MODESTUM
OPEN ACCESS

EURASIA Journal of Mathematics, Science and Technology Education, 2018, 14(9), em1585
ISSN:1305-8223 (online)
Research Paper https://doi.org/10.29333/ejmste/92553

"… and therefore in a Remote Sense Abduction Rests upon Diagrammatic Reasoning"

Michael H. G. Hoffmann [1*]

[1] Associate Professor for Philosophy, School of Public Policy, Georgia Institute of Technology, Atlanta, GA, USA

Received 21 March 2018 ▪ Revised 9 May 2018 ▪ Accepted 24 May 2018

ABSTRACT

Peirce developed two different concepts—"abduction" and "diagrammatic reasoning"—that are interesting for theories of creativity in mathematics, the sciences, and in learning. He defined "abduction" as the "inference" from surprising, or unexplained, observations to an explanatory hypothesis. However, he does not provide much to explain how the process of creating new hypotheses might be possible. In this contribution, I start from a remark by Peirce claiming that diagrammatic reasoning might somehow be the foundation of abduction. Using an example from astronomy, I argue that at least one form of abduction is indeed based on diagrammatic reasoning: theoretic model abduction.

Keywords: abduction, consistency, creativity, diagrammatic reasoning, representational system, scientific discovery

INTRODUCTION

After the failure of Logical Positivism to provide what Carnap called a "rational reconstruction of the concepts of all fields of knowledge"[1] became apparent in the previous century, the attention in the philosophy of science shifted more and more to the question of how to understand scientific progress. Since the procedural character of knowledge development and the problem of scientific creativity has been discussed by Charles S. Peirce more than a century ago, it does not come as a surprise that there is an increasing interest in the conceptual tools that Peirce developed to describe these processes, and the methods he suggested to actually perform the creation of new knowledge.

Within the philosophy of science—broadly conceived—the rising interest in Peirce can be divided into two camps: On the one hand, there are authors who focus on Peirce's pragmatism which—although primarily designed as a theory of meaning—is closely related to his "doubt-belief" conception of knowledge development: doubts that arise in certain situations are settled by newly developed beliefs. On the other hand are those who are interested in Peirce's "abductive inference" which he introduced as a third form of reasoning besides deduction and induction.

With regard to abduction, the main problem is that many things Peirce himself wrote about this form of reasoning are not very helpful when it comes to *explaining how* "the process of forming an explanatory hypothesis," as he defines abduction, might be possible (Peirce CP 5.171 [1903]). He mainly hints at an "instinctive" power of "guessing rightly"[2] and at "the uncontrolled part of the mind" to answer this question.[3] But since none of this is sufficiently elaborated, the possibility of abductive creativity remains at the end unexplained.

To gain more clarity with regard to the question of how in particular the *creative* dimension of abductive hypothesis formation is possible, this contribution focuses on a somewhat strange remark by Peirce that relates abduction to another concept that is crucial for his thinking about knowledge development and learning: diagrammatic reasoning. Concluding a discussion about diagrammatic reasoning, Peirce writes in 1906 that "in a

---

1 Carnap 1967, v. See Quine 1969, 1971 <1951> for a description of this failure.
2 See Rescher 1995; Fann 1970, pp. 35–38; Sami Paavola 2005; Sami Paavola & Hakkarainen 2005, and the criticism by Thagard 2010.
3 CP 5.194. See Semetsky 2005, and Burton 2000.

✉ m.hoffmann@gatech.edu **(*Correspondence)**

Contribution of this paper to the literature

- This is the first, comprehensive discussion of what seems to be the only passage in Peirce's writings in which he explicitly relates abduction and diagrammatic reasoning.
- An argument that diagrammatic reasoning is not only a foundation of "theoretic model abduction," but of all forms of abduction that aim at an "explanation" in the sense of the deductive-nomological model of explanation.
- An argument that the possibility of diagrammatic reasoning depends on (1) knowing and accepting the rules, conventions, and ontology of a chosen system of representation and (2) on the logical consistency of these systems.

remote way Abduction rests upon diagrammatic reasoning" (Peirce NEM IV 320). The goal of this paper is to investigate two questions: Can diagrammatic reasoning indeed be conceived as a foundation of abductive creativity? And: What could be the relationship between abduction and diagrammatic reasoning?

Peirce made this remark in MS 293, a manuscript whose pages he counted with the running title "PAP," referring to "Prolegomena for an Apology to Pragmaticism." In 1906, he published an article with this title in *The Monist* (C. S. Peirce 1906). MSS 292 and 293 are thought to be two drafts of this publication (Peirce SEM III, p. 75). MS 293 is now easily accessible in Volume IV of Carolyn Eisele's edition of *The New Elements of Mathematics by Charles S. Peirce* (Peirce NEM IV 313-330).

In this manuscript we find an extensive discussion of diagrammatic reasoning that is clearly separated from the rest of the text by two clearly drawn lines.[4] The first one, on page 6 (NEM IV 314), concludes a discussion of anthropomorphism in "logic as a science of signs." The section after this line starts with Peirce's well-known thesis: "All necessary reasoning is diagrammatic." After a lengthy justification of this thesis (which, as we will see, contains important considerations about his notion of diagram and diagrammatic reasoning), Peirce adds a related discussion on the question whether "non-necessary reasoning" is also diagrammatic (Peirce MS 293 CSP 17 = NEM IV 319). As non-necessary forms of reasoning he mentions "probable deduction," induction understood as "experimental reasoning," and "abduction" which he defines here as "processes of thought capable of producing no conclusion more definite than a conjecture" (Peirce NEM IV 319).

Without discussing the relationship between probable deduction and diagrammatic reasoning, Peirce turns immediately to a certain revision of his thinking regarding the "general principle of the validity of Induction" which he summarizes as follows:

> *The validity of Induction consists in the fact that it proceeds according to a method which though it may give provisional results that are incorrect will yet, if steadily pursued, eventually correct any such error. (Peirce NEM IV 319)*

With regard to this method, Peirce then argues "that Induction, separated from the deduction of its validity, makes no essential use of diagrams. But instead of experimenting on Diagrams it experiments upon the very Objects concerning which it reasons" (Peirce NEM IV 320).

This quote is immediately followed—just before the second line concludes this section on diagrammatic reasoning—by the passage that is of interest here:

> *The third mode of non-necessary reasoning … is Abduction. Abduction is no more nor less than guessing, a faculty attributed to Yankees.*[5] *Such validity as this has consists in the generalization that no new truth is ever otherwise reached while some new truths are thus reached. This is a result of Induction; and therefore in a remote way Abduction rests upon diagrammatic reasoning. (Peirce MS 293 CSP 21-22 = NEM IV 320)*

It is important to note that the "generalization" Peirce talks about towards the end of the quote can only refer to the following two claims: (1) that there is no other way to "reach" a new truth than by abduction, and (2) that—obviously as a matter of historical fact—"some new truths are thus reached." If we would assume that this generalization is absolutely true, then the talk about "some" new truths in (2) would not make sense because: if no other way of finding new truths is possible, as claimed in (1), then *all* truths are reached this way, and not only

4 Images of the manuscript pages are published in the Digital Peirce Archive, https://rs.cms.hu-berlin.de/peircearchive/.
5 The asterisk has been written by Peirce in the manuscript. After it, he drew a line under which he wrote: "[Footnote] *In point of fact, the three most remarkable, because most apparently unfounded, guesses I know of were made by Englishmen. They were Bacon's guess that heat was a mode of motion, Dalton's of chemical atoms, and Young's (or was it Wallaston's) that violet, *green* (and not yellow, as the painters said) and red were the fundamental colors." On the next sheet he continues as quoted above.

"some." This means that the generalization in question can only be an inductive generalization, as Peirce says explicitly in the quote, to which applies what he wrote above: There might be cases in the future that invalidate this generalization so that it needs to be "corrected."

The reason why these few words should be analyzed in such detail is to avoid the possible misunderstanding that the generalization mentioned refers to a specific method of abductive reasoning itself. It does not. It only refers to what I distinguished as (1) and (2) above. This, however, means that there is—at least in this passage—no justification (at least none that would be easily identifiable) of what is clearly presented as the conclusion of an argument at the end: "…and therefore in a remote way Abduction rests upon diagrammatic reasoning." This statement remains unjustified in spite of its apparent presentation as the conclusion of an argument. Besides having no justification, there is also nothing that could help us to understand what the claim actually means (in contrast to what S. Paavola 2011, p. 301 claims). What does it mean that abduction, "in a remote way," "rests upon diagrammatic reasoning"?[6]

However, Peirce's remark about at least a "remote" relationship between abduction and diagrammatic reasoning is interesting enough to justify some further considerations. Although Peirce himself does not seem to have connected these two forms of creative thinking outside of this passage in MS 293, it would enrich our understanding of abduction if it could be related to diagrammatic reasoning. At the same time, a clarification of this relationship can contribute to current discussions about the role of various external representations in theories of scientific discoveries. That "signs and other mediating artifacts" play a role in a methodologically oriented approach to abduction has already been emphasized by Sami Paavola (Paavola 2007; 2011; Paavola, Hakkarainen, & Sintonen 2006; see also Skagestad 1999, on "the externality of genuine reasoning processes" in Peirce). Moreover, it would be interesting to see whether a better understanding of the cognitive role of diagrammatic reasoning can contribute to discussions on "distributed cognition" (Hutchins 1995) and the "extended mind" (Clark 2007; Clark & Chalmers 1998; Hoffmann 2007), as well as to those on model-based reasoning (Magnani 2001, 2009, 2010; Magnani & Nersessian 2002; Magnani, Nersessian, & Thagard 1999; Nersessian 2008).

But what exactly is diagrammatic reasoning? Trying to answer this question is the goal of the next section. A crucial point of this answer will be the thesis that an essential precondition of diagrammatic reasoning has not yet been taken seriously enough. As I will show, a necessary condition for learning something by means of diagrammatic reasoning is knowing and accepting the rules, conventions, and ontology of a chosen system of representation; a system by means of which diagrams can be constructed. Such a system, which has to be well-defined, constrains reasoning in a way that our cognitive energy gets focused on points that are crucial for creativity—just like a fireman's jet of water will be the more focused the more it is constrained. Without any constraints there would be no direction for our reasoning.

After clarifying Peirce's concept of "diagram" and its dependence on a "consistent system of representation" in the next section, the third one will analyze the significance of these systems for diagrammatic reasoning. Based on this preparation, I will then argue in Section 4 that diagrammatic reasoning is indeed very closely related to a particular form of abduction, "theoretical model abduction," in which the explanatory hypothesis of an abductive inference is a theoretical model. This argument will be based on an example—Ptolemy's explanation of the retrograde motion of Mars—that can show how theoretical model abduction "rests upon diagrammatic reasoning."

DIAGRAMMATIC REASONING AND DIAGRAMS

As far as I can see, it was John Venn who coined the term "diagrammatic reasoning" in his article "On the Diagrammatic and Mechanical Representations of Propositions and Reasoning" (Venn 1880a). When Venn evaluated in another paper three different accounts of what we call today categorical logic, he used as a criterion their ability to "yield itself readily to any accurately correspondent diagrammatic system of illustration." A "transparent clearness of illustration" is in itself a "great merit," he writes, because this way we can "*intuite* a proposition" (Venn 1880b, p. 349). Venn achieved this goal by using overlapping circles, ellipses, and other figures to represent the relationships among terms, shaded areas to indicate empty sets, and asterisks to designate particulars. This enabled him to visualize elegantly propositions containing up to five terms.

The use of geometrical figures to represent syllogisms is documented already in Ancient comments on Aristotle's logic. A more systematic approach has been developed by Leibniz, although his approach of using circles became known only through Leonard Euler's independently developed work.[7] Venn developed his graphical illustrations of universal, particular, affirmative, and negative propositions based on a detailed criticism of Euler's

---

6 Paavola cites another passage from MS 296 in which Peirce writes that the Existential Graphs "are equally capable of representing the creations of explanatory conjectures" (S. Paavola 2011, 301). For him that indicates that Peirce "would have wanted to include abduction to diagrammatic reasoning." This passage, however, talks only about representing the *results* of abductive reasoning, not the reasoning itself.

7 See Euler 1768, and Bochenski 1970 <1956>, 24.34, 36.13-14.

approach. After Venn, it was Peirce who enlarged and revolutionized the study of diagrammatic representations of logical relations, first in his system of "Entiative Graphs," then with his "Existential Graphs."[8] Describing the purpose of this "diagrammatic syntax" (C. S. Peirce 1909, p.10), he writes in our MS 293 immediately after the line that concludes his discussion of diagrammatic reasoning:

> *Let us call the collective whole of all that could ever be present to the mind in any way or in any sense, the Phaneron. Then the substance of every Thought (and of much beside Thought proper) will be a Constituent of the Phaneron. The Phaneron being itself far too elusive for direct observation, there can be no better method of studying it than through the Diagram of it which the System of Existential Graphs puts at our disposition. (Peirce NEM IV 320)*

What this says is that the System of Existential Graphs provides diagrammatic means to visualize what "could ever be present to the mind" for "direct observation." In spite of this extraordinary broad claim, the system is designed as a *logical* notation; it "greatly facilitates the solution of problems of Logic," as Peirce writes in the *Monist* version of his "Prolegomena to an Apology for Pragmaticism" (C. S. Peirce 1906 CP 4.571 [1906]). We know already that for Peirce, "All necessary reasoning is diagrammatic," but such necessary reasoning is not only present in logic but also in mathematics. Peirce himself states that he developed the notion of diagrammatic reasoning to describe the specific nature of "The Reasoning of Mathematics." In his so-called "Carnegie Application," he writes about the relevance of his discovery, and defines "diagrammatic reasoning," as follows:

> *The first things I found out were that all mathematical reasoning is diagrammatic and that all necessary reasoning is mathematical reasoning, no matter how simple it may be. By diagrammatic reasoning, I mean reasoning which* constructs a diagram *according to a precept expressed in general terms,* performs experiments *upon this diagram, notes their results,* assures itself *that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and* expresses *this in general terms. This was a discovery of no little importance, showing, as it does, that all knowledge without exception comes from observation. (Peirce NEM IV 47-48 [1902]; my emphasis)*[9]

This way, "diagrammatic reasoning" is defined as a process in which constructing diagrams and "experimenting" with them plays a fundamental role. But what exactly is a "diagram"? It is important to note that Peirce's definition of "diagram" differs from our usual understanding of diagrams as "pictorial" or "spatial" representations as we encounter it in the common distinction between "diagrammatic" and "sentential representations."[10] Such a distinction does not exist for Peirce. According to Peirce, even *sentences* are diagrams. The essential feature of Peirce's "diagrams" is that they represent *relations*. "Many diagrams resemble their objects not at all in looks; it is only in respect to the relations of their parts that their likeness consists" (Peirce EP II 13 [1895]). "Diagrams are restricted to the representation of a certain class of relations; namely those that are intelligible."[11] This is the reason why sentences like "Ezekiel loveth Huldah" and algebraic equations like $x=y^2$ are considered "diagrams" as well.[12]

But that is not all. If we look for a comprehensive definition of "diagram" in Peirce's writings, the most important candidate seems to be a definition that he formulated to introduce the notation of his "existential graphs" in a manuscript written in 1903 (Peirce MS 492):

> *A diagram is a representamen which is predominantly an icon of relations and is aided to be so by conventions. Indices are also more or less used. It should be carried out upon a perfectly consistent system of representation, founded upon a simple and easily intelligible basic idea. (Peirce CP 4.418)*

8 See C. S. Peirce 1909; Roberts 1973; Shin 2002.

9 Note that this definition defeats the thesis developed by Campos 2007 that "diagramming," for Peirce, is primarily a mental process (as also Kent 1997, p. 445, claims). It is hard to perform experiments on something that exists only in our imagination unless it is simple enough. In MS 293, Peirce is absolutely clear about this point: "A Diagram, in my sense, is in the first place a Token, or singular Object used as a Sign; for it is essential that it should be capable of being perceived and observed" (NEM IV 315 Fn. [1906]; see also CP 2.216 [1901]). Of course, in less complex cases it is possible to perform diagrammatic reasoning mentally. -- For similar definitions of diagrammatic reasoning see also CP 1.54 [1896] and 2.778 [1901].

10 This distinction defines the term "diagrammatic reasoning" in cognitive science; see, for instance, Glasgow, Narayanan, & Chandrasekaran 1995; Cheng & Simon 1995; Nersessian 2008.

11 NEM IV 316 Fn. [1906]. As I will argue below, the intelligibility of diagrams that is mentioned here rests on the rationality of the representation system by means of which diagrams must be constructed.

12 EP II 13, 17 [1895]. That representing relations is the primary function of diagrams is documented also in CP 2.778 [1901-02]; EP II 274 [1903]; and CP 4.530 [1906]. This central feature of diagrammatic reasoning remains unfortunately underdeveloped in Stjernfelt 2007 otherwise excellent *Diagrammatology* (2007); see Hoffmann 2009.

This definition is important because it not only refers to representing relations as the function of diagrams, but it also emphasizes that any construction of a diagram has to be performed by means of "a perfectly consistent system of representation."[13] In other contexts, Peirce also talks about a "diagrammatical syntax" (NEM III 162 [1911]; C. S. Peirce 1909, p.10) or a "system of expression" (NEM III 1120 [1903]). But he does not provide any further details on what these formulations mean. If we look at the Existential Graphs, however, it is clear that such a "system of representation" can be defined by three elements: first, an *ontology* that defines the entities (including relations) which can be represented by means of the system; second, *conventions* that prescribe how to construct a diagram and how to read it; and third, *rules* that determine how to transform diagrams.

There is no question that one needs "a perfectly consistent system of representation" for representing logical implications as in Peirce's Existential Graphs. The challenge for designing such a representation system is to define the ontology, conventions, and rules in a way that the system is sound and complete. Perfectly consistent systems of representation are not only a precondition for deductive logic, but also—in the form of systems of axioms—a foundation of mathematics.

In a manuscript that is part of his Lowell Lectures, Peirce reflects on the need to design diagrammatic systems to analyze "the reasoning of mathematical demonstrations." He argues that the delay in providing such a system

> *has been partly due to many writers entirely missing the point and directing their energies to ascertain the sequence of mental phenomena in reasoning instead of the logical sequence of argument, which need not be closely related to the psychological sequence. (Peirce NEM III 1119 [1903])*

To study the logical basis of mathematical reasoning, Peirce concludes, it

> *is necessary to devise a system of expression for the purpose which shall be competent to express any proposition whatever without being embarrassed by its complexity, which shall be absolutely free from ambiguity, perfectly regular in its syntax, free form all disturbing suggestions, and come as near to a clear skeleton diagram of that element of the fact which is pertinent to the reasoning as possible. ... If you learn this system and will then train yourselves to the use of it, I can promise that it will help you much to unravel tangles of thought.*
>
> *Only, let not its aim be mistaken. I wish to declare distinctly and once for all that it is not intended to furnish a speedy or ready way by which to pass from premisses to conclusion. It aims in the diametrically opposite direction, namely, to break up reasoning into the greatest possible number of distinct steps, so that the constitution of reasonings may be studied. If we wished to obtain speedy passage from premisses to conclusion, we should, on the contrary, seek to make the steps as few and as large we could. In short this system is meant not as an aid in reasoning but as an aid in the minute analysis of reasonings. Practice with it, however, will make thought clearer, and will so conduce indirectly to skill in reaching conclusions.*
>
> *This system is a system of diagrams. A diagram has the advantage of appealing to the eye, and to that adds others due to the prominence it gives to conventional signs. ... The special system of diagrams that I am about to describe is called the* Method of Existential Graphs. *(Peirce NEM III 1120 [1903]; his emphasis)*

However, it is not only in logic and mathematics that we are confronted with representation systems. These are only the two most prominent domains in which clearly defined and consistent systems of representation with their respective ontology, conventions, and rules are absolutely necessary. Consider our everyday language: Although hardly "perfectly consistent," there is no question that communication would be extremely difficult if we had no grammar. An expression like "John friend give past I apple" can mean a lot of different things while "John's friend gave me an apple" pretty much specifies what is ambiguous in the first expression. Although we hardly think about it, our ability to communicate by spoken or written language depends heavily on our—mostly implicit—knowledge of the ontology, conventions, and rules of our language. One of the main functions of grammar is to specify relations between words and thus reduce ambiguity.

For graphical language systems like Peirce's Existential Graphs it is crucial that what is implicitly known in everyday language needs to be specified and defined explicitly. The problem that Peirce was facing—as anybody who develops graphical systems to visualize reasoning—is the following: Since there is no established "grammar"

---

13 Peirce mentions this requirement only a few times explicitly (see also CP 4.530 [1906]; 5.166 [1903]; 4.430; NEM IV 318 [1906]). But it is obvious that any logical notation such as the Existential Graphs need to be realized as a perfectly consistent system of representation.

for constructing and reading visualizations, every visualization system has to establish its own rules and conventions and can only be used efficiently when users are familiar with them (see Hoffmann 2011a).

THE ROLE OF "CONSISTENT" REPRESENTATION SYSTEMS IN DIAGRAMMATIC REASONING

After this clarification of Peirce's concept of "diagram" and its dependence on what he describes as a "system of representation" or "diagrammatical syntax," we can return to our initial definition of diagrammatic reasoning that I quoted from Peirce's Carnegie-Application at the beginning of the previous section. The process by which diagrammatic reasoning is *defined* according to this quote can now be summarized as a sequence of the following five steps:

1. Construct a diagram by means of a consistent system of representation, either mentally or by external means.
2. Perform experiments[14] upon this diagram according to the rules of the chosen system of representation.
3. Note the results of those experiments.
4. Assure yourself of the generality of these results.
5. Express these results "in general terms."

When Peirce writes at the end of the passage quoted above that the discovery of diagrammatic reasoning showed him "that all knowledge without exception comes from observation" (Peirce NEM IV 48), it is clear that the primary *function* of diagrammatic reasoning is the creation of knowledge. "Diagrammatic reasoning is the only really fertile reasoning."[15] That means, understanding diagrammatic reasoning can be key to understanding the possibilities of creativity, learning, and cognitive change.

But how exactly can we learn something through diagrammatic reasoning? My thesis is that the creation of new knowledge depends on the *normative* role of the chosen system of representation in the five-step process of diagrammatic reasoning. Any system of representation is normative in so far as its rules and conventions are norms that determine how to construct, read, and transform or manipulate diagrams. The rules and conventions determine what is permissible in diagrammatic reasoning, and what is not (NEM IV 318 [1906]).

We will pretty much *feel* the normative force of representation systems when we buy a book for $15 and a magazine for $5 and the clerk at the register demands $50. In a similar vein, Peirce tells the story of "one extremely bright man in MS 293 who could not, for the life of him, perceive any fault in this reasoning:

It either rains or it doesn't rain;

It rains;

Therefore, it doesn't rain (Peirce NEM IV 315 [1906])

For diagrammatic reasoning, the normativity of representation systems plays a central role in all five of the steps listed above.

1. Before we *construct* a diagram, we have to choose an appropriate system of representation. For example, if the goal is to prove that the side and the diagonal of a square are incommensurable, we can choose either a geometrical proof or an algebraic one. Such a choice can have important implications. Whereas, in this case, the geometrical proof would show—indeed in the sense of making visible—that the process of determining a common measure of side and diagonal can never be completed (i.e., there cannot be a common measure, fulfilling thus the condition of incommensurability), the famous algebraic proof that is documented in Euclid's *Elements* demonstrates the same by means of a *reductio ad absurdum* argument (see Hoffmann & Plöger 2000). There are significant differences between both approaches. For example, modern mathematics would criticize the geometrical proof as depending on intuition; we have to "see" that the procedure of determining a common measure will never end although we can perform only a limited set of steps. The algebraic proof, on the other hand, is hard to understand if we are not used to prove something indirectly; being rather "unintuitive," it suffers just from what its rival provides abundantly.

While both proofs depend on the normativity of the chosen representation system—since without it we could never be sure that we have proved anything at all—the acceptability of these norms themselves depends on the context. What might be acceptable in school settings is not acceptable in science; what was acceptable in the past, might no longer be acceptable today.

---

14 It would be anachronistic to assume that Peirce already used the notion of "experiment" in the sense of testing a hypothesis in a controlled setting with which we are familiar. Clarifying what "experiment" means for Peirce in this context is one of the goals of this section.

15 Peirce CP 4.571 [1906]; see also 4.530f. [1906]; 3.559f. [1898].

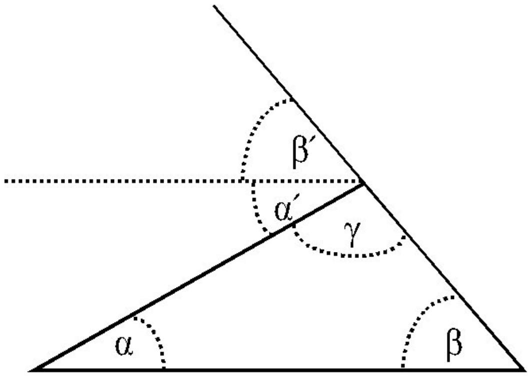

Figure 1. Proving that the sum of α, β, and γ is 180° by means of an auxiliary line (the dotted line)

After having chosen a certain system of representation, it is clear that the ontology provided by this system and the rules and conventions defined in it determine and constrain the construction of diagrams. The possibilities of representing something are predetermined and constrained by the representational means available, be it concepts, graphical means, variables, theories, or whatever.

2. The second step of diagrammatic reasoning is "experimenting" with those diagrams. This is now the essential step for preparing the possibility of discovering something new, and to motivate creativity. First of all, it should be clear that this concept of an experiment should not be confused with the testing of a hypothesis in a set up that attempts to isolate some real-world phenomenon in the lab. Since Peirce's experiments refer to diagrams here, the activity in question can only be a transformation or manipulation of a given diagram. Secondly, any such operation on a diagram is determined by the rules defined in the chosen representation system. Again, there are permissible steps and those that are simply forbidden, and since it must be clear in every step what is what, we need representation systems whose rules are defined as clearly as possible. This means, thirdly, that the *outcome* of any transformation or manipulation is determined by these rules. While this seems to indicate that there can hardly be something new coming out of experimenting with diagrams, the crucial point is—as Peirce writes regarding the proofs a geometer performs—that by observing the results of experimenting with diagrams, the geometer "is able to synthesize and show relations between elements which before seemed to have no necessary connection" (CP 1.383 [c.1888]).

At another opportunity, Peirce observes that in mathematics we often "meet with a surprising result" because of "loose reasoning" (CP 5.166). Although "the certainty of pure mathematics and of all necessary reasoning is due to the circumstance that it relates to objects which are the creations of our own minds"—what is particularly true of the representation systems that we create for the very purpose of performing necessary reasoning—it is clear that we can never have a complete overview of all the implications of what we know. Only operating on representations of our knowledge reveals what is implicitly given in what we know. Thus, diagrammatic reasoning can reveal things that appear surprising, but that are indeed implications of what we already know.

The crucial point regarding creativity is the following: Whereas the outcome of diagrammatic experiments is predetermined, within the possibilities provided by a certain system of representation there is no limitation regarding the question *which* experiments we perform. We are free to do whatever seems appropriate in a certain context. It is a genuine creative act to come up with an idea what to do. Among the set of possible transformations there might be *one* approach, one transformation of a diagram, that leads the experimenter—based on the normative force of the chosen representation system—to a representation she never saw before; a transformation of the original diagram that "compels" her to perceive a new relation, a new necessary connection, or an organizing structure of a set of data that she did not see before.

An example that already Kant used to show that mathematical knowledge is knowledge gained through "the construction of concepts" in space[16] is the proof that the sum of the inner angles in a triangle is 180° (Kant CPR B 741-745). Kant hints at a geometrical proof that can be diagrammed as in **Figure 1**. The crucial creative step in this proof is to come up with the idea of using a certain auxiliary line. If we draw a parallel to the triangle's base through its apex (the dotted line), the proof can be performed in a sequence of steps whose outcome is logically necessary based on the rules of Euclidean geometry. These rules determine that angle α equals angle α´, and angle β equals angle β´, hence α+β+γ= α´+β´+γ´=180°.

16 Peirce makes it clear that his ideas on diagrammatic reasoning are part of a tradition in the philosophy of mathematics that goes back to this Kantian definition of mathematical knowledge (CP 3.556 [1898]). This tradition can be traced back even further to Proclus's commentary on Euclid where he reports on a debate between those who define the task of mathematics as discovering theorems and those who define it as creating knowledge through constructions (see Hoffmann 2005, ch. 4).

It is just this combination of creativity and logical necessity that is indispensable to perform the proof. Without creativity we would never come up with the idea of using exactly this auxiliary line, and without the normative force of the chosen representation system we could never be sure whether the reasoning described above proves anything. To accept these operations as a proof, we have to accept the rules of Euclidean geometry as determining the outcome of experiments with logical necessity.

Whatever the accepted rules are, the important point is that any "experimenting" with a diagram will always—based on the normative force of the chosen system of representation—lead to certain kinds of inevitable experiences and observations. The rationality immanent in systems of diagrammatization firstly defines the *limits* of possible transformations, and secondly it constrains a set of necessary *implications* of operations on diagrams.

3 to 5. The three final steps of diagrammatic reasoning after experimenting with a diagram are *noting* the results of these experiments, *assuring* oneself "that similar experiments performed upon any diagram constructed according to the same precept would have the same results," and *expressing* this in general terms (Peirce NEM IV 47-48 [1902]). The final outcome, thus, is a new rule which is, on the one hand, a logical implication of the rules applied in its discovery and, on the other, an addition to this set of rules. With regard to the proof mentioned above, for example, we can say that the rules we are using for the diagram's construction and the experimentation with it include the definition of a triangle and a few propositions regarding angles that can be derived from Euclid's fifth axiom about parallel lines. The "new" rule that we discover by means of these prior rules is the theorem that the sum of the inner angles in a triangle equals 180°.

Since it is possible that such a "surprising observation" which results from certain experiments with diagrams occur only based on arbitrary circumstances—like the shape of the concrete triangle in **Figure 1**—it is necessary that we do not only "note" the results of experiments, but that we prove the generality of our new discovery and represent it "in general terms," that is in form of a new rule. This way, the normativity of the chosen representation system allows us—through experimenting with a diagram that has been constructed by the means of this system—to perform experiments that again, if their outcome can be proven to be necessary, leads to an enlargement or further specification of the chosen normative system of representation.

To summarize these considerations about the crucial role of the normativity of representation systems for the five steps of diagrammatic reasoning, we can distinguish the following points:

- Since the outcome of any experiment with a diagram is determined by the rules of the chosen representation system, diagrammatic reasoning confronts us with necessary implications of our original assumptions.
- This way, the outcome of diagrammatic reasoning seems to be always a chain of necessary reasoning. After coming up with the idea of using an auxiliary line to prove the theorem about the triangle's inner angles, the proof itself will be a piece of necessary or deductive reasoning. This means: The goal of diagrammatic reasoning is being able to formulate a deductively valid argument.[17]

Although the outcome of diagrammatic reasoning will be a piece of necessary reasoning, creativity is needed at two points: first, one needs to come up with an idea of how to diagram a problem so that a deduction is possible and, second, one must find those experiments that lead to a necessary conclusion—to find, for example, an adequate auxiliary line or to shift the perspective on a diagram in a way that the possibility of a proof becomes evident.[18]

ABDUCTION

Creativity is required both for diagrammatic reasoning and for abduction. Even though Peirce presents abduction as a particular form of inference, the explanatory hypothesis that is mentioned in one of the premises and in the conclusion has to be created, first of all. While the inference has a logical form, there is no logical process of hypothesis creation (Hoffmann 1999).

In the quote that inspired the present considerations, however, the interesting point is not that both forms of reasoning require creativity, but that Peirce claims that abduction—at least in a "remote" sense—"rests upon diagrammatic reasoning." Research on abduction suggests that different forms of this type of inference can be distinguished (Boden 2004 <1990>; Hoffmann 2011b; Magnani 2001; Schurz 2008). This means, however, that there might be differences regarding the significance of diagrammatic reasoning, depending on the particular form of abduction.

In order to see whether the role of diagrammatic reasoning varies for different forms of abduction, let me start with the classification of 15 different forms of abduction that I proposed a few years ago (Hoffmann 2011b, in

17 This corresponds to Peirce's repeated claim that "All necessary reasoning without exception is diagrammatic" (CP 5.162 [1903]) and that "All mathematical reasoning is diagrammatic" (NEM IV 47 [1902]).

18 Such a shift of perspective is the necessary "theoric transformation" that allows the proof of Desargues's theorem that I described in Hoffmann 2011b.

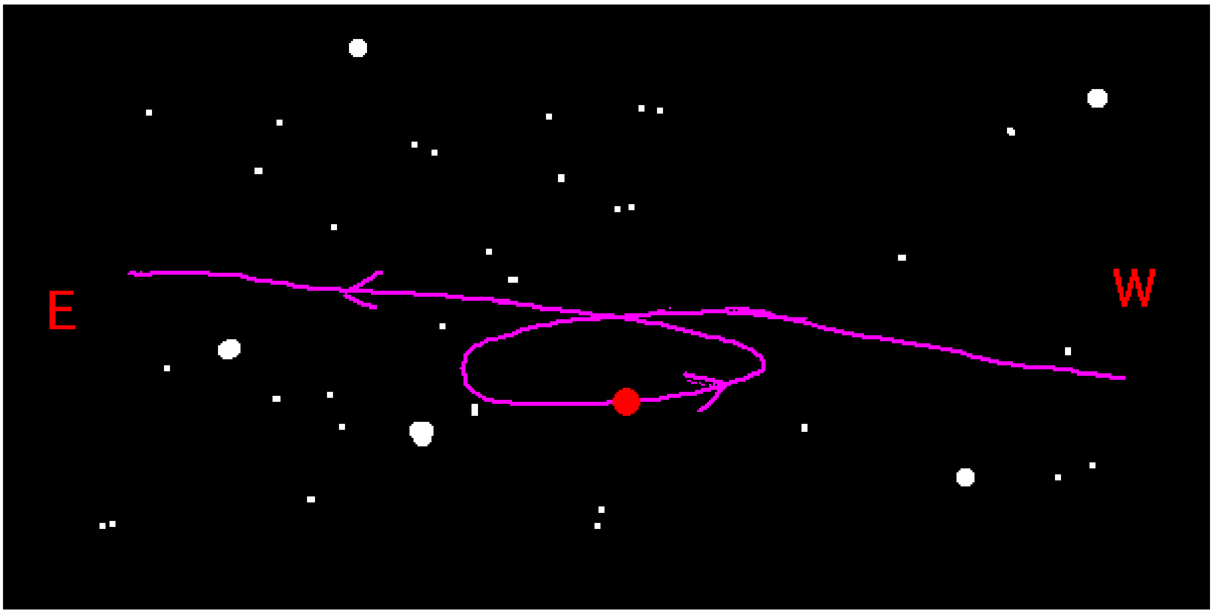

Figure 2. For months, Mars moves eastward through the signs of the zodiac, "but then it slows, comes to a stop against the background stars, brightens, and, now quite conspicuous, moves westward for several weeks before stopping, fading, and finally resuming its direct motion" (Gingerich, 1993, pp. 7-8). Ptolemy called this loop a "retrograde motion." The picture has been produced by Davison E. Soper, Institute of Theoretical Science, University of Oregon: http://pages.uoregon.edu/soper/Orbits/eudoxus.html (accessed April 5, 2018)

particular p. 585). This typology of 15 forms results from combining a distinction of five different things that we might infer abductively with another one of three modes of inferring an explanatory hypothesis: Such a hypothesis can either (1) be selected from a given set hypotheses that is already present in our mind (as in reading when we infer the meaning of a particular word based on a given mental "dictionary"), or it can be newly created in two different forms, namely either (2) by individuals who recreate in learning what is already part of their cultural knowledge, or (3) in creating something that is historically new.

For the question whether the role of diagrammatic reasoning differs, only the distinction of what can be created (or selected) abductively seems to be relevant. The following are the five forms that I distinguished based on this criterion:

1. **Fact abduction**: a singular fact is inferred as an explanatory hypothesis; for example when we explain a person's disease by a particular cause.
2. **Type abduction** (or the abduction of theoretical concepts), as when we explain something by a general concept such as "inertia" or "energy."
3. **Law abduction**, as when we explain the behavior of a gas by Boole's law (Schurz 2008, pp. 211-212).
4. **Theoretical-model abduction**: when we infer a theoretical model as an explanatory hypothesis, that is, a certain combination of facts, types, and/or laws. In contrast to the other forms listed above, in pure theoretical model abduction everything that is part of the model—facts, concepts, and laws—is given; new is only a particular combination in a model (Schurz 2008, pp. 213-216).
5. **Meta-diagrammatic abduction**: when an explanation becomes possible by "inferring" a certain system of representation. Since a theoretical model and its elements need to be represented by the means available in a certain system of representation, and since the creation of new knowledge is often dependent on the creation of new representation systems—as can be seen, for instance, in the shift from Euclidean to non-Euclidean geometries—the creation of new representation systems (or their selection in a given situation) can be counted as a form of abduction.

Given the Peircean notion of "diagram" discussed in Section 2 above, diagrammatic reasoning seems to be closest to theoretic model abduction. To get a clearer sense of what it could mean to claim that theoretical model abduction "rests upon diagrammatic reasoning," it should be helpful to discuss an example.

Already the Babylonians noticed the odd motion of Mars that looks like as depicted in **Figure 2**. They knew, as Owen Gingerich 1993) writes in *The Eye of Heaven*, "that in 79 years Mars made almost exactly 42 complete revolutions through the zodiac, and that it moved 40% faster when it was in Capricorn than when it was opposite in the sky in Cancer" (p. 8).

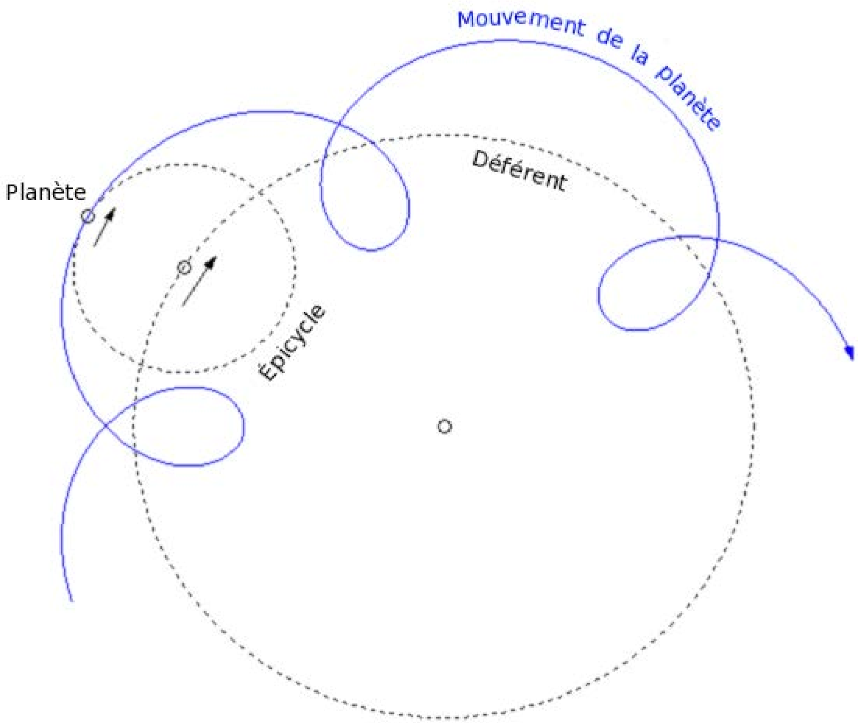

Figure 3. A planet's orbit, including retrograde motions, can be explained as resulting from the combined movement of an epicycle and a deferent. The Earth would be in the center of the deferent. From http://commons.wikimedia.org/wiki/File:Epicycle_et_deferent.png (accessed April 5, 2018).

Reacting to Plato's famous challenge to find "uniform and ordered movements by the assumption of which the phenomena in relation to the movements of the planets can be saved," Eudoxus of Cnidus was the first who created a "hypothesis" to explain the phenomenon of retrograde motion at least in principle.[19] Eudoxus tried "to save the phenomena" (*sozein ta phenomena*), as Plato's call is known today, by creating the theoretical model of the so-called "homocentric spheres." This is a model of nested spheres, all circling around the earth as the center, but each sphere rotating around an axis whose poles are in various angles connected to a slightly bigger sphere. The largest sphere was the sphere of the fix stars which was thought to be moved by the Earth in the center. To describe the motion of the Sun and the Moon, Eudoxus introduced three additional nested spheres for each of the two, and four for each of the planets (Linton 2004, p. 26; Neugebauer 1975, pp. 677-685, and Fig. 27 on p. 1358).

Although it is possible to explain Mars's retrograde motion by such a system of connected, nested, and concentric spheres (Neugebauer 1975, p. 684), Eudoxus's model fails to account for the varying distance between Mars and Earth. Since the planet's brightness changes significantly through its retrograde motion, Mars obviously changes its distance to the Earth. However, in a concentric model all the distances from the center to the planets and stars remain of course unchanged.

The first really convincing explanation of Mars's retrograde motion has been provided about five centuries later by Ptolemy. Although still based on the wrong assumption that all the stars and planets revolve around the Earth, Ptolemy's explanatory model achieves the same degree of accuracy regarding the celestial phenomena as Copernicus's heliocentric model 1400 years later (Gingerich 1993, p. 6; only Kepler was able to take a huge step forward). Based on the limited space available here, I will focus on Ptolemy whose reasoning is sufficient to illustrate a possible relationship between diagrammatic reasoning and theoretic model abduction.

Facing the phenomenon of Mars's retrograde motion that is not really satisfactorily explained in Eudoxus's model of homocentric spheres, Ptolemy's crucial abductive step was "forming the explanatory hypothesis" that the orbit of Mars is the result of two cyclic movements that are combined as depicted in **Figure 3**. In what has become famous as the "epicycle theory," the planet would move on an "epicycle" whose center is located on the "deferent" that again revolves around the center of the universe.

19 Simplicius on *De caelo*. Quoted from Linton 2004, p. 26.

If we try to imagine the situation Ptolemy was facing—the unexplained retrograde motion of Mars, its changing distance from the Earth, and its varying speed in its course through the zodiac—it seems to be impossible to create the sophisticated theoretical model of the epicycles without what Peirce defined as diagrammatic reasoning. We need to imagine the whole system—the deferent, the epicycle, and the course that Mars would take if the movements of both are combined—to explain the phenomenon depicted in **Figure 2**. And we need to "experiment" with such a mentally or externally produced representation as we find it in **Figure 3** to "see" whether the outcome of diagrammatic manipulations fits to the observation of the retrograde motion. Moreover, we need to rely on the rules of Euclidean geometry as the chosen system of representation, because only these rules can guarantee that the resulting movement will indeed look like as depicted in **Figure 3**. The normativity of the representation system determines the outcome of the movement, and the outcome of any transformation we perform with the model. We need reasoning, as Peirce defines diagrammatic reasoning, that "*constructs a diagram* according to a precept expressed in general terms, *performs experiments* upon this diagram, *notes their results, assures itself* that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and *expresses* this in general terms" (Peirce NEM IV 47-48 [1902]; my italics).

The normativity of the representation system that I discussed in Section 3 as a condition of diagrammatic reasoning is crucial for Ptolemy's discovery. Because it is just this normativity that forced him to acknowledge that the theoretical model represented in **Figure 3** is still not sufficient to explain the observed phenomenon.

Whereas the model—after the size of the epicycle in relation to that of the deferent has been defined based on a theorem provided by Apollonius—determines with necessity a retrograde motion, the model fails to explain two details of the observations: First, as already mentioned, "Mars appeared to move (after averaging out the effect of the retrogression) 40 percent faster on one side of the orbit compared with the other. Second, the retrograde loops themselves varied in size from one retrogression to another" (Gingerich 1993, p. 9). It is only because Ptolemy took the rules of geometry seriously that he saw the necessity to modify the epicycle theory in a fashion that a more satisfactory explanation becomes possible.

While the process of abductively selecting the theoretical model represented in **Figure 3** seems to be intimately connected to diagrammatic reasoning, that is, to constructing and experimenting with diagrams according to the rules of the chosen system of representation, the same diagrammatic reasoning challenges Ptolemy now to perform another abductive inference. Based on Hipparchos idea to place the orbit of the sun eccentrically with respect to the Earth to explain the Sun's non-uniform motion over the year, Ptolemy created a new epicycle model in which the center of the deferent was "placed eccentric by 20 percent, so that the planet would be 20 percent closer (and faster) than the average in one direction and 20 percent farther (and slower) than the average in the other direction" (Gingerich 1993, p. 10). This way, the 40 percent faster speed of Mars on one extreme of the orbit could be perfectly explained.

However, this new approach—again determined by the normativity of geometry as the chosen system of representation—did not produce the sizes of the retrograde loops that were observed. At this point, Ptolemy finally performed another creative type abduction that led to an explanation that "turned out to be both elegant and unexpectedly accurate" (Gingerich 1993, p. 10). He introduced the theoretical concept of the "equant" that allowed him to separate the geometrical center of the deferent from the equant as the center of uniform angular motion. This way it was possible to explain Mars's *orbit* perfectly accurately by reducing the eccentricity of the deferent's center to 10 percent instead of 20, while at the same time taking care of the planet's varying *speed* which could be claimed, now, to be uniform from the equant's point of view, even though not from the center's perspective.

Again, although the creation of the equant and both its and the eccentricity's exact location in the model are clearly abductive steps, these abductions are inseparably connected with diagrammatic reasoning. At each point of the reasoning process, the creativity of coming up with new ideas is constrained—but also "scaffolded"—by the possibilities of diagrammatic representations. The possibilities and impossibilities of certain diagrams guide the search for explanatory hypotheses.

It should be mentioned, at this point, that the arguments presented here are all visual arguments. Diagrams like the one depicted in **Figure 3** show the *possibility* of certain retrograde motions, but they do not determine a specific orbit as it would be possible with a deductive argument that derives the orbit from certain initial conditions. For our purposes, those visual arguments are sufficient. However, the real work that astronomers like Ptolemy and Kepler performed was aimed at mathematical explanations that allowed exact calculations and predictions of celestial events. With regard to this goal not only Kepler was highly successful, but also Ptolemy (see, for example, Gingerich 1993; Linton 2004; Neugebauer 1975).

As this example shows, there is not only a "remote" relation between abduction and diagrammatic reasoning, as Peirce remarks, but a very substantial one. The manipulation of diagrams according to Peirce's five-step process of diagrammatic reasoning seems to be precondition for developing new ideas—new theoretical models, new concepts, and new laws—that can explain certain phenomena.

CONCLUSION

The example of Ptolemy's repeated attempts to explain the retrograde motion of Mars by creating more and more sophisticated models of epicycles as explanatory hypotheses can teach us something more general about the relationship between abduction and diagrammatic reasoning. As we saw with this example, the insufficiency of a particular theoretical model became visible to Ptolemy in the discrepancy between what a specific model predicted and what was known based on observations. He experienced explanatory success only after he perceived a perfect "fit" between predictions and observations; "perfect" in the sense that there were no observations that remained unexplained.

While this talk about "explanatory success" might be convincing with regard to this concrete example, the question is whether it can be generalized. It seems to me that it can, at least in the context of one particular understanding of "explanation" which is known as the deductive-nomological model of explanation. To reformulate its core idea in Peirce's language of diagrammatic reasoning, we can say that such an explanation requires the following: The relation between a possible explanans and the explanandum (i.e., the relation between what does the explaining and what gets explained) needs to be represented by means of a consistent system of representation so that—based on certain laws (*nomoi* in Greek)—the explanandum is a deductively necessary implication of certain starting conditions.

Since this applies to any deductive-nomological explanation, it provides a success criterion for all forms of abduction—or at least for those for which "explanatory hypothesis" is conceptualized in the deductive-nomological sense. An abductively inferred hypothesis—be it an inferred fact, concept, law, or model—is successful in this sense of explanation if it can "explain" phenomena in a very specific sense, namely, as necessitated by the normativity of the used system of representation and certain factual assumptions—for example the assumption that the Earth is at the center of the universe. If a phenomenon can be reconstructed as a necessary outcome of certain rule-based operations on a certain diagram, then the search for a better explanation can stop.[20] It seems that Peirce would have shared what has later been called the deductive-nomological sense of explanation because in 1901 he wrote that an abductively inferred "explanation must be such a proposition as would lead to the prediction of the observed facts, either as necessary consequences or at least as very probable under the circumstances" (CP 7.202).

But if such necessity is the success criterion for abductive creativity in general, then it is impossible to create an explanatory hypothesis without the manipulation of certain diagrams that are constructed by means of a certain system of representation. If success consists in the ability to demonstrate a necessary relation between explanans and explanandum, there is no way to be successful without operating from the very beginning within one consistent system of representation. We can conclude, thus, that all those forms of abduction in which "explanation" is understood according to the deductive-nomological model do not only "in a remote sense" rest "upon diagrammatic reasoning," but in a very direct sense.

REFERENCES

Bochenski, J. M. (1970 <1956>). *Formale Logik* (3. ed.). Freiburg i. Br.: Alber.

Boden, M. A. (2004 <1990>). *The Creative Mind. Myths and Mechanisms* (2nd ed.). London; New York: Routledge.

Burton, R. G. (2000). The Problem of Control in Abduction. *Transactions of the Charles S. Peirce Society, 36*(1), 149-156.

Campos, D. G. (2007). Peirce on the Role of Poietic Creation in Mathematical Reasoning. *Transactions of the Charles S. Peirce Society, 43*(3), 470-489.

Carnap, R. (1967). *The logical structure of the world [and] pseudoproblems in philosophy; translated (Der logische Aufbau der Welt)* (R. A. George, Trans. 2nd ed.). London: Routledge & K. Paul.

Cheng, P. C.-H., & Simon, H. A. (1995). Scientific Discovery and Creative Reasoning. In S. M. Smith, T. B. Ward & R. A. Finke (Eds.), *The Creative Cognition Approach* (pp. 205-228). Cambridge, Mass.: MIT Press.

Clark, A. (2007). Curing cognitive hiccups: A defense of the extended mind. *Journal of Philosophy, 104*(4), 163-192. https://doi.org/10.5840/jphil2007104426

Clark, A., & Chalmers, D. (1998). The extended mind. *Analysis, 58*(1), 7-19. https://doi.org/10.1093/analys/58.1.7

Euler, L. (1768). *Lettres à une Princesse d'Allemagne*. St. Petersburg: l'Academie Imperiale des Sciences.

Fann, K. T. (1970). *Peirce's Theory of Abduction*. The Hague: Nijhoff. https://doi.org/10.1007/978-94-010-3163-9

Gingerich, O. (1993). *The eye of heaven: Ptolemy, Copernicus, Kepler*. New York, NY: American Institute of Physics.

Glasgow, J., Narayanan, N. H., & Chandrasekaran, B. (Eds.). (1995). *Diagrammatic Reasoning: Cognitive and Computational Perspectives*. Menlo Park, CA: AAAI Press / The MIT Press.

20 In Hoffmann 2011b, I discussed this as the "stopping rule" that defines "abductive insight."

Hoffmann, M. H. G. (1999). Problems with Peirce's Concept of Abduction. *Foundations of Science, 4*(3), 271–305. https://doi.org/10.1023/A:1009675824079

Hoffmann, M. H. G. (2005). *Erkenntnisentwicklung. Ein semiotisch-pragmatischer Ansatz*. Frankfurt am Main: Klostermann.

Hoffmann, M. H. G. (2007). Learning from people, things, and signs. *Studies in Philosophy and Education, 26*(3), 185-204. https://doi.org/10.1007/s11217-007-9027-5

Hoffmann, M. H. G. (2009). Über die Bedingungen der Möglichkeit durch diagrammatisches Denken etwas zu lernen: Diagrammgebrauch in Logik und Arithmetik. *Zeitschrift für Semiotik, 31*(3-4), 241-274.

Hoffmann, M. H. G. (2011a). Cognitive conditions of diagrammatic reasoning. *Semiotica, 186*(1/4), 189-212. https://doi.org/10.1515/semi.2011.052

Hoffmann, M. H. G. (2011b). "Theoric Transformations" and a New Classification of Abductive Inferences. *Transactions of the Charles S Peirce Society, 46*(4), 570-590.

Hoffmann, M. H. G., & Plöger, M. (2000). Mathematik als Prozess der Verallgemeinerung von Zeichen: Eine exemplarische Unterrichtseinheit zur Entdeckung der Inkommensurabilität. *Zeitschrift für Semiotik, 22*(1), 81–114.

Hutchins, E. (1995). *Cognition in the wild*. Cambridge, MA: MIT Press.

Kant, I. (CPR). *Critique of pure reason* (P. Guyer & A. W. Wood, Trans.). Cambridge 1998: Cambridge Univ. Pr. (quoted according to the first edition—A: 1781—or the second—B: 1787).

Kent, B. (1997). The Interconnectedness of Peirce's Diagrammatic Thought. In N. Houser, D. D. Roberts & J. van Evra (Eds.), *Studies in the Logic of Charles Sanders Peirce* (pp. 445–459). Bloomington and Indianapolis: Indiana University Press.

Linton, C. M. (2004). *From Eudoxus to Einstein: A history of mathematical astronomy*. Cambridge, UK; New York: Cambridge University Press. https://doi.org/10.1017/CBO9780511541452

Magnani, L. (2001). *Abduction, Reason, and Science. Processes of Discovery and Explanation*. New York: Kluwer Academic / Plenum Publishers. https://doi.org/10.1007/978-1-4419-8562-0

Magnani, L. (2009). *Abductive Cognition. The Epistemological and Eco-Cognitive Dimensions of Hypothetical Reasoning*. Berlin: Springer. https://doi.org/10.1007/978-3-642-03631-6

Magnani, L. (Ed.). (2010). *Model-Based Reasoning in Science & Technology*. Berlin Heidelberg: Springer-Verlag. https://doi.org/10.1007/978-3-642-15223-8

Magnani, L., & Nersessian, N. J. (Eds.). (2002). *Model-Based Reasoning: Science, Technology, Values*. New York: Kluwer Academic / Plenum Publisher. https://doi.org/10.1007/978-1-4615-0605-8

Magnani, L., Nersessian, N. J., & Thagard, P. (Eds.). (1999). *Model-based Reasoning in Scientific Discovery*. New York: Plenum Publishers. https://doi.org/10.1007/978-1-4615-4813-3

Nersessian, N. J. (2008). *Creating scientific concepts*. Cambridge, Mass.: MIT Press.

Neugebauer, O. (1975). *A history of ancient mathematical astronomy*. Berlin; New York: Springer-Verlag. https://doi.org/10.1007/978-3-642-61910-6

Paavola, S. (2005). Peircean Abduction: Instinct or Inference? *Semiotica, 153*(1/4), 131-154. https://doi.org/10.1515/semi.2005.2005.153-1-4.131

Paavola, S. (2007). Abductive logic of discovery with distributed means. In O. Pombo & A. Gerner (Eds.), *Abduction and the Process of Scientific Discovery* (pp. 47-62). Lisboa: CFCUL/Publidisa.

Paavola, S. (2011). Diagrams, iconicity, and abductive discovery. *Semiotica, 186*(1-4), 297-314. https://doi.org/10.1515/semi.2011.057

Paavola, S., & Hakkarainen, K. (2005). Three Abductive Solutions to the Meno Paradox – with Instinct, Inference, and Distributed Cognition. *Studies in Philosophy and Education, 24*(3–4), 235–253. https://doi.org/10.1007/s11217-005-3846-z

Paavola, S., Hakkarainen, K., & Sintonen, M. (2006). Abduction with dialogical and trialogical means. *Logic Journal of the IGPL, 14*(2), 137-150. https://doi.org/10.1093/jigpal/jzk010

Peirce. (CP). *Collected Papers of Charles Sanders Peirce* (Volumes I-VI, ed. by Charles Hartshorne and Paul Weiss, 1931–1935, Volumes VII-VIII, ed. by Arthur W. Burks, 1958; quotations according to volume and paragraph). Cambridge, Mass.: Harvard UP.

Peirce. (EP). *The Essential Peirce. Selected Philosophical Writings. Vol. 1 (1867–1893), Vol. 2 (1893–1913)*. Bloomington and Indianapolis 1992 +1998: Indiana University Press.

Peirce. (MS). *The Charles S. Peirce Papers. Manuscript Collection in the Houghton Library, Harvard University*. Available in the Peirce Microfilm edition. Pagination: CSP = Peirce / ISP = Institute for Studies in Pragmaticism.

Peirce. (NEM). *The New Elements of Mathematics by Charles S. Peirce* (Vol. I-IV). The Hague-Paris / Atlantic Highlands, N.J., 1976: Mouton / Humanities Press.

Peirce. (SEM). *Charles S. Peirce, Semiotische Schriften* (Vol. I-III). Frankfurt a.M. 1986-1994: Suhrkamp.

Peirce, C. S. (1906). *Prolegomena to an Apology for Pragmaticism*: *The Monist* 16: 492-546 (quoted from Peirce, CP, 4.530-572). https://doi.org/10.5840/monist190616436

Peirce, C. S. (1909). *(Fragments on Existential Graphs)*: MS 514; Transcription by Michael Balat, with commentary by John F. Sowa at: http://www.jfsowa.com/peirce/ms514.htm

Quine, W. V. O. (1969). Epistemology Naturalized. In W. V. O. Quine (Ed.), *Ontological relativity and other essays* (pp. 69–90). New York Columbia Univ. Press.

Quine, W. V. O. (1971 <1951>). Two Dogmas of Empiricism *from a Logical Point of View: 9 Logico-Philosophical Essays* (2. ed., pp. 20–46). Cambridge, Mass.: Harvard Univ. Press.

Rescher, N. (1995). Peirce on Abduction, Plausibility, and the Efficiency of Scientific Inquiry. In N. Rescher (Ed.), *Essays in the History of Philosophy* (pp. 309–326). Aldershot Avebury.

Roberts, D. D. (1973). *The Existential Graphs of Charles S. Peirce*. The Hague Mouton.

Schurz, G. (2008). Patterns of abduction. *Synthese, 164*(2), 201-234. https://doi.org/10.1007/s11229-007-9223-4

Semetsky, I. (2005). Learning by abduction: A geometrical interpretation. *Semiotica, 157*(1/4), 199–212. https://doi.org/10.1515/semi.2005.2005.157.1-4.199

Shin, S.-J. (2002). *The Iconic Logic of Peirce's Graphs*. Cambridge, MA: MIT Bradford Books.

Skagestad, P. (1999). Peirce's inkstand as an external embodiment of mind. *Transactions of the Charles S Peirce Society, 35*(3), 551-561.

Stjernfelt, F. (2007). *Diagrammatology: An Investigation on the Borderlines of Phenomenology, Ontology, and Semiotics*. Dordrecht, NL: Springer. https://doi.org/10.1007/978-1-4020-5652-9

Thagard, P. (2010). How Brains Make Mental Models. In L. Magnani (Ed.), *Model-Based Reasoning in Science & Technology* (pp. 447–461). Berlin Heidelberg: Springer-Verlag. https://doi.org/10.1007/978-3-642-15223-8_25

Venn, J. (1880a). On the Diagrammatic and Mechanical Representations of Propositions and Reasoning. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 9*, 1-18.

Venn, J. (1880b). On the Forms of Logical Proposition. *Mind, 5*(19), 336-349. https://doi.org/10.1093/mind/os-V.19.336