\frac{1dy}{2y}=\int\frac{x}{x}\,dx$. This is a sign of absolute misconception of basic ideas by PST 1. PST 23 is confusing x^2+y^2 with the difference of two squares $[(x^2-y^2)=(x-y)(x\ y)]$, whilst PST 56 wrongly split the denominator. PST 57 is also confusing x^2+y^2 with $(x+y)^2$. In addition to these excerpts, PST 8 simplified $\frac{dy}{dx}=4x+y$ to get $\boldsymbol{\frac{dy}{y}=4x+\frac{y}{y}\,dx}$. These errors show that failure to separate the variables hindered the entire process of integration in order to solve the ODEs. This poor performance denotes comprehension errors which, if not remediated, might have a devastating effect on the PSTs' understanding of ODEs.

Misapplication of the rules of natural logarithms (ln) also caused some hitches and glitches in solving ODEs. The study showed that PSTs, because of their failure to simplify natural logarithms, had challenges solving ODEs up to the final expected answers. **Figure 2** is an example of the PSTs' reflections on natural logarithms.

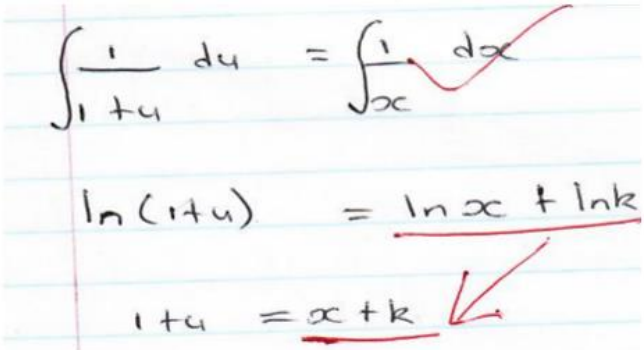

Figure 2. PST 36 question 2

The last statement on the excerpt exhibits the PST's poor understanding of the rules of logarithms. The final answer generated could have been confused with the concept $\ln(xy)=\ln x+\ln y$. The PST might also have thought of this question as if its equivalent to: $\ln(1+u)=\ln(x+k)$, where in this case $1+u=x+k$. PST 36 therefore erred in considering $\ln(x+k)=\ln x+\ln y$ as true. This is a comprehension error because the statement is not true for every value of x and y.

A lack of relational skills, that is, failure to link ideas together in order to demonstrate understanding was reflected when some PSTs, after being asked to obtain a first order differential equation that does not contain the arbitrary constant A for $y=3x^2+Ax$ (question 1), they had to find the logarithms both sides of the equation. Even though the method would not take them anywhere, the worrisome issue is how they failed to adhere to the rules of natural logarithms. Participants 15, 19, and 4 could have confused the problem with the rule: $\ln(xy)=\ln(x)+\ln(y)$ as given in the examples below.

PST 15 wrote that given $y=3x^2+Ax$, $\Rightarrow\ln(y)=\ln(3x^2+Ax)$ $\boldsymbol{\Rightarrow\ln(y)=\ln(3x^2)+\ln(Ax)}$. This was also a worrisome output because, in addition to failing to identify the nature of the ODE, the participants exhibited their weaknesses in laws of logarithms. In addition, some students like PST 28, in trying to solve the linear equation $\frac{dy}{dx}=\frac{x+2y}{x}$, went through the process of finding the I.F.$=e^{\int\frac{-2}{x}dx}=e^{-2\ln(x)}$, where $p=\frac{-2}{x}$, but could not

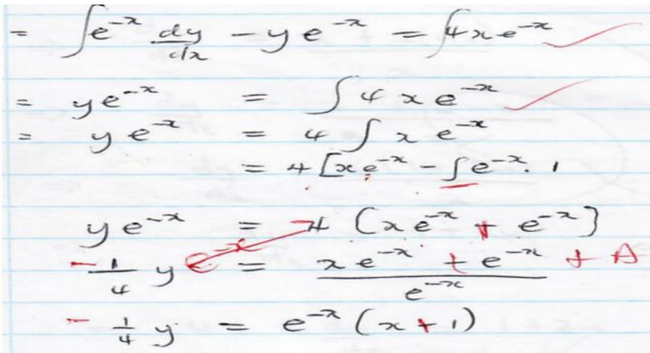

Figure 3. PST 50 question 3

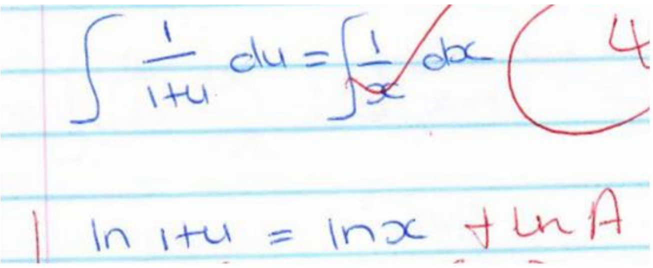

Figure 4. PST 9 question 2

simplify $e^{-2\ln(x)}$, to get x^{-2}, a reflection of a lack of knowledge of the rules of logarithms which denotes a process skill error. All these errors regarding the manipulation of the rules of natural logarithms could have contributed to the difficulties in solving ODEs. Some participants, despite being knowledgeable about the procedures to follow when solving ODEs, jammed in the process because of the hiccups experienced in processing natural logarithms.

Failure to add a constant after integration of an indefinite integral, giving an answer without a constant was another challenge that the PSTs faced. Sometimes participants may integrate and process answer without a constant and then introduce the constant in the final answer. **Figure 3** and **Figure 4** are evidence of this error.

The error of not writing the integration constant was very prominent among PSTs, implying that they did not value its existence. By definition, the indefinite integral $\int f(x)dx$ is the function $F(x)$ such that $\frac{d}{dx}F(x)=f(x)$. The problem is due to the multiple (in fact infinitely many) such functions $F(x)$. So, to get around this, the *constant of integration C* is introduced (Blatter, 2011). The indefinite integral $\int f(x)dx$ is also defined to be the general class of *functions (F(x))* whose derivatives are $f(x)$, (Blatter, 2011). For example, given that $F(x)=(x^2+4)$ or (x^2+2) or (x^2+8), then $\frac{d}{dx}F(x)=f(x)$ is equal to $2x$ for all the three functions. However, finding the antiderivative $\int f(x)dx$ would give $F(x)=x^2$ only. So, to cater for the general family of functions of $F(x)$, the constant is required so that $\int f(x)dx=x^2+C$.

A different example to show some misconceptions of a constant is exhibited by PST 19 who, after integrating

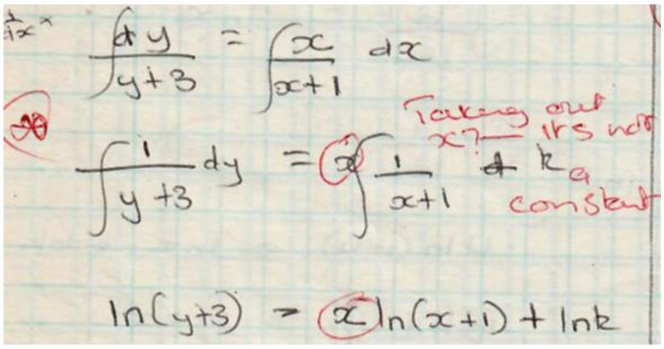

Figure 5. PST 8 question 4

the ODE $x^2 \cos u\frac{du}{dx}+2x \sin u=\frac{1}{x}$ successfully to get $x^2\sin u=\ln x+C$, proceeded to simplify the solution wrongly as given in the subsequent example: $x^2\sin u=\ln x+C$, $\Rightarrow \sin u=\frac{1}{x^2}\ln x+Cx^{-2}$ (procedurally correct). Then she continued: let $\boldsymbol{A = Cx^{-2}}$, therefore, $\sin u=\boldsymbol{\frac{1}{x^2}\ln x+A}$, where A is a constant. This is a comprehension challenge where the PST does not understand what a constant is. x^{-2} is not a constant but a variable. This could have been left as $x^2\sin u=\ln x+\ln C$ or $x^2\sin u=\ln xC$ or $e^{x^2\sin u}=xC$, where C is an integration constant.

The other challenge faced by PSTs was that they could not distinguish between integrating a constant and integrating a variable. PST 8's answer on question 4 is representative of the other PSTs' responses (**Figure 5**).

The PST in this example factored out x and integrated the remaining fraction to get $x\ln(x+1)$. Whilst the PST knew that she should integrate the function from a certain stage of the problem, she erred by not linking all the rules of integration to demonstrate the relational skills in ODEs as described in the SOLO taxonomy. This was consistent with what most of the PSTs wrote. They could not express the fractions as partial fractions to enable and/or facilitate integration. In the example above, the PST is treating the variable x as a constant by factoring it out.

A lack of knowledge in calculus, integration in particular, was another factor that impacted on the PSTs' ability to solve ODEs. These problems involved the application of integration by parts to solve ODEs. The following examples from question 4 show this weakness.

As PST 7 wrote that $ye^{-x}=\int 4x.e^{-x}dx$, $\boldsymbol{\Rightarrow ye^{-x}=\frac{4x^2}{2}e^{-x}+C}$, $\boldsymbol{\Rightarrow ye^{-x}=2x^2e^{-x}+C}$. PST 7 failed to apply the concept of partial integration. This wrong use of formula or method reflects a conceptual error as described by the SOLO taxonomy.

The misapplication of integration rules was also perceived in PST 20's solution to question 7. This is what the PST 20 wrote: $\int\frac{1}{x^2}=\ln x^2$. The PST is confusing this with the standard integration rule $\int\frac{1}{x}=\ln x$, implying that he does not know how $\int\frac{1}{x}=\ln x$ was deduced, instead he

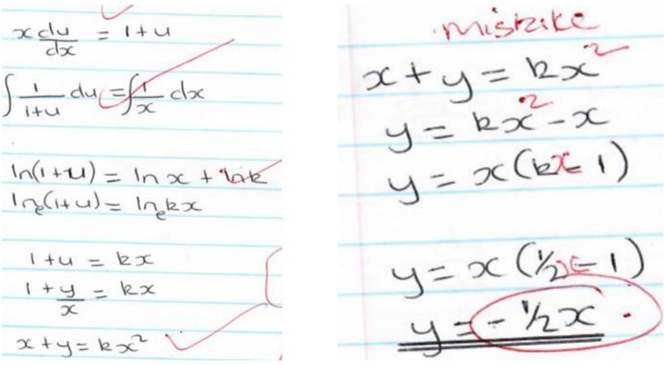

Figure 6. PST 13 question 2

might have just memorized the concept without understanding. This is a conceptual error which resulted in his inability to apply the concept in a different context.

Some technical errors/slips, which proved costly on the PSTs' performance, were also noticed in the participants' work. For example, participant 13 presented her answer as in **Figure 6**.

This error looks like it is just a slip where, instead of writing, say "kx^2", the participant writes "kx", leaving out the squared sign. The error looks simple and minor but could distort the entire process of solving the ODE. Some made technical errors of copying the question wrongly and some of these wrongly written expressions were not integratable. The most common mistake was also seen on question 8 where PSTs, in trying to multiply the differential equation by the I.F. *sinx*, instead of writing *ysinx* at one point in the process of solving the equation, they wrote *sinxy*, which they ultimately confused with *sin(xy)* instead of y multiplied by *sinx*. The resulting solution was therefore wrong.

Generally, the errors made by participants in this study were mainly a result of the comprehension or conceptual errors. This implies that the PSTs lacked the relational skills to solve ODEs.

DISCUSSION OF FINDINGS & CONCLUSION

This study explored the errors in ODEs made by PSTs. The errors were examined through ODE test items that the PSTs answered after a two-week period of instruction. The study identified some themes or categories that included the types of errors made, the causes of the errors as well as the possible remediation of the errors amongst the PSTs.

By and large, the study found that PSTs have idiosyncratic difficulties in learning ODEs which emanated from various causes. Some errors were just ordinary mistakes (slips) that were not conceptual but rather misapplication of some concepts that they knew. However, the rest of them were comprehension and/or conceptual errors revealing inadequacy of knowledge and limitations of imaginations in particular contexts (Brown et al., 2016).

Generally, the participants' performance in the test showed that they were still at the lower cognitive skill level that include the pre-structural, uni-structural and the multi-structural levels of ability of the SOLO taxonomy as described by Tarigan et al. (2019). The research subjects rarely reached at least the relational level in most of the questions. This conclusion was reached because of the PSTs' dismal performance in the test. The dismal performance resonates with Maat and Zakaria's (2011) study which found that many learners have registered difficulties in learning ODEs and related topics. A few errors were individualistic whilst the rest were common across all levels of participants. The performance reveals that the subjects lacked high mathematical abilities in solving ODEs as described by Yarman et al. (2020).

In all the test items, a significant number of participants were leaving out some questions unanswered. This could be a sign of giving up as a result of difficulties. According to Herholdt and Sapire (2014), questions not attempted could be an indication of area of weakness, hence full attention would be required. For example, question 1 was fully answered (but not necessarily getting the correct answers) by about 40% of the PSTs. The rest never attempted to answer the question.

In this study, it was established that most PSTs made conceptual/comprehension errors in ODEs because they lacked basic knowledge of pre-requisite concepts such as algebra to solve ODEs. According to Legutko (2008), a conceptual error reveals inadequacy of knowledge and is closely connected with limitations of imagination and creativity in new situations. This result implies that the PSTs portrayed an insufficient mastery of basic facts, concepts and skills in the ODEs test.

Making use of wrong formulae to solve ODEs was one of the major difficulties that manifested in the PSTs. This was due to the reason that they could hardly identify the type of ODE being solved, which was important for their choice of method to solve the ODE. For example, if a PST identifies a linear ODE in the form $\frac{dy}{dx}+p(x)y=Q(x)$, then they are aware that the integrating factor should be $e^{\int pdx}$, to make the equation exact. Nykamp (2015) asserts that solving an ODE is more complicated than simple integration because several mathematical skills are required to get the right solution.

Furthermore, Nykamp (2015) contends that the most difficult part in solving ODEs is the decision learners take on choosing the method of integration that is needed to solve the ODE, for example, whether it is integration by substitution or integration by parts or integration by partial fractions. Since ODEs are a calculus driven topic (Luneta & Makonye, 2010) and calculus is a feeder topic of first order differential equations (Chikwanha, 2021), thin integration skills dominated in contributing to poor performance in solving ODEs.

Errors regarding the manipulation of the rules of natural logarithms, a lack of knowledge in calculus, integration in particular, failure to distinguish between integrating a constant and integrating a variable and inability to write the integration constant, were also prominent errors among PSTs, implying that this could have contributed to the difficulties in solving ODEs (**Figure 2**, **Figure 3**, **Figure 4**, and **Figure 5**). According to Herholdt and Sapire (2014), a prevalence of certain errors could be a sign of general confusion which might reflect the need for further clarification of the concept. Errors in algebra made by the PSTs in the test were quite extensive and obtrusive that they stifled and incapacitated the whole process of solving the ODEs in the test as reflected in **Figure 1**.

Considering that the participants are PSTs who are training to teach these basic concepts in secondary school, if not addressed early, the problem can turn out to be professionally tragic. Such performance validates Luneta and Makonye's (2010) study which established that performance in calculus is underpinned by weak pre-calculus skills such as algebra. The ZIMSEC's examiners' reports (November 2010, November 2013, June 2016, June 2017) also concur that learners failed ODEs due to poor algebraic skills. The reports highlighted failure to separate variables in solving ODEs, the challenges that emanated from weak algebraic skills. This algebraic incapacitation hence presented conspicuous obstacles that impacted on learning ODEs among PSTs.

In conclusion, the PSTs performance in ODEs was generally undermined by conceptual/comprehension errors such as weak algebraic skills, incorrect use of formulas, a thin knowledge of the rules of natural logarithms, a lack of knowledge in solving simple linear equations (which became obtrusive in separating ODE variables) and technical errors such as omissions and miscalculations. This untenable situation implies the need to equip PSTs with pre-calculus skills as described in the study before the introduction of first order differential equations.

Limitations of the Study

Since this study was limited to PSTs from one institution of higher learning, conclusions cannot be generalized. Data from a larger group of participants from various institutions and various environments is therefore recommended for further studies to confirm the findings of this study. The conclusions reached were also based on participants' performance in the test without taking cognizance of the conditions under which the test was written. As such, a qualitative research instrument where participants' perceptions about the topic could be solicited after the test, would have been necessary. Furthermore, the study mainly focused on the misconceptions and difficulties encountered by the participants in ODEs with little attention to remedial strategies in order to rescue the situation. It is therefore recommended that further studies may objectively look at both the misconceptions and the remedies to make the study complete. In fact, another research instrument could have been used to glean data that would assist in correcting the misconceptions.

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