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An inclusive hypothetical learning trajectory for systems of linear equations for engineering students

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Abstract

Engineering students show persistent difficulties with the solution sets of systems of linear equations (SLE) and with the coordination of their representations, difficulties that are accentuated in heterogeneous classrooms. This study presents the first implementation and the retrospective analysis of a hypothetical learning trajectory (HLT) for SLE that integrates Gravemeijer's (1999, 2007) emergent models, universal design for learning, and Duval's (2006) registers. The HLT was implemented once with 25 first-year students. The hypothetical process was contrasted with the observed one on the basis of four data sources. The analysis documents the forms of activity elicited by each task, the enactment of six inclusive practices fixed a priori, and three discrepancies concerning empty-set notation, parameterized SLE, and row operations. The results are offered as partially tested design conjectures and redesign orientations.

Keywords: hypothetical learning trajectory, systems of linear equations, linear algebra, engineering education, universal design for learning, design-based research

INTRODUCTION

Systems of linear equations (SLE) are foundational in undergraduate engineering mathematics. They underpin later topics in linear algebra and are used in engineering contexts such as circuit analysis and network flow. However, research shows that many students solve SLE procedurally without fully interpreting the solution set. Tatira (2023) documents that students successfully reduce SLE with a unique solution but have difficulties reducing and interpreting those whose solution is not unique. Rodríguez Jara et al. (2019) and Rozas-Torres et al. (2023, 2024) identify persistent difficulties in conceptualizing solution sets, particularly in systems with infinitely many solutions or no solution. Bianchini et al. (2019), in a review of Latin American studies, note that engineering students often privilege algorithmic memorization over conceptual understanding and have difficulty coordinating algebraic, geometric, and tabular registers.

These difficulties are also shaped by the diversity of first-year engineering classrooms. Students enter university with different secondary-school experiences, levels of mathematical preparation, and degrees of familiarity with formal mathematical language (UNESCO-IESALC, 2020). For this reason, the teaching challenge consists in moving students from geometric intuition toward formal algebraic reasoning and in designing this transition so that it offers multiple points of access without lowering mathematical demands.

Three lines of research are relevant to addressing this challenge. The first concerns hypothetical learning trajectories (HLTs), which organize a learning goal, a sequence of tasks, and an anticipated learning process (Simon, 2020). HLTs have been used productively at the university level in topics such as spanning sets, linear differential equations, and the kernel of a linear transformation (Cárcamo et al., 2018, 2026; Rasmussen & Blumenfeld, 2007). The second concerns Gravemeijer's (1999, 2007) emergent-modeling heuristic, which describes how students may progress from situational activity toward referential, general, and formal reasoning through the shift from models-of to models-for. The third concerns inclusive mathematics education. In particular, universal design for learning (UDL) (Alba Pastor, 2016; CAST, 2024) provides principles for anticipating learner variability through multiple means

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Contribution to the literature

- This study extends research on university-level linear algebra education by integrating Gravemeijer's (1999, 2007) emergent-modeling heuristic and UDL principles in the design of a single HLT for teaching SLE.
- It documents, through written protocols and classroom records, the forms of mathematical activity that each task of the HLT elicited, from geometric interpretation to modeling, together with the difficulties that persisted.
- It offers a traceable example of the contrast between the hypothetical learning process (HLP) and the observed process and derives from that contrast three verifiable redesign orientations for the next design-based research (DBR) cycle.

of representation, action and expression, and engagement.

Although these lines offer relevant tools, few studies have brought them together in the design and empirical analysis of an HLT for SLE in undergraduate engineering. Existing HLT studies in linear algebra (Cárcamo et al., 2018, 2021) mainly emphasize conceptual progression. Two recent HLTs from the same research program incorporate inclusive mathematics education, for the kernel of a linear transformation (Cárcamo et al., 2026) and for vector subspace (Cárcamo & Fuentealba, 2026); in both, however, the inclusive strategies are assigned to the learning tasks or enacted through instructional decisions during implementation, without being derived from an explicit set of principles and without a distinction between the practices anticipated in the design and those observed in the classroom. Studies on SLE that emphasize multiple representations (Henriques & Martins, 2022; León del Carmen et al., 2024; Smith et al., 2022; Tatira, 2023) do not usually embed this coordination within an inclusive learning trajectory. Likewise, research on inclusive higher-education mathematics (Moriña, 2017; Prediger & Buró, 2024) often addresses participation and access at a general pedagogical level, without focusing on the specific conceptual demands of linear algebra. This leaves a gap at the intersection of conceptual progression, inclusive design, and topic-specific instruction for SLE, together with the need to make the inclusive dimension of an HLT analyzable in the retrospective analysis rather than only declarable in its design.

The present study addresses this gap through one DBR cycle (Bakker, 2018; Molina González et al., 2011). Following Komatsu et al. (2025), the theoretical referent of the program is a local instructional theory, a notion whose structure is developed in the theoretical framework and which is refined across successive cycles; the present study reports the first of them. In this study, the HLT was designed to support students' transition from geometric interpretations of SLE toward matrix-based and formal reasoning about solution sets, while incorporating potential inclusive practices aligned with UDL principles.

The HLT was implemented with 25 first-year engineering students at a public university in southern Chile over five 60- to 90-minute sessions. Drawing on written protocols, audio recordings, classroom observations, and teacher feedback, we retrospectively analyzed the HLT through the four levels of the emergent-modeling heuristic and the contrast between the HLP and actual learning process (ALP). The study was guided by three research questions:

(1) What forms of mathematical activity (situational, referential, general, and formal) did each learning task of the HLT elicit in the students, and to what extent do they correspond to those anticipated in the HLP?

(2) How were the inclusive practices anticipated in the design implemented at each level of the HLT, and what evidence documents their contributions and their limits for students' participation and conceptual access?

(3) What discrepancies between the hypothetical and the observed learning processes arose during the implementation, and what redesign orientations do they specify for subsequent DBR cycles?

THEORETICAL FRAMEWORK

This section develops the perspectives that support the design and retrospective analysis of the HLT: HLTs, the emergent-modeling heuristic, the registers of semiotic representation, and UDL. In addition, it states the relationship between rank and solution set, as this is the central mathematical content of learning task 3. The final subsection brings these perspectives together and introduces the notion of a local instructional theory (Komatsu et al., 2025), to which this first cycle contributes partially tested design conjectures.

Hypothetical Learning Trajectories as a Design Device

The notion of an HLT, introduced by Simon (1995) and developed in subsequent work (Simon, 2020; Simon & Tzur, 2004), characterizes the work of teachers and designers as the formulation of three interdependent components: a learning goal, a set of learning tasks, and

an HLP that anticipates how students' thinking will evolve as they engage with those tasks. An HLT is hypothetical in two senses: it expresses a prediction about cognitive change that is open to empirical revision, and it operates as a working hypothesis that organizes instructional decisions during implementation. The epistemic value of an HLT derives from the contrast between the anticipated and the observed trajectories, which is why HLTs are particularly suited to DBR (Bakker, 2018).

For the present study, the HLT was structured to move students from geometric interpretations of SLE (intersections of planes in $\mathbb{R}^3$) toward matrix-based and formal reasoning about solution sets. The HLP anticipated that students would first rely on familiar geometric meanings and would later use augmented matrices, row reduction, rank, and set notation as more general tools. The retrospective contrast between this anticipation and the ALP organizes the analysis of the tasks.

Emergent Models and the Four Levels of Activity

The emergent-modeling heuristic (Gravemeijer, 1999, 2007) characterizes mathematical learning as a process in which formal mathematical reasoning emerges from students' situated activity rather than being externally imposed. Gravemeijer (1999) characterizes four qualitatively distinct levels of mathematical activity. At the *situational* level, students reason within a meaningful situation and mathematical actions are interpretations of that situation. At the *referential* level, students use models that still refer to the situation but begin to organize it mathematically. At the *general* level, students work with models that no longer depend on the initial context. At the *formal* level, students reason with conventional symbolism without intermediate referential support.

For SLE, this heuristic organizes the transition from geometric to algebraic reasoning. The intersection of planes in $\mathbb{R}^3$ provides a situational entry point that visually distinguishes among cases of unique, infinite, and no solutions. Augmented and reduced matrices function as referential and general models that preserve information about the system while detaching from the geometric configuration. The formal description of the solution set as a subset of $\mathbb{R}^n$, together with the notions of rank and free variables, corresponds to the formal level. The purpose of using emergent models is to employ geometric interpretation, without replacing algebraic work, as the starting point of a sustained transition toward structural meaning, a transition that Dorier and Sierpinska (2001) identified as the central difficulty of undergraduate linear algebra.

Registers of Semiotic Representation and the Coordination of Representations

The transition described by the emergent-modeling heuristic requires students to operate with representations of a different nature. To analyze it with precision, we adopt Duval's (2006) theory of registers of semiotic representation, which distinguishes register, treatment, and conversion. A register is a semiotic system that makes it possible to represent mathematical objects and to transform them internally. The geometric configuration of planes, the SLE, the augmented matrix, the verbal description, and set-builder notation constitute different registers. A treatment is a transformation that remains within a single register, such as reducing a matrix through elementary row operations. A conversion is a transformation that changes register while preserving the denoted object, such as passing from a configuration of planes to an SLE, or from a reduced matrix to a solution set expressed in set-builder notation.

As Duval (2006) shows, the understanding of a mathematical object is manifested in the capacity to convert between registers and to recognize the same object under heterogeneous representations. Therefore, in this study the expression coordinating registers designates the effective performance of conversions between the representations present in a student's production.

The Relationship Between Rank and Solution Set

Learning task 3 of the HLT is organized around the Rouché-Frobenius theorem, also known as the rank theorem in the reference textbooks used, which we state here because it is the mathematical content that this task seeks to make accessible. Given an SLE $Ax = b$ with m equations and n unknowns, the SLE is consistent if and only if the rank of the coefficient matrix A coincides with the rank of the augmented matrix $[A \mid b]$. When the SLE is consistent, it admits a unique solution if that common rank equals the number n of unknowns, and infinitely many solutions if it is smaller than n; in the latter case, the number of free parameters of the solution set is the difference between n and the common rank. If the two ranks differ, the SLE is inconsistent and its solution set is the empty set (Grossman & Flores Godoy, 2012; Poole, 2011).

This relationship is of didactical interest because it links objects that students usually treat separately, that is, a procedure (row reduction), an invariant (the rank), and an object (the solution set as a subset of $\mathbb{R}^n$). Learning task 3 proposes that students come to formulate this relationship from the systematic analysis of reduced matrices, instead of receiving it as a stated result, which corresponds to the general level of the emergent-modeling heuristic.

Universal Design for Learning as an Inclusive Design Lens

The UDL framework (Alba Pastor, 2016; CAST, 2024; Meyer et al., 2014) assumes that learner variability is expected and should be considered from the design stage. In this study, UDL is used as an inclusive design lens for a heterogeneous first-year engineering classroom. Its three principles (multiple means of representation, multiple means of action and expression, and multiple means of engagement) are pertinent for SLE because students must coordinate geometric, algebraic, matrix, verbal, and set-theoretic representations.

UDL was adopted because it operates at the level of curriculum and task design, without being reduced to an individualized adaptation, which makes it compatible with an HLT for the whole class. In addition, its principles align with the emergent-modeling heuristic, since multiple representations support the transition from the situational to the formal, diverse forms of expression allow students to externalize their evolving models, and the engagement principles sustain participation during that transition. This adoption does not assume that the effectiveness of UDL is established. Boysen (2024) shows that the evidence supporting the CAST (2024) guidelines is scarcer than their dissemination suggests, so in this study UDL is employed as a design lens and its decisions are justified by their coherence with the demands of the HLT, not by a demonstrated effect.

From Design Conjectures to a Local Instructional Theory

The four frameworks operate jointly to define what this study calls an inclusive HLT for SLE. In the complete HLT, the learning goal and the sequence of learning tasks form the macro-structure, the emergent-modeling heuristic defines the cognitive progression that this macro-structure seeks to support, and UDL establishes the equity criteria that it must satisfy at each transition. In each task, the frameworks act jointly, since the task must function as a model-of or model-for at a specific level, offer multiple representations and means of engagement, and have its cognitive demand anticipated within the HLP. In the retrospective analysis, the frameworks define the categories of the contrast between the HLP and ALP, that is, the progression across levels according to emergent models, the conversions between registers according to Duval (2006), participation according to UDL, and the discrepancies according to the HLT. Following Komatsu et al. (2025), this integration of frameworks defines the goal of the research program, a local instructional theory, understood as a structured account comprising a learning goal, design features, an explanation of the connection between them, and the conditions of success under which that connection can materialize, all situated in a particular context and refined across successive cycles. Since this article reports the first cycle, its results constitute a set of partially tested design conjectures and redesign orientations that subsequent cycles will have to put to the test.

METHODOLOGY

This study adopted a qualitative DBR approach to design, implement, and retrospectively analyze an HLT for SLE. DBR (Bakker, 2018) was appropriate because the purpose was to examine how a theoretically grounded instructional design supported students' learning in a real classroom and how it should be refined for subsequent cycles.

With respect to the scope of the study, a single implementation is reported, in one course, with 25 participants, five sessions, and retrospective analysis. The design makes it possible to contrast, in a traceable way, an HLP with the process actually observed, and to derive redesign orientations from that contrast. Reporting the first cycle with this scope is a contribution proper to DBR. The a priori specification of the conjectures, their contrast with the evidence, and the derivation of a traceable redesign constitute the starting point of subsequent cycles (Bakker, 2018; Komatsu et al., 2025; Sandoval, 2014).

Context and Participants

The teaching experiment was conducted in a first-year linear algebra course within the common engineering program of a public university in southern Chile. The course is compulsory for engineering students and includes vectors, matrices, SLE, vector spaces, linear transformations, eigenvalues, and eigenvectors. The intervention took place during the unit on SLE.

Twenty-five first-year engineering students participated in the study. All students were enrolled in the same course section. Their previous mathematical background included solving 2×2 SLE through substitutions, elimination, and graphical methods during secondary education. They had also studied equations of lines and planes in $\mathbb{R}^3$ in a prerequisite course in analytic geometry for engineering. At the time of the implementation, students had worked with matrices, echelon forms, reduced echelon forms, rank, and determinants, and began the formal study of SLE.

The group was selected purposefully because the students had the prior knowledge needed to engage with the geometric entry point of the HLT, in particular with the intersections of lines and planes, and because the group reflected the academic heterogeneity commonly found in first-year engineering classrooms, with diverse educational trajectories and different levels of familiarity with formal mathematical language. This

Table 1. Design of the learning tasks and their correspondence with emergent-modeling levels, mathematical contexts, anticipated inclusive practices, and UDL principles

Task/level	Mathematical context	Anticipated inclusive practices	UDL principles activated
Task 1. *Situational*	Intersection of two planes in $\mathbb{R}^3$	• Activation of prior knowledge (geometric experiences) • Multiple representations (graphical–algebraic) • Meaningful contextualization (plane intersections as familiar situation)	• Multiple means of representation • Multiple means of engagement
Task 2. *Referential*	Visual analysis of three planes in various relative positions + matrix-based resolution of 3× 3 SLE	• Multiple representations (graphical–algebraic–matrix transition) • Structured collaboration (group discussion on graphical interpretations) • Differentiated feedback (according to individual difficulties)	• Multiple means of representation • Multiple means of action and expression • Multiple means of engagement
Task 3. *General*	Reduced augmented matrices, Rouché–Frobenius relationship, parameterized SLE	• Graphic organizers (tables for ranks, unknowns, solution types) • Multiple representations (matrix analysis independent of geometry) • Structured collaborative work (discussion on identified patterns)	• Multiple means of representation • Multiple means of action and expression
Task 4. *Formal*	Network-flow modeling in engineering	• Meaningful contextualization (engineering problems, with GeoGebra support for row reduction) • Multiple representations (modeling–resolution–interpretation) • Differentiated feedback (in modeling and interpretation)	• Multiple means of representation • Multiple means of action and expression • Multiple means of engagement

made the group suitable for examining how an HLT integrating emergent models and UDL principles could support conceptual progression and participation in a heterogeneous classroom.

Before data collection, institutional authorization was obtained and students received written and oral information about the aims of the study, the voluntary nature of participation, the use of written protocols and audio recordings for research purposes, and the anonymization of all reported data. Students who agreed to participate provided informed consent, and pseudonyms (S1, S2, …, S25) were used throughout the analysis and reporting.

Design of the Hypothetical Learning Trajectory

The HLT was designed following the model proposed by Cárcamo and Fuentealba (2023), which integrates three sources: a critical review of the specialized literature on the teaching and learning of SLE, an analysis of widely used linear algebra textbooks for engineering (Grossman & Flores Godoy, 2012; Larson, 2015; Lay, 2012; Poole, 2011), and a characterization of the student population, particularly with respect to prior conceptions and recurrent difficulties (Rozas-Torres et al., 2023, 2024). This triangulated foundation grounded the three components of the HLT (learning goal, learning tasks, and HLP) in disciplinary, didactical, and contextual considerations simultaneously. The following subsections present each component for the HLT designed in this study.

Learning goal

The learning goal of the HLT is formulated as "to understand the concept of an SLE of size $m \times n$ and to solve it using matrix methods in order to obtain its solution set expressed as a subset of $\mathbb{R}^n$ or $M_{n\times 1}(\mathbb{R})$". This formulation entails three operational requirements: students should be able to

(a) identify the essential features of an SLE (unknowns, multiple representations, types of solutions, leading and free variables),

(b) establish connections with other concepts of linear algebra, and

(c) apply this knowledge in various contexts, particularly in mathematical modeling.

Learning tasks

Four tasks were designed, each corresponding to one level of activity in the emergent-modeling heuristic and aimed at operationalizing the three principles of UDL through specific inclusive practices. Task 1 introduces SLE through the intersection of two planes in $\mathbb{R}^3$. Task 2 advances toward 3 × 3 systems through the visual analysis and matrix-based resolution of three planes in various relative positions. Task 3 directs students to interpret reduced augmented matrices and to deduce the rank-solution relationship, including its application to parameterized SLE. Task 4 applies SLE to a network-flow modeling problem drawn from engineering practice. **Table 1** relates each task to its level of activity, its mathematical context, the anticipated inclusive

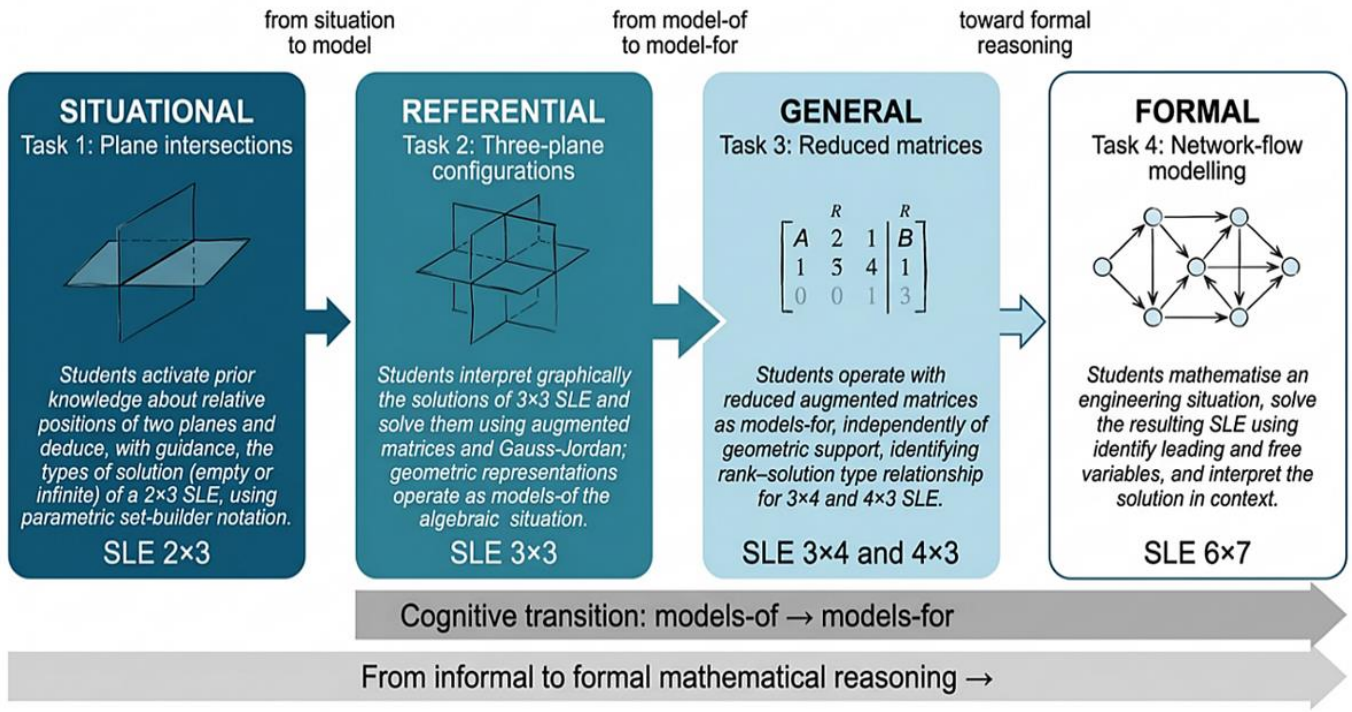


Figure 1. HLP: Progression across the four levels of emergent-modeling activity (The color gradient from left [saturated] to right [light/outline] represents the transition from concrete-geometric to abstract-formal mathematical reasoning) (the authors' own elaboration)

practices, and the UDL principles, and constitutes the operational specification of the design features.

Hypothetical learning process

HLP anticipates how students' cognitive activity is expected to evolve across the four tasks, from situational engagement with geometric configurations to the formal use of SLE as mathematical tools for engineering modeling. Its construction is guided by four orienting questions: what actions are students expected to develop, at what point of the trajectory, through which cognitive strategies, and for what conceptual purpose?

For task 1, students are expected to recognize the intersection of two planes as a line with infinitely many points and to express it using parametric set-builder notation. For task 2, geometric representations operate as models-of, since students are expected to identify the three types of solution from the configuration of three planes and to reproduce them through matrix procedures, writing the solution set as a subset of $\mathbb{R}^3$. For task 3, students are expected to operate with reduced matrices as models-for the analysis of solution sets, independently of geometric support, identifying the structural relationship between rank and solution type and extending it to parameterized SLE. For task 4, students are expected to mathematize an engineering situation, solve the resulting SLE, and interpret the solution in context. **Figure 1** represents the HLP and the transition from models-of to models-for that characterizes the HLT.

The HLP indicators, together with the task specification of **Table 1**, were fixed before the implementation and constitute the prospective conjectures of the design in the sense of Sandoval (2014), since they state which features of the design should mediate which processes and what observable evidence would make it possible to recognize them.

Implementation

The teaching experiment consisted of five 60- to 90-minute sessions, conducted in the regular classroom of the course over two consecutive weeks. The implementation followed the sequence of the HLT with one session per task, except for task 3, which required two sessions owing to the complexity of the rank-solution relationship and parameterized systems. During the sessions, students produced individual written protocols and engaged in collaborative or whole-class discussions. The instructor acted as a facilitator, requesting explanations, comparisons between representations, and justifications of procedures.

To maintain fidelity to the HLT and respond to emerging difficulties, the instructor followed the sequence and intended focus of each task, adapting prompts and feedback to students' productions. After each task, time was devoted to individual and whole-class feedback, with two purposes, addressing the conceptual errors identified and institutionalizing the mathematical ideas that emerged during the activity.

Table 2. Structure of the contrast matrix

Indicator of the HLP	Empirical evidence	Reconstructed ALP	Degree of correspondence
Expected action, reasoning, or representation for each task	Written excerpt, oral episode, or observed classroom action	Interpretation based on triangulated data	Full/partial/weak/not observed

The inclusive practices were implemented throughout the sequence rather than added as separate supports. In task 4, GeoGebra was used after students had formulated the network-flow SLE to reduce the operational burden of matrix computation, thereby allowing students to focus on the interpretation of leading variables, free variables, and contextual constraints.

Data Collection and Analytic Strategy

Data were collected from four sources. Written protocols from all students were collected for each task. These protocols were the main source for identifying strategies, representations, procedures, errors, and forms of solution-set interpretation. In addition, all sessions were audio-recorded to capture students' verbal reasoning, peer discussion, and classroom interaction. Likewise, classroom observations were documented through field notes, with attention to participation, difficulties, teacher interventions, and the use of anticipated inclusive practices. Finally, teacher feedback, both oral and written, was systematized as evidence of the pedagogical adaptations made during the implementation.

Written protocols were treated as the primary evidence of students' mathematical production, whereas audio recordings, field notes, and teacher feedback were used as complementary sources to contextualize students' written responses, clarify the reasoning behind incomplete productions, and triangulate interpretations of participation and conceptual access.

The retrospective analysis contrasted the HLP with the ALP observed during implementation. The main analytic tool was a contrast matrix (**Table 2**) adapted from Dierdorp et al. (2011). The matrix included four columns: the indicator of the HLP, representative empirical evidence from written or oral student productions, the reconstructed ALP, and a qualitative assessment of correspondence between the expected and observed process, using a four-level scale (full, partial, weak, or not observed).

The six inclusive practices analyzed in this study were specified before the implementation (**Table 1**) by operationalizing the three UDL principles in each task. The analysis documented how the anticipated inclusive practices were enacted and what evidence makes it possible to assess their contributions and their limits. For this purpose, explicit evidence criteria were established. A practice was considered effectively implemented when there was an identifiable record of its enactment, such as a material handed to the students, an audio episode, a field note, or a documented teacher intervention. In addition, its contribution was assessed only when it was possible to link that enactment to a subsequent written or oral production by the student. Likewise, the cases in which the support proved insufficient and those in which a practice was not implemented as anticipated were recorded. As there was no comparison group and no controlled variation of the supports, these associations are reported descriptively and do not sustain claims of effectiveness.

To assess the relationship between each practice and the available evidence, a retrospective classification into four categories was adopted: supported (when the triangulated evidence linked the enactment of the support to students' sustained, expanded, or recovered participation or to a mathematically pertinent production, without a reduction of the conceptual demand of the task); qualified (when that evidence coexisted with relevant errors or limits); limited (when the evidence partially contradicted the design conjecture); and insufficient evidence (when the sources did not allow the relationship to be assessed). No practice was classified as limited in this cycle. In this coding, participation was recorded through observable actions, such as handing in a production, explaining or justifying orally, revising an answer after a teacher intervention, collaborating in the discussion, persisting in the task after a difficulty, or using one of the means offered to express reasoning, whereas conceptual access required evidence of work with mathematical meaning, such as a pertinent production even when the notation was still unstable, the coordination of two or more registers, the justified revision of a production, or the recognition of a structural relationship. The result of this classification is reported practice by practice in the Results section. When the evidence on participation and the evidence on conceptual access led to different categories, the assessment is recorded separately for each of the two dimensions instead of being summarized in a single label.

The coding scheme combined theory-driven and data-driven categories. Theory-driven categories were derived from the four emergent-modeling levels, the HLP indicators associated with each task, the UDL-informed design features specified in **Table 1**, and the registers of semiotic representation (Duval, 2006). In the protocols selected as analytic exemplars, the conversions between registers were also coded, in particular the conversion from the geometric configuration to the SLE and from the algebraic conclusion to set-theoretic notation, together with the points at which those conversions broke down. The results of this coding are

reported in the cases of S1, S2, and S3. Data-driven codes were added when students' productions revealed recurrent difficulties or strategies not fully anticipated in the HLP.

The analysis was conducted in three phases. In the first, each student's protocols were analyzed individually across the four tasks, reconstructing the ALP of each participant and comparing it with the HLP. In the second, a cross-case analysis identified patterns, recurrent strategies, and shared difficulties, and supported the identification of the inclusive practices associated with the forms of activity at each level. In the third, the four data sources were triangulated to corroborate the patterns and delimit the discrepancies between the hypothetical and the observed process, from which the redesign orientations are derived.

To strengthen the credibility of the analysis, two members of the research team independently coded a stratified subset of thirteen complete student protocol sets, corresponding to 52% of the 25 participants. Each set included the student's written productions across the four tasks, and the procedure served to refine the analytic criteria. Percentage agreement was used as a descriptive check of coding convergence, and disagreements were resolved through iterative discussion until consensus was reached. The refined codes were then applied systematically to the remaining 12 protocol sets. The contrast between the design and its enactment examined, for each task, which features were implemented as anticipated, with which productions they coincided, and which contextual factors operated as facilitators or constraints.

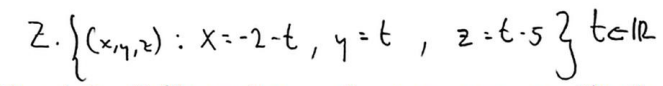
After the cross-case analysis, the protocols of S1, S2, and S3 were selected as analytic exemplars because they represented recurrent or contrasting forms of progression identified in the data. They are therefore used in the results section to illustrate broader empirical patterns.

RESULTS

This section presents the findings in relation to the three research questions, that is, the forms of activity across the four levels through the contrast between the HLP and ALP, the implementation of the six anticipated inclusive practices, and the discrepancies with their redesign challenges. **Table 4**, presented at the end of this section, summarizes the correspondence between the HLP and the ALP for each task.

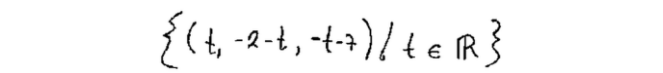
Forms of Activity Elicited by the Tasks Across the Four Levels of the Heuristic

The retrospective analysis showed a progressive shift in students' activity across the four tasks. Each task was constructed to operate at one level of activity and the four tasks were applied in the order of the heuristic, so this shift reflects what the sequence demands and is not attributed here to learning, a limit that the discussion examines. At the beginning, they relied on geometric interpretations; later they coordinated geometric, algebraic, matrix, and set-theoretic representations, and in the final task several used SLE as formal modeling tools. The observed activity was not uniform across students. The HLP/ALP contrast documents the extent to which what was observed in each task corresponded to what was anticipated in the design, as well as the points where it did not.



Trancription: $Z.\{(x,y,z): x=-2-t, y=t, z=t-5\}t\epsilon\mathbb{R}$

Figure 2. Protocol of student S1 for learning task 1 (Set-builder notation for the line of intersection between two planes, through conditions expressed coordinate by coordinate on a parameter) (study data collected by the authors)



Trancription: $\{(t,-2-t,-t-7)/t\epsilon\ \mathbb{R}\}$

Figure 3. Protocol of student S2 for learning task 1 (Set-builder notation for the line of intersection between two planes, through the direct expression of the generic vector as a function of the parameter) (study data collected by the authors)

Situational level: Plane intersections as the entry point to solution sets (task 1)

Twenty-two of the 25 students recognized, without prompting, the intersection of two planes as a line with infinitely many points, and 11 expressed this set through parametric set-builder notation. S1 wrote $\{(x,y,z): x = -2-t, y = t, z = t-5, t \in \mathbb{R}\}$ (**Figure 2**), a form that makes the parameter explicit coordinate by coordinate, whereas S2 wrote $\{(t,-2-t,-t-7): t \in \mathbb{R}\}$ (**Figure 3**), a form that directly presents the generic vector. Both parameterizations describe the same line. They share the direction vector, since $(1,-1,-1) = -(-1,1,1)$, and the point $(0,-2,-7)$, which S1 produces at $t = -2$ and S2 at $t = 0$; the reparameterization connecting them is $t = -2 - s$. Both expressions are transcribed from the original protocols. The difference between the two productions is one of register. S1 keeps the structure of conditions on coordinates, closer to the initial geometric statement, whereas S2 operates on the generic vector as a single object. Among those twenty-two, three students, including S3, showed conceptual access without a stable formal notation. S3 wrote:

> Considering $\mathbb{R}^3 = \{(x,y,z) / x,y,z \in \mathbb{R}\}$, the set of all points belonging to the intersection of both planes must satisfy the form (x,y,z) with $(13-t, 5+t, t)$ and $t \in \mathbb{R}$ (**Figure 4**).

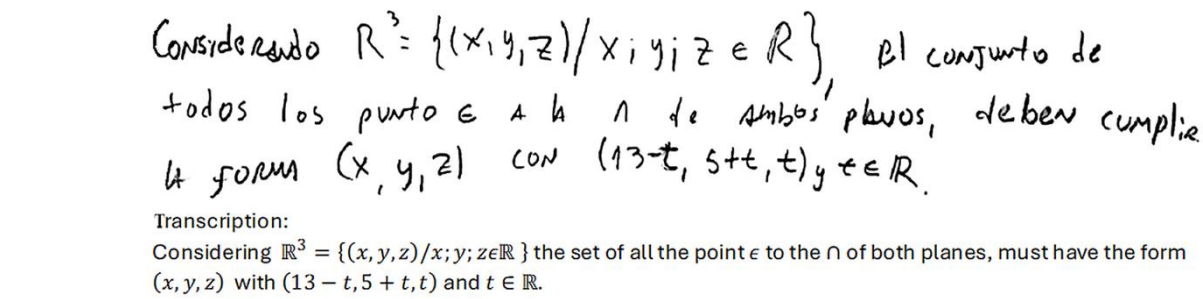


Figure 4. Protocol of student S3 for learning task 1 (Natural-language description combined with partial mathematical symbols of the intersection between two planes, evidencing conceptual access to the meaning of the solution set, together with an arithmetic error in solving for the first component that produces a line parallel to the intended one) (study data collected by the authors)

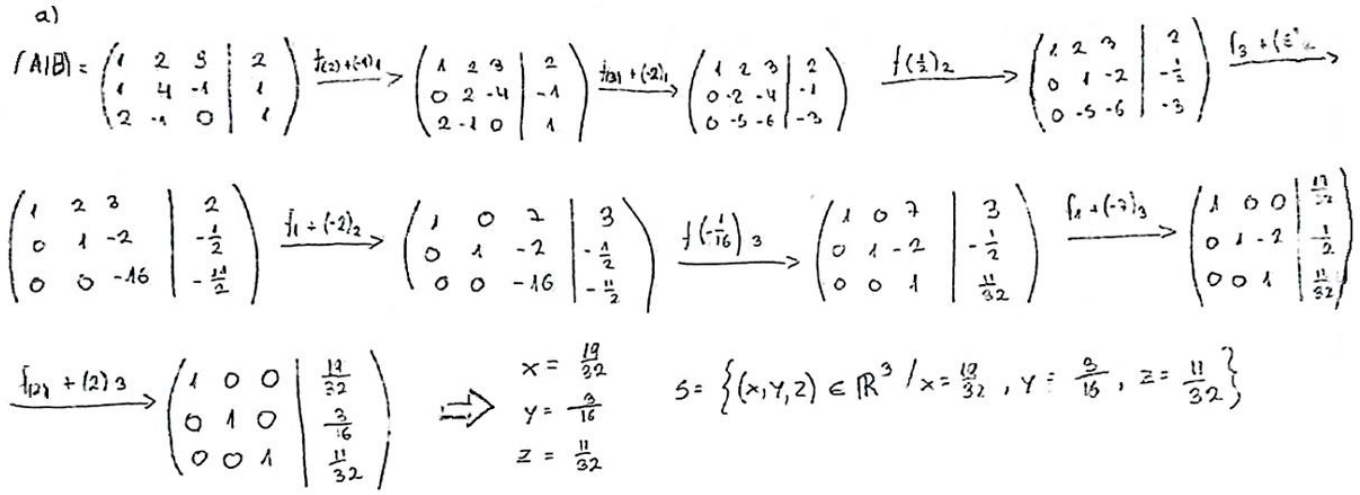


Figure 5. Protocol of student S2 for learning task 2 (Matrix-based resolution of a 3 × 3 SLE with a unique solution and expression of the solution set in parametric set-builder notation) (study data collected by the authors)

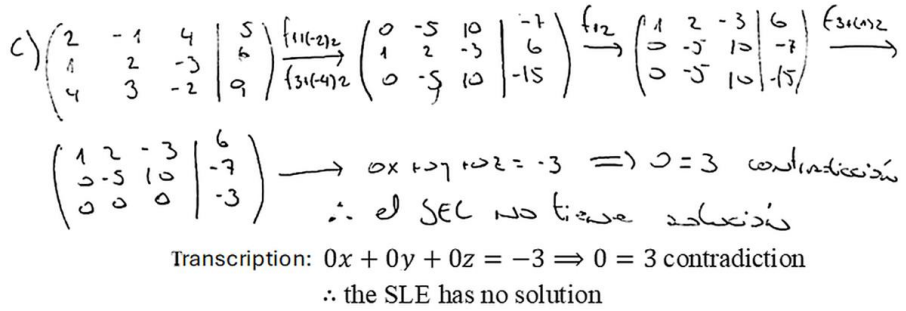
This production combines natural language with mathematical symbolism and adopts the expected parametric structure, with the correct direction vector. However, the constant term of the first component is incorrect. The line of intersection, parameterized with $z = t$, is $(-7 - t, 5 + t, t)$, so S3's expression describes a line parallel to the intended one, distinct from it. The error is arithmetic, located in solving for the first coordinate, and does not affect the parametric structure or the direction. S3 has the meaning of the solution set and the way to represent it, although the symbolic execution fails at one step. Therefore, the situational level was reached at the level of meaning, while the symbolic externalization remained fragile. This dissociation between conceptual access and reliability in notational execution reappears in the subsequent tasks.

Referential level: From plane configurations to augmented matrices (task 2)

Task 2 extended the geometric work to systems represented by three planes in different relative positions and then introduced the matrix-based resolution of 3 × 3 SLE. The visual phase functioned as expected: students associated configurations of three planes with solution types and began to interpret the cardinality and structure of the solution set through the geometry. S1, observing three planes intersecting in a line, concluded that the system had infinitely many solutions, arguing that "*the three planes intersect in a line of infinitely many points*". S2 produced an analogous argument: "*there is an intersection between the three planes where a line of infinitely many points (solutions) is formed*". These verbalizations provide evidence that the geometric configuration was still operating as a model-of and as the primary anchor of meaning, while students were beginning to focus on the algebraic property encoded by the configuration.

The subsequent matrix work showed the referential character of the activity. S2 wrote the augmented matrix for a 3 × 3 system with a unique solution, performed elementary row operations, derived an equivalent reduced system, and expressed the solution set using parametric set-builder notation as $S = \left\{(x,y,z) \in \mathbb{R}^3 | \, x = \frac{19}{32}, y = \frac{3}{16}, z = \frac{11}{32}\right\}$ (**Figure 5**).

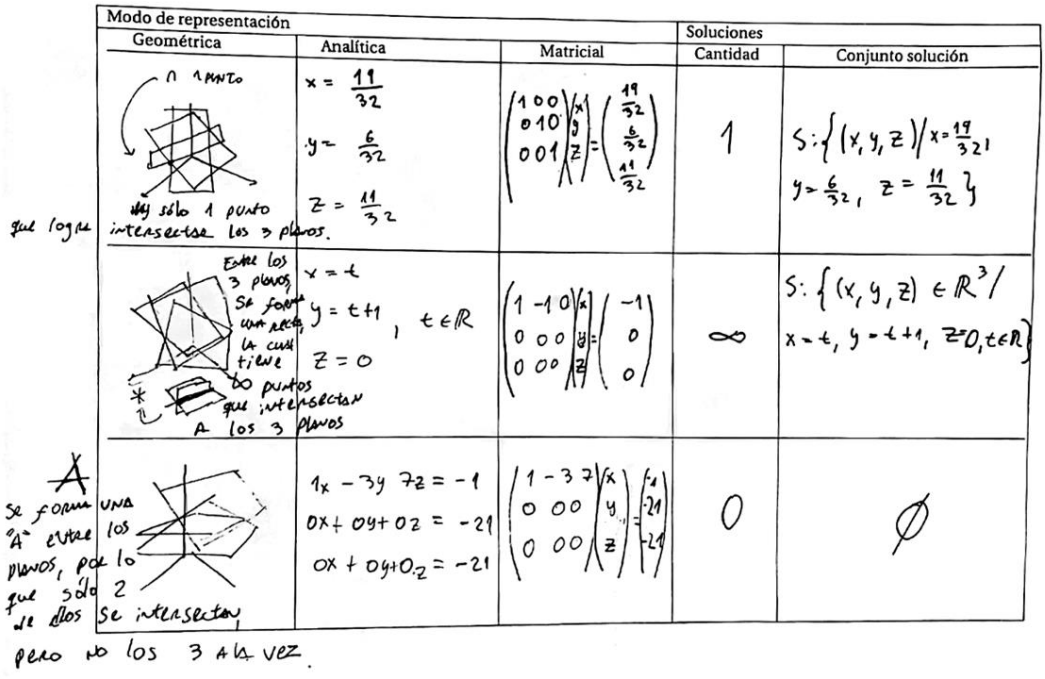
The systematic use of set-builder notation even for unique solutions indicates that students were treating the solution set as a mathematical object rather than as the answer to a procedure.



Transcription: $0x + 0y + 0z = -3 \Rightarrow 0 = 3$ contradiction

$\therefore$ the SLE has no solution

Figure 6. Protocol of student S1 for learning task 2 (Detection of an algebraic contradiction (0 = -3) in the row-echelon form and inference of an empty solution set; The constant term of the last row comes from an arithmetic error in the final step, which does not alter the contradictory character of the row) (study data collected by the authors)



English transcription:

Headings: Mode of Representation: Geometric / Analytical / Matrix-based; Solutions: Number / Solution Set

Column One: ∩ 1 point, there is only 1 point that manages to intersect the 3 planes; Among the 3 planes a line is formed, which has ∞ points that intersect the 3 planes; An "A" is formed between the planes, so only 2 of them intersect, but not the 3 at the same time

Figure 7. Protocol of student S3 for learning task 2 (Analysis of the three types of solution [unique, infinite, empty], with a final matrix of rank 1 in the infinite-solutions case and the restriction z = 0 added without justification) (study data collected by the authors)

S1, upon obtaining an echelon matrix whose last row read "0 = -3", wrote "contradiction" and concluded that "the system has no solution" (**Figure 6**), using the contradiction in the algebraic form as the signal for an empty solution set. The final step of the reduction contains an arithmetic error that alters the constant term of that last row. The resulting row is still contradictory, so the conclusion of inconsistency is correct even though the recorded value is not. The geometric configuration was no longer necessary as direct support, although it remained available as a referential resource for interpreting algebraic results.

Differentiated feedback was directed at students who had faced difficulties in task 1, whose productions at this level are described next. S3 organized the three types of solution set in a tabular structure that coordinates the geometric, analytic, and matrix representations, identified the case of a unique solution and the inconsistent case, and in the latter wrote the empty set with the conventional notation. In the case of infinitely many solutions, two difficulties converge. The reduced matrix recorded in the protocol (**Figure 7**) has rank 1, with two null rows, and cannot come from valid elementary operations on the given SLE, whose rank is 2; this reduction is taken up again among the procedural

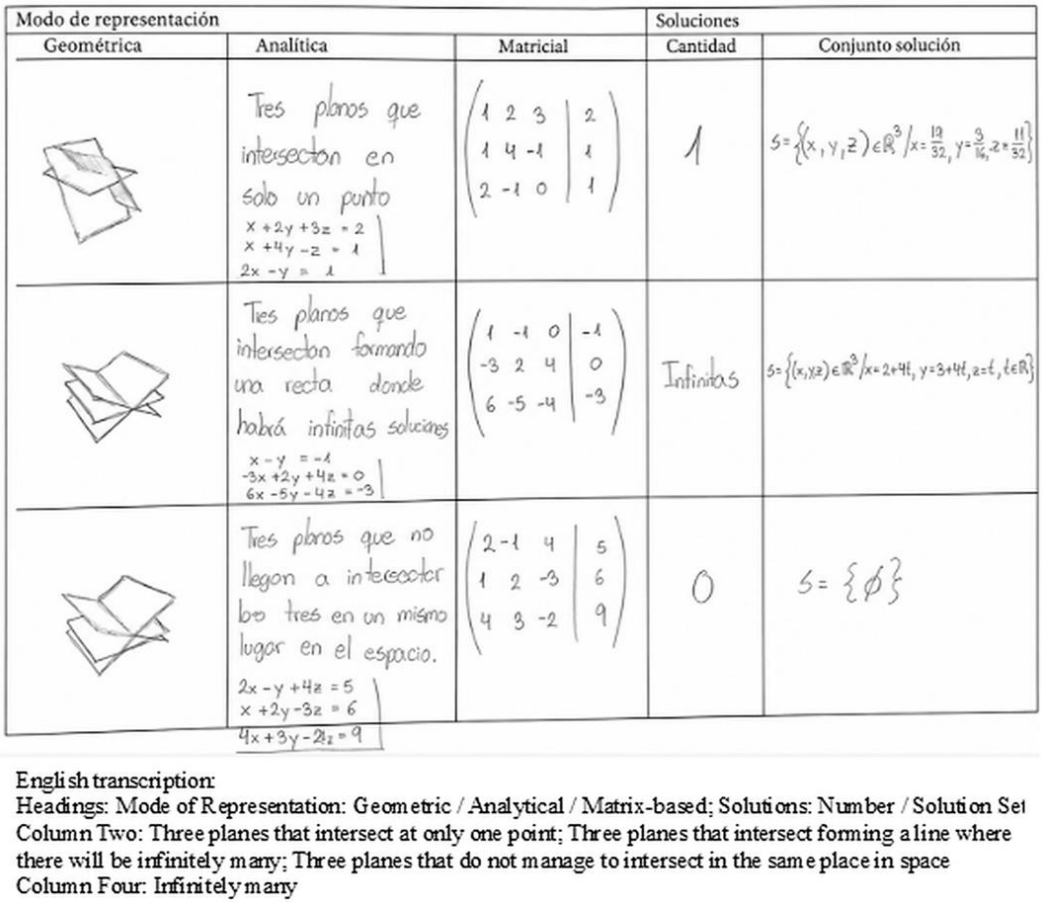
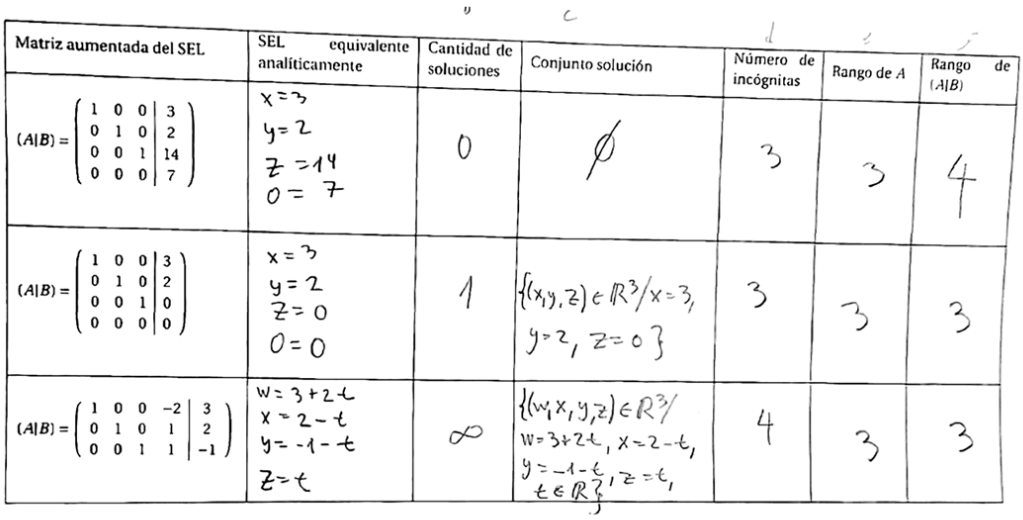


Figure 8. Protocol of student S2 for learning task 2 (Analysis of the three types of solution [unique, infinite, empty] through set-builder notation; The sets recorded in the unique-solution and infinite-solutions cases are correct for the SLE of the protocol; in the inconsistent case, the writing S = {∅} records the discrepancy analyzed in the text) (study data collected by the authors)

difficulties. On this matrix of his own, which leaves two free variables, S3 wrote x = t, y = t + 1, z = 0, with the restriction $z = 0$ added without justification, and obtained a line that is not the solution set of the given SLE either. The relationship among rank, number of unknowns, and free variables did not stabilize; that relationship is the object of task 3 and reappears there as a difficulty.

Within task 2, S2's protocol documents the contrast (**Figure 8**). S2 organized the three cases in the same tabular structure and coordinated the geometric, analytic, and matrix representations. In the unique-solution case, S2 solved the SLE and wrote $S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 19/32, y = 3/16, z = 11/32\}$. In the case of infinitely many solutions, S2 interpreted three planes intersecting in a line and expressed the solution set in set-builder notation as $S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 2 + 4t, y = 3 + 4t, z = t, t \in \mathbb{R}\}$, a correct parameterization consistent with the rank 2 of the coefficient matrix, which leaves one free variable. In the inconsistent case, S2 identified that the number of solutions is zero, but wrote $S = \{\emptyset\}$. The contrast between the two protocols delimits the referential level attained. In particular, S2 adjusts the number of parameters to the rank and fails in writing the empty set, whereas S3 converts among the three registers and leaves the relationship between rank and free variables unstable.

Procedural difficulties nevertheless persisted. For example, S1 made an arithmetic error in the final step of a row reduction (**Figure 6**), and S3 chained row operations that did not preserve the system in the infinite-solutions case of task 2, since the final matrix of the protocol (**Figure 7**) has rank 1 while the given SLE has rank 2, and valid elementary operations preserve the rank, so the form obtained was correct, but its content did not correspond to the original system. The rank argument can be checked in **Figure 7**. In the infinite-solutions case, the SLE given in the task, which S2 also transcribes in **Figure 8**, is x - y = -1, -3x + 2y + 4z = 0, 6x - 5y - 4z = -3. Adding three times the first equation to the second gives -y + 4z = -3 and subtracting six times the first equation from the third gives y - 4z = 3, so the coefficient matrix has rank 2 and the solution set is the line {(2 + 4t, 3 + 4t, t): t ∈ ℝ}. The final matrix of S3 keeps the first equation and replaces the other two by null rows, and the line x = t, y = t + 1, z = 0 written on it meets the solution set only at the point (2, 3, 0). The same pattern appears in the inconsistent case of the protocol. Its first row is the difference between the first two



English transcription:
Headings: Augmented matrix of the SLE/Analytically equivalent SLE/ Number of solutions/ Solution set / Number of unknowns/ Rank of A/ Rank of (A|B)
Row 3, solution set: $\{(w,x,y,z): w = 3 + 2t, x = 2 - t, y = -1 - t, z = t, t \in \mathbb{R}\}$

Figure 9. Protocol of student S3 for learning task 3 (Tabular analysis of three reduced augmented matrices and identification of the corresponding solution types [empty, unique, infinite], with a dimensional inconsistency in declaring in $\mathbb{R}^3$ a solution with four components) (study data collected by the authors)

equations of the given SLE, 2x - y + 4z = 5 and x + 2y - 3z = 6, and the other two rows have null coefficients, whereas the coefficient matrix of that SLE has rank 2. There the conclusion of inconsistency is not affected, because a single row of the form 0 = c with c ≠ 0 suffices to establish it. A notational difficulty of a different nature also persisted. Nine students wrote the empty solution set as S = {∅} instead of S = ∅. The difference is set-theoretic. The set ∅ contains no elements, whereas {∅} contains exactly one, which is ∅; the nine protocols do not distinguish membership from inclusion and treat the empty set as an element of the solution set instead of the solution set itself. Writing S = {∅} attributes to the SLE exactly one solution, which reverses the result the student had just obtained. In the nine protocols, the conversion from the algebraic conclusion of inconsistency to the set-theoretic register breaks down at that point.

General level: Reduced matrices, ranks, and parameterized systems (task 3)

Task 3 asked students to interpret reduced augmented matrices without performing row reduction themselves and to deduce the relationship between the ranks of the coefficient and augmented matrices and the type of solution set. S3's protocol illustrates the transition toward general activity (**Figure 9**). S3 organized three cases in a tabular structure that supported the systematic analysis of the reduced forms. For the first case, whose last row was (0 0 0 | 7), S3 correctly concluded that there was no solution and wrote the empty solution set. In the second case, S3 initially interpreted the row (0 0 1 | 0) as a contradiction, but after the differentiated feedback wrote the equivalent SLE x = 3, y = 2, z = 0, 0 = 0, noted "there is no contradiction here", and concluded that there was a unique solution. In the third case, S3 identified the four unknowns and the ranks, both equal to 3, and parameterized correctly with w = 3 + 2t, x = 2 - t, y = -1 - t, z = t, where z = t operates as the free variable. However, S3 wrote the solution set as a subset of $\mathbb{R}^3$, declaring as an element of that space a quadruple that belongs to $\mathbb{R}^4$. The discrepancy is conceptual and concerns the correspondence between the number of components of the solution and the dimension of the ambient space. S3 masters the reading of the reduced matrix, the identification of the ranks, and the parameterization, and fails in situating the object in its space; the reduced matrix functioned here as a model-for, without requiring the geometric situation as support.

The second part of task 3 organized the information about ranks, unknowns, and solution types in tables, which supported the deduction of the rank-solution relationship. Most students formulated in their own words a partial version of this relationship, without reaching the complete statement of the Rouché-Frobenius theorem. S2 wrote:

> the relationship is that if the rank of A is a smaller number than the rank of the augmented matrix, then there will be no solutions; and if both ranks are equal, there may be one solution or infinitely many.

This verbalization evidences the construction of a model-for that allowed reasoning about the structural features of SLE independently of specific systems.

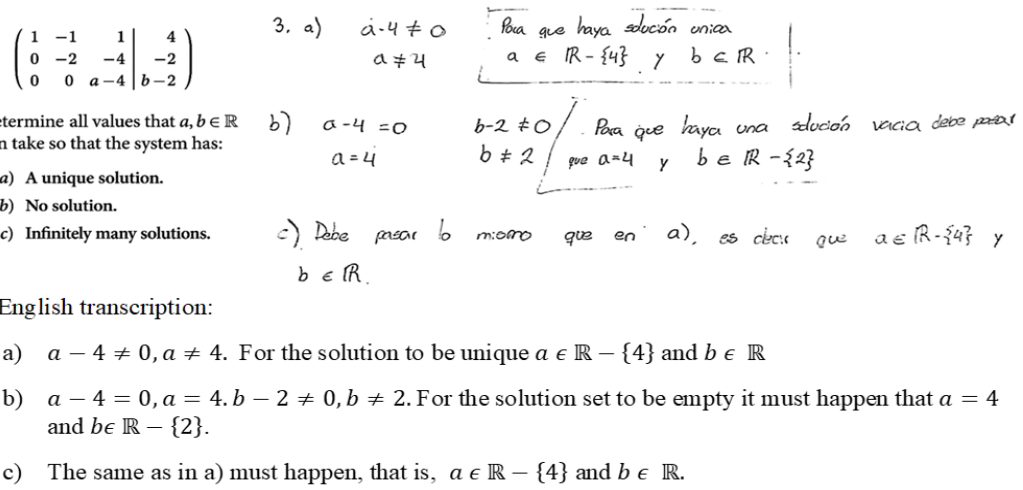


Figure 10. Protocol of student S2 for learning task 3 (Application of the rank–solution relationship to a parameterized SLE, showing the difficulty in differentiating the parameter values that produce unique and infinite solutions; The correct conditions are a unique solution if a ≠ 4, no solution if $a = 4$ and b ≠ 2, and infinitely many solutions if $a = 4$ and $b = 2$) (study data collected by the authors)

However, the application of this relationship to parameterized SLE gave evidence of a limitation of the HLT. S2 correctly determined the parameter values that produced an empty solution set but did not differentiate between the values producing a unique solution and those producing infinitely many solutions, indicating the same values for both cases (**Figure 10**). The correct conditions are recorded in the caption of **Figure 10**; S2 noted a ≠ 4 both for the unique solution and for the infinitely many solutions. Ten students exhibited analogous difficulties. The HLT supported the deduction of the rank-solution relationship, but the autonomous application of this relationship to parameterized SLE required supports additional to the four tasks. This finding is analyzed in the third subsection as a redesign challenge.

Formal level: Network-flow modeling and contextual interpretation (task 4)

Task 4 required students to use SLE as tools to model a network-flow problem related to engineering. S1's work documents the modeling cycle (**Figure 11**), which comprises representing the network through the labelled nodes A, B, C, D, E, and F and seven variable flows x_1, x_2, …, x_7; tabulating the incoming and outgoing flows at each node and applying the flow-conservation rule to formulate an SLE of six equations with seven unknowns, with its augmented matrix of order 6 × 8; and reducing that matrix with GeoGebra to obtain an equivalent SLE. The six nodal equations and the equivalent SLE after the reduction appear in **Figure 11** and its typographic transcription; the rank is 5 over seven unknowns, so the solution set depends on two parameters and the free variables are x_6 and x_7. The row 0 = 0 reflects that one nodal equation is a linear combination of the others, consistent with the global balance of the network. The formal character of the activity is seen in the coordination among modeling, matrix representation, and computation, without recourse to intermediate referential supports. Twenty-one of the 25 students completed the modeling cycle; the difficulties of the remaining students were related mainly to the size of the SLE, more than to the underlying concept, a difficulty that the use of GeoGebra sought to mitigate. Differentiated feedback during this task supported the interpretation of leading and free variables.

Implementation of the Inclusive Practices Anticipated in the Design

The six inclusive practices (**Table 1**) anticipated before the implementation, that is, the activation of prior knowledge, the integration of multiple representations, the use of graphic organizers, structured collaborative work, differentiated feedback, and meaningful contextualization, were enacted across the four tasks. No new practice was incorporated during the sessions and none of the six was withdrawn. We describe next how each was implemented at the level where it was most visible in the data, together with the observed limits. These descriptions document the enactment of the design and the productions associated with it.

At the situational level, the activation of prior knowledge and meaningful contextualization framed task 1 in a situation that allowed students to connect with their previous study of plane intersections in analytic geometry. This connection offered a common entry point to students with different secondary-school trajectories, as suggested by the high proportion (22 of 25 students) who recognized the intersection of two planes as a line with infinitely many points (**Figure 2**, **Figure 3**, and **Figure 4**). Within that framework, students with weaker symbolic backgrounds, such as S3, constructed meaning from the geometric situation even with a fragile formal notation.

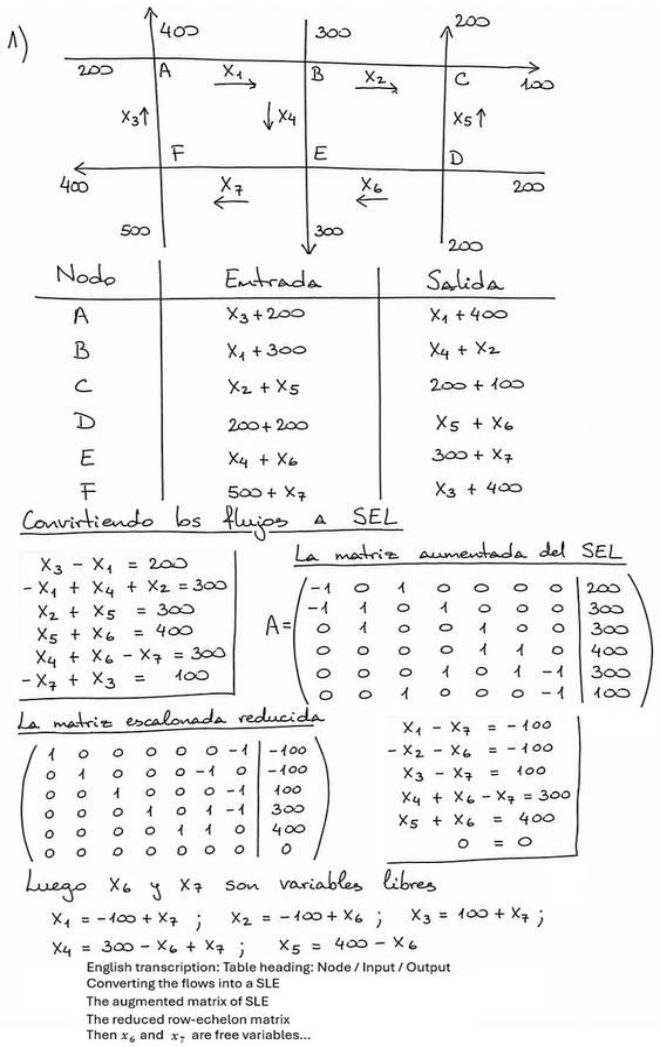


Figure 11. Protocol of student S1 for learning task 4 (Modeling cycle of a network-flow problem: (a) network diagram with labelled nodes and the flows x_1, …, x_7; (b) flow-conservation table for each node, formulation of the 6 × 7 SLE, and augmented matrix; and (c) reduced echelon matrix obtained with GeoGebra and equivalent system, from which the free variables x_6 and x_7 follow) (study data collected by the authors)

At the referential level, the integration of multiple representations, structured collaborative work, and differentiated feedback supported the coordination of the geometric, algebraic, and matrix registers during task 2. The case of S3 illustrates their enactment, since after an incomplete notation in task 1 (**Figure 4**), in task 2 S3 expressed the solution set in set-builder notation in two of the three cases and performed conversions among the three registers; the change coincides with the change of task and cannot be attributed to the practices. According to the field notes, group discussions gave room for students with incomplete symbolic productions to intervene in collective reasoning, although no interaction excerpts linkable to individual productions are preserved. At this level, the three practices addressed the first two UDL principles (multiple means of representation and multiple means of action and expression).

At the general level, graphic organizers were used systematically in task 3. The tabular structures for analyzing ranks, unknowns, and solution types operationalized the second UDL principle, and their use is documented in S3's protocol (**Figure 9**). The organizer was handed to the whole group and everyone completed it, although with errors, such as the dimensional inconsistency recorded in that same protocol (**Figure 9**). In addition, the organizers functioned as visible records that supported the teacher's institutionalization of the rank-solution relationship.

At the formal level, meaningful contextualization situated task 4 within engineering practice (network-flow modeling), and differentiated feedback supported the interpretation of leading and free variables. GeoGebra reduced the operational burden of computation to concentrate the work on modeling and interpretation (**Figure 11**; 21 of 25 students completed the cycle). These elements anchored abstract symbolic procedures in a meaningful disciplinary situation and operationalized the third UDL principle. The level-by-level description does not imply exclusivity. Differentiated feedback, for example, operated across the four levels, with greater visibility at the referential and general levels. **Table 3** synthesizes, for each practice, its enactment, the associated evidence, the counter-evidence or observed limits, and the resulting retrospective assessment.

Table 3. Enactment, evidence, counter-evidence, and retrospective assessment of the inclusive practices anticipated in the design

Anticipated practice (**Table 1**)	Enactment and associated evidence	Counter-evidence or limit	Retrospective assessment
Activation of prior knowledge	22 of 25 students recognized the intersection of two planes as a line with infinitely many points (task 1); broad participation at the geometric entry point	No independent measure of the levelling across prior trajectories	Supported for participation; qualified for conceptual access
Integration of multiple representations	Documented conversions among the geometric, matrix, and set-theoretic registers (S1, S2; **Figure 2**, **Figure 3**, **Figure 5**, and **Figure 6**) and between the verbal and the parametric register (S3; **Figure 4**)	Persistent errors: arithmetic in solving for one component (S3, task 1), empty-set notation (9 of 25), and dimensionality (S3, task 3)	Qualified

Table 3 (Continued).

Anticipated practice (**Table 1**)	Enactment and associated evidence	Counter-evidence or limit	Retrospective assessment
Use of graphic organizers	Complete appropriation associated with the systematic recording of the three solution types (S3, **Figure 9**); support for the institutionalization of the rank–solution relationship	Handed to the whole group, so no comparison is possible; it coexisted with the dimensional error and with the incomplete application of the rank–solution criterion	Supported as an externalization resource; effect not established
Structured collaborative work	Group discussions recorded in field notes and audio recordings	No excerpts linkable to individual productions	Insufficient evidence
Differentiated feedback	Documented revision episodes, such as S3's reinterpretation of the row (0 0 1 | 0) after the teacher's intervention (task 3)	The unjustified restriction z = 0 (task 2, **Figure 7**) and the dimensional error (task 3) persist after the feedback	Qualified
Meaningful contextualization (with GeoGebra in task 4)	21 of 25 students completed the modeling cycle; S1's original protocol (**Figure 11**)	4 of 25 did not complete the cycle; no records that allow the contribution of the software or the context to be isolated	Supported for the implementation; mechanism not established

Table 4. Synthesis of the HLP/ALP contrasts across the four emergent-modeling levels

Task	HLP indicator	ALP synthesis	Correspondence	Main difficulty
Task 1	Students interpret the intersection of planes as a solution set and express it through parametric set-builder notation.	22 of 25 students recognized the intersection as a line of infinitely many points; 11 expressed this set through parametric set-builder notation (S1, S2).	Full for geometric interpretation; partial for formal notation	Set-builder notation among students with a weaker symbolic background; arithmetic error in solving for a coordinate that produces a parallel line (S3)
Task 2	Students connect plane configurations with solution types and use augmented matrices to solve 3× 3 SLE.	Students associated visual configurations with solution types; S2 wrote and reduced matrices for unique-solution systems; S1 detected algebraic contradictions and inferred no solution.	Full/partial	Empty-set notation; arithmetic error in the final step of a row reduction (S1, Figure 6); unjustified restriction of a free variable in the infinite-solutions case (S3, Figure 7)
Task 3	Students operate with reduced matrices as *models-for*, deduce the rank–solution relationship, and apply it to parameterized SLE	Students read solution types from reduced matrices and verbalized the rank–solution relationship; 10 students did not differentiate parameter values for unique or infinite solutions	Partial/partial	Autonomous application of the rank–solution relationship to parameterized systems; correspondence between the number of components of the solution and the ambient space (S3, **Figure 9**)
Task 4	Students mathematize a network-flow situation, solve the SLE with GeoGebra, identify leading and free variables, and interpret the solution in context	21 of 25 students completed the modeling cycle; S1 used GeoGebra and identified x_6 and x_7 as free variables (**Figure 11**); students needed support to coordinate free parameters with contextual constraints	Full/partial	Coordination of free parameters with contextual interpretation

Discrepancies Between the Hypothetical Learning Process and the Actual Learning Process and Redesign Challenges

The HLP/ALP contrast identified three classes of discrepancy between the design and its implementation, synthesized in **Table 4**.

The difficulty with empty-set notation was evidenced in the protocols of nine students who wrote S = {∅} instead of S = ∅ in task 2 and task 3. The autonomous application of the rank-solution relationship to parameterized SLE was evidenced in S2's protocol (**Figure 10**) and replicated in ten students. Reliability in the execution of elementary row operations was observed in S1's protocol for task 2 (**Figure 6**), where an arithmetic error in the final step alters the constant term of the last row; the conclusion of inconsistency is correct, although the episode shows that the execution of the algorithm was not consolidated. Their theoretical

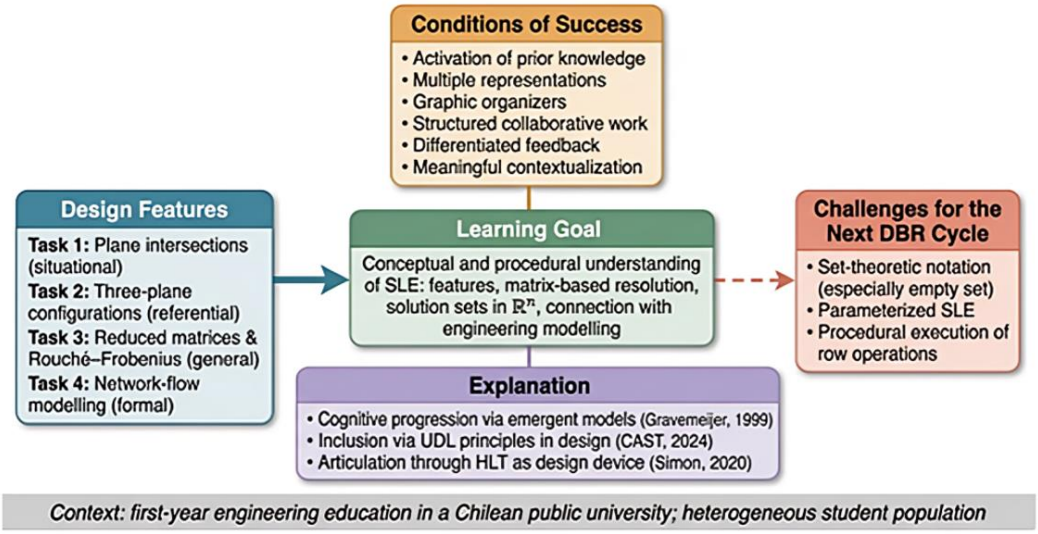


Figure 12. Provisional synthesis of the design conjectures and the redesign orientations for the inclusive learning of SLE in undergraduate engineering (the authors' own elaboration)

interpretation and the redesign orientations are developed in the discussion. **Table 4** summarizes the correspondence between the HLP and the ALP for each task and synthesizes the patterns described in the preceding subsections. The contrast was applied to the HLP indicators fixed before the implementation. **Table 4** recapitulates the patterns exemplified in the text, without introducing new evidence.

DISCUSSION

This section interprets the findings of the study in relation to the three research questions, delimits the scope of the design conjectures tested in this DBR cycle, and analyzes the redesign challenges that orient the next cycle.

Contrasted Design Conjectures and Their Scope

In line with the conceptualization adopted in the theoretical framework, the empirical findings of this DBR cycle are organized according to the conceptual structure of local instructional theories (Komatsu et al., 2025), without implying that such a theory has been established. The four elements of that structure, situated in the context of first-year engineering education at a Chilean public university, are represented in **Figure 12** and developed in this subsection.

The learning goal consists in constructing a conceptual and procedural understanding of SLE, from their essential features and matrix-based resolution to the representation of the solution set in $\mathbb{R}^n$ and their use in modeling. The design features are organized as the sequence of four tasks aligned with the levels of the heuristic (Gravemeijer, 1999) and with the three UDL principles, operationalized through the six inclusive practices. The theoretical explanation of how the design features lead to the learning goal holds that cognitive progression is supported by the transition from models-of to models-for, that inclusion is favored through the operationalization of the UDL principles in each learning task, and that the integration of cognitive progression and inclusion is structured through the HLT as a design device. The six practices of **Table 1** were deployed at the level corresponding to each task and were documented in the group's productions, without this cycle allowing it to be established that they were necessary or sufficient conditions for the recorded activity.

In addition, provisional conditions of success are derived from the retrospective analysis, understood as the circumstances under which the design features showed their function. Among them are that the graphic organizer be appropriated by the student to compare systematically ranks, number of unknowns, and solution types, and that the teacher take advantage of it to institutionalize the rank-solution relationship; that differentiated feedback address the obstacle actually manifested, occur before a new resolution opportunity, and leave the student the possibility of revising the production; that the situational entry rest on prior familiarity with analytic geometry; and that GeoGebra be used after the formulation of the SLE, so that the reduction of the operational burden does not replace the modeling work. These conditions are observational conjectures of this cycle and remain subject to testing in subsequent cycles.

The design of this study imposes an interpretive limit. Each learning task of the HLT was constructed to operate at a given level of activity, and the four were applied in the order of the heuristic. Consequently, the fact that students' activity shifts from geometric interpretations toward formal reasoning cannot simply be attributed to learning, because the task sequence itself demands that shift. Separating it from learning would require a comparison group, a variation of the order, or

tasks that did not signal the expected level, conditions that this first implementation does not meet. Therefore, the data document the forms of mathematical activity that each learning task elicited; the design does not allow a progression with an established cause to be affirmed. Within that limit, some regularities of the data are informative because they are not explained by position in the sequence, although they do not establish learning either. One of them is persistence. The notation S = {∅} appears in task 2 and reappears in task 3, and the dissociation between conceptual access and notational reliability observed in S3 runs through the first three learning tasks in different forms. Another regularity is failure within the level itself. Students verbalize in task 3 a version of the rank-solution relationship and, in that same task, ten of them fail to apply it to parameterized SLE. This dissociation is the regularity with the strongest support, because the mismatch occurs within a single learning task. The protocols do not allow it to be established whether the ten students who did not apply the relationship are the same ones who had previously verbalized it. Intra-task change constitutes an additional regularity. In the only documented episode, also in task 3, S3 reformulates the reading of the row (0 0 1 | 0) after the differentiated feedback. Persistence and failure within the level show that advancing in the sequence is not enough for the corresponding level to be attained; intra-task change occurs without a new learning task intervening. In addition, the dissociation observed at the situational level admits Sierpinska's (2000) reading, since students operated in a synthetic-geometric mode while the transition to the analytic-arithmetic mode remained fragile; for this reason, bridges between modes of thinking should be designed from the situational entry. Finally, the protocols record no reversions to earlier levels, because the four tasks always signal the expected level; the next cycle will incorporate at least one mixed task, without level indication, in which the choice of representation is left to the student.

Inclusive Practices as Constitutive Design Decisions

The six inclusive practices anticipated in the design should be interpreted as design decisions, without being reduced to remedial accommodations. Their function was to diversify the ways in which students could access, express, and progressively formalize their reasoning about solution sets, without simplifying the mathematical goal. In this sense, inclusion operated within the mathematical architecture of the HLT: multiple representations were associated with the transition from models-of to models-for, graphic organizers offered a visible record of structural relationships, and differentiated feedback accompanied the revision of symbolic expression without reducing the conceptual demand of the tasks. This treatment differs from that of the previous HLTs of the program (Cárcamo & Fuentealba, 2026; Cárcamo et al., 2026), in which the inclusive strategies were associated with each learning task or with the teachers' decisions during implementation, and their contribution was inferred from students' progression. Here, the six practices are derived from the three UDL principles, and the distinction between anticipated and implemented practices makes it possible to report which of them operated in the classroom and how, instead of assuming that what was planned coincided with what occurred.

Learner variability was addressed at the level of the design of the learning tasks, without recourse to subsequent remediation, so the inclusive dimension of the HLT was constitutive of the conceptual design. It should be noted, however, that UDL was employed here as a design lens to anticipate learner variability, without treating it as a framework whose effectiveness has been demonstrated in this study. Boysen (2024) notes that the empirical basis of the CAST (2024) guidelines is more limited than their dissemination suggests, so the decisions inspired by UDL are justified in this work by their coherence with the demands of the HLT, without appealing to an accredited effectiveness.

These practices also respond directly to a recurrent obstacle identified in Latin American studies on engineering students' learning of linear algebra (Bianchini et al., 2019): the privileging of algorithmic memorization over conceptual understanding and the limited coordination of algebraic, geometric, and tabular registers. The integration of multiple representations across the four tasks, together with the use of graphic organizers in task 3 and meaningful contextualization in task 1 and task 4, addressed this obstacle by design rather than through post-hoc adaptation.

Persistent Challenges as Zones of Proximal Development for the Next Cycle

The three classes of discrepancy identified in the Results require a theoretical interpretation that orients the next DBR cycle. The notational difficulty concerning the empty set consists in the lack of distinction between an object and the set that has it as its only element and turns the conclusion of inconsistency into a singleton set. Bagni (2006) documents that representations of the empty set generate stable cognitive difficulties when the symbol is read as the name of an object, without denoting a set devoid of elements, and the observed breakdown also admits a reading as a failed conversion between the algebraic and the set-theoretic registers (Duval, 2006). Therefore, the redesign orientation consists in incorporating learning tasks that contrast the empty set, the singleton set, and the parameterized infinite set through multiple representations, treating set-theoretic notation as content of the HLT.

The difficulty in autonomously applying the rank-solution relationship to parameterized SLE can be interpreted, following Dorier and Sierpinska (2001), as a

manifestation of the obstacle of formalism: the structural relationship was within reach as a general statement, but its application to a specific parameterized system required a level of formal manipulation beyond what the four tasks supported. This interpretation reframes the difficulty in a productive direction, since parameterized SLE constitutes a zone of proximal development within the scope of an HLT for SLE, whose treatment requires additional design work in the next cycle.

The difficulties observed in reducing and interpreting parameterized SLE coincide with what Tatira (2023) reports, who documents that the reduction of systems whose solution is not unique, and notably that of augmented matrices with variable entries, demands process and object conceptions that his students had not attained. In our case, students recognized the purpose of the procedure and the structure of the solution set but executed the algorithm unreliably and did not autonomously apply the rank-solution relationship to parameterized SLE. The available evidence, an identifiable arithmetic error in one protocol, does not allow it to be decided whether the execution difficulty also reaches what each elementary operation preserves; the next cycle must collect evidence directed at that question. Consequently, the corresponding redesign orientation concerns complementary practice and examples worked between sessions, outside the four conceptual learning tasks. Treating these three challenges as constitutive elements of the local instructional theory under construction is consistent with the perspective of Komatsu et al. (2025), who conceive local theories as constructions in a state of flux, refined through successive cycles in which empirical findings interact dialectically with theoretical commitments.

Boundary Conditions and Transferability

The design conjectures developed in this study are grounded in a specific context, a public university in southern Chile, a first-year engineering program, and a heterogeneous student population. The transferability of the design to other contexts depends on examining which features are conditioned by the context and which are likely to operate similarly. The choice of plane intersections as the situational entry depends on students having previously studied analytic geometry in $\mathbb{R}^3$; network-flow problems depend on their disciplinary relevance for the program in question; and the operationalizations of UDL reflect the linguistic and cultural composition of the participating classroom. By contrast, the four-level structure of the emergent-modeling heuristic, the integration of cognitive progression and inclusion at the level of design, and the conceptualization of the empirical output through the structure of local instructional theories constitute context-independent elements that other DBR studies in mathematics education can adopt.

CONCLUSIONS

This study reports the first cycle of a DBR effort on an inclusive HLT for SLE in undergraduate engineering, integrating Gravemeijer's (1999, 2007) emergent-modeling heuristic, UDL principles, and the conceptual structure of local instructional theories (Komatsu et al., 2025). The retrospective analysis supports three conclusions. The integration of emergent models and UDL within a single HLT for SLE proved viable in a heterogeneous first-year engineering classroom. This result extends the previous HLTs of the same program that combined emergent models with inclusive strategies for the kernel of a linear transformation and for vector subspace (Cárcamo & Fuentealba, 2026; Cárcamo et al., 2026): here the inclusive dimension is derived from an explicit set of principles and, through the distinction between anticipated and implemented practices, becomes an object of the retrospective analysis rather than a feature declared in the design. In addition, the HLT elicited the anticipated forms of activity in most of the indicators of the hypothetical process, unevenly across students and with three difficulties that the sequence did not resolve by itself; the design employed does not allow it to be determined whether that shift is due to learning or to the order of the learning tasks. Finally, the contrast between the HLP and the ALP made it possible to formulate partially tested design conjectures and traceable redesign orientations, which constitute the starting point of the next cycle, without yet configuring an established local theory.

The study has limitations that delimit the scope of its claims and open orientations for future research. The findings come from a single implementation, analyzed retrospectively and without a comparison group, so they do not sustain claims of effectiveness or generalization; the development of a more refined local theory requires successive cycles with new groups, specializations, and contexts. The analysis did not assess long-term retention, so the question remains open about whether the documented forms of activity are sustained in subsequent units. Finally, the study did not examine the conditions under which other instructors, with different training backgrounds, could implement the HLT with equivalent fidelity. Future research can address these limitations and the three redesign orientations identified in this cycle: an explicit treatment of set-theoretic conventions from the situational entry, an extension task on parameterized SLE as a zone of proximal development for the general level, and complementary practice on row-reduction procedures between sessions. Each orientation is formulated as a testable conjecture for the next cycle. The task contrasting the empty, singleton, and parameterized sets should reduce the writing $S = \{\emptyset\}$, present here in 9 of the 25 protocols. The extension task on parameterized SLE should make it possible to differentiate the parameter values that produce a unique

solution from those that produce infinitely many, a distinction that ten students did not make. And the complementary row-reduction practice should eliminate the arithmetic errors in the final step of the algorithm. If those indicators do not vary in the predicted direction, the corresponding conjecture is refuted.

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