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Bridging mathematics and physics: From theoretical frameworks to classroom practice

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Abstract

This article reports on the design and classroom implementation of interdisciplinary worksheets aimed at strengthening the relationship between mathematics and physics in upper-secondary education. Grounded in complementary theoretical perspectives—including the structural role of mathematics in physics, interdisciplinarity, covariational reasoning, and specific disciplinary topics—we developed six research-based activities on vectors, derivatives, and integrals and implemented them within a teacher learning community. Data from 203 grade-13 students include a baseline quantitative-reasoning assessment, a post-test on calculus and vectors, and item-level confidence ratings, complemented by teacher focus-group evidence. Results show improved alignment between confidence and performance and suggest a possible threshold effect, with gains emerging primarily for students who completed all six activities. Teachers reported that the worksheets acted as concrete boundary objects that made mathematics-physics connections instructionally actionable, while the community structure supported adaptation, reflection, and sustained interdisciplinary practice.

Keywords: physics education, mathematics education, interdisciplinarity in STEM, confidence, teacher learning community, upper-secondary education

INTRODUCTION

The integration of mathematics and physics represents a long-standing challenge in science education. Although the two disciplines are historically and epistemologically intertwined, students often experience them as disconnected domains, with mathematics perceived as abstract and procedural, physics as empirical and contextual. This fragmentation can hinder conceptual understanding and the ability to transfer reasoning across contexts. Recent research has therefore emphasized the need for instructional designs that explicitly address the structural role of mathematics in physics—that is, mathematics as a way of reasoning and a language that both shapes and expresses physical meaning, rather than merely serving as a computational tool (Pospiech, 2019, 2023; Uhden et al., 2012).

Despite broad consensus on this need, the translation of such theoretical perspectives into classroom practice remains limited. As Turşucu (2020) argue, genuine integration requires tasks that are simultaneously concept-oriented and meaningfully connected, enabling students to navigate between disciplinary representations. Similarly, Pospiech (2023) highlights the importance of scaffolding mathematization at multiple levels, supporting students in constructing and interpreting formal models. Yet, many curricular implementations remain procedural, with insufficient opportunities for reflective and interdisciplinary reasoning.

The present research responds to this gap by designing research-based interdisciplinary worksheets that operationalize the structural role of mathematics in physics, and by piloting them within a community of in-service teachers. The worksheets also incorporate

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Contribution to the literature

- This article offers a theory-driven yet classroom-tested model for designing interdisciplinary materials that make both the structural role of mathematics in physics and the reciprocity of the mathematics-physics relationship instructionally visible.
- It integrates multiple frameworks—on modelling, interdisciplinarity, covariational reasoning, and specific disciplinary topics—into concrete worksheet designs on vectors, derivatives, and integrals, and documents their enactment within a teacher learning community (TLC).
- The study provides empirical evidence on how these materials relate to students' performance and confidence calibration. It also illuminates how research-based worksheets can function as boundary objects that support teachers in implementing and sustaining mathematics-physics integration in real school settings.

confidence assessment, acknowledging the metacognitive and affective dimensions of learning in addition to cognitive ones. Students' ability to evaluate their own reasoning has been linked to engagement and learning persistence and can foster more inclusive and equitable learning environments. This perspective aligns with international policy frameworks, such as the United Nations (2015) 2030 sustainable development goals, in particular goal 4 and goal 5, which emphasize inclusive, reflective, and equitable learning opportunities.

The present work contributes to the research agenda on mathematics-physics integration in two ways. First, it examines how theoretical frameworks on the structural role of mathematics and interdisciplinarity can be translated into the design of practical teaching materials. Second, it investigates how these materials, when implemented within a TLC, can impact students' performance and confidence calibration, and how teachers perceive and enact them in their classroom practice.

Accordingly, the study addresses the following research questions (RQs):

(RQ1) How can theoretical frameworks on the structural role of mathematics and interdisciplinarity be used to design concrete instructional activities that explicitly strengthen the relationship between physics and mathematics?

(RQ2) How do the interdisciplinary worksheets, implemented within a TLC, affect students' performance and confidence calibration, and how do teachers perceive and enact them in their classroom practice?

Theoretical Background

This study builds on a set of complementary theoretical perspectives: Uhden et al.'s (2012) model; the family resemblance approach (FRA) (Erduran & Dagher, 2014) and the boundary crossing framework (Akkerman & Bakker, 2011). Uhden et al.'s (2012) model offers a way to examine the role of mathematics in physics, with particular attention to the processes of mathematization and interpretation. To adopt a more symmetrical perspective on the relationship between two historically intertwined disciplines, we also draw on the FRA to support reflection on the epistemic nature of mathematics and physics, highlighting both similarities and discipline-specific features, and Boundary Crossing theory to bridge and connect the two domains. These perspectives are further enriched, at a more specific level, by the covariational reasoning in physics (CoRP) framework (Olsho et al., 2023), which foregrounds the coordination of changing quantities as central to quantitative literacy and to the learning of calculus, and by topic-specific research on students' conceptual difficulties with vectors, derivatives, and integrals (e.g., Barniol & Zavala, 2016; Deprez et al., 2019; Jones, 2015b, 2019). Together, these frameworks guided the development of instructional materials to generate concrete, classroom-ready activities, thereby bridging research and practice. A schematic overview of the references is shown in **Figure 1**.

The Interplay Between Mathematics and Physics

The relationship between mathematics and physics is deeply intertwined, both historically and epistemologically. In physics, mathematics serves not merely as a computational tool but as a structural component that provides formal coherence, inferential power, and meaning-making resources.

Uhden at al. (2012) emphasized the *structural* rather than merely *technical* role of mathematics in physics, highlighting how it shapes the representation, modelling, and interpretation of physical phenomena and inviting educators to orient teaching toward practices that make this role explicit. Building on Uhden et al.'s (2012) modelling cycle, mathematical reasoning in physics can be understood as an iterative process of mathematization and interpretation, in which physical situations are gradually abstracted into formal models and these models are subsequently reconnected with empirical meaning.

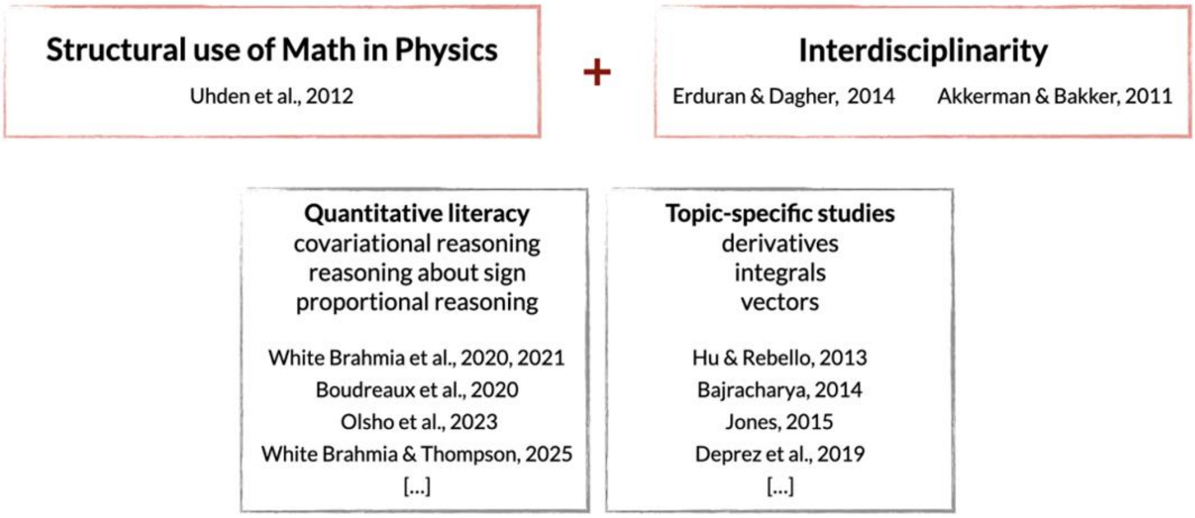

Figure 1. Overview of the theoretical framework used to design the instructional materials (the authors' own elaboration)

Integrating these phases into the design of learning materials supports the development of *representational fluency*—the ability to navigate and translate among symbolic, verbal, and graphical forms—recognized as a hallmark of expert reasoning in physics (Etkina et al., 2006).

Uhden et al.'s (2012) model, however, is focused on the role of mathematics in physics education and therefore contains an asymmetry between the two disciplines. A more balanced perspective is offered by *interdisciplinarity* theoretical frames, that we articulate based on the integration of two complementary perspectives: the FRA (Erduran & Dagher, 2014) and Akkerman and Bakker's (2011) *boundary crossing*. A similar combination of these perspectives has been proposed in recent work on mathematics-physics integration (Satanassi et al., 2023).

The FRA model conceives science simultaneously as a cognitive-epistemic enterprise, involving aims, values, practices, methods and methodologies, and knowledge, and as a social institution, involving professional activities, ethos, social values, and organizational and political structures. This set of categories allows for a fine-grained analysis of what constitutes a discipline and how it operates in both theoretical and practical terms. The idea of "family resemblance" explains how different scientific disciplines can share some, but not all, of these categories and characteristics, forming overlapping patterns that give science both unity and diversity.

From the FRA perspective, disciplines such as mathematics and physics share a network of overlapping cognitive-epistemic features—such as modelling, reasoning, and use of representations—while maintaining distinct characteristics, including validation norms and specific practices. Recognizing these "family resemblances" allows educators to design tasks that make both the unity and diversity of scientific reasoning explicit, avoiding the reduction of one discipline a mere toolkit for the other.

In parallel, Akkerman and Bakker's (2011) notion of *boundary crossing* conceptualizes interdisciplinarity as a process of navigating and negotiating disciplinary boundaries. These boundaries are not barriers but productive interfaces where *identification*, *coordination*, *reflection*, and *transformation* can occur. *Identification* concerns the recognition and articulation of disciplinary identities and differences. *Coordination* involves establishing minimal yet functional exchanges between disciplines to support smoother collaboration. *Reflection* entails viewing one's disciplinary practice from the perspective of another, fostering epistemic humility. *Transformation* goes a step further, suggesting that sustained interdisciplinary engagement can lead to the development of new hybrid practices or boundary practices in which the traditional boundaries are redefined or even dissolved. Interdisciplinary learning arises through engagement with boundary objects, that is, shared representations, concepts, or models that function across domains while preserving disciplinary specificity.

Combining the FRA and boundary-crossing perspectives thus provides a dual theoretical lens: the FRA clarifies *which* features can be shared or contrasted between disciplines, while boundary crossing explains *how* such sharing and transformation occur in practice. Together, they support a view of interdisciplinarity as a dynamic and epistemically balanced process, where mathematics and physics maintain their integrity while enriching each other through structured interaction.

Derivatives, Integrals, and Vectors Between Mathematics and Physics

The specific conceptual domains addressed in this research—derivatives, integrals, and vectors—are at the core of the physics-mathematics relationship and form the foundation of most introductory physics courses. Despite this pivotal role, students' understanding of these tools often remains procedural and fragmented, limiting flexible transfer across contexts (Jones, 2017, 2019; Feudel & Biehler, 2021; Sijmkens et al., 2022).

For derivatives, students may recall the phrase "rate of change" yet struggle to anchor it in concrete situations or to coordinate slope, concavity, and physical meaning (Jones, 2013; Zimmerman et al., 2025). Instruction that systematically links graphical and algebraic reasoning (including explicit work with signs and units) has been shown to strengthen interpretive skills and reduce purely symbolic manipulation (Wille, 2020).The ability to reason with rates and interpret them as relationships between quantities—rather than merely as symbolic manipulations—is critical in bridging mathematical formalism and physical meaning (White Brahmia & Thompson, 2025). Rate relates varying quantities over intervals (not only instantaneously) and naturally pairs with accumulation (Samuels, 2022; White Brahmia, 2023; White Brahmia & Thompson, 2025). In physics, this coupling is cognitively demanding because it must be reconciled with measurement, units, and vector structure, and with the tension between continuous models and discrete data (Thompson & Carlson, 2017; White Brahmia et al., 2021). Treating differentials as infinitesimal changes with physical interpretation (rather than as purely symbolic entities) aligns instruction with how physicists read velocity, acceleration, and work (Ely & Jones, 2023).

For integrals, difficulties extend beyond the formal definition to coordination with the fundamental theorem of calculus (FTC) (Bajracharya, 2014). Students often rely on "area under the curve" conceptions and/or analytical procedures without connecting to Riemann sums and accumulation of products, which impairs application (e.g., displacement from velocity-time graphs) (Jones, 2013, 2015a, 2015b). Textbook emphases can reinforce formalism at the expense of reasoning in context (Martínez-Sierra et al., 2019). Interpreting the definite integral as "summing up parts"—a rate times a small interval yielding accumulated change—supports modelling consistency with units and measurement (Jones, 2015a; Zimmerman et al., 2025). This perspective extends cleanly to vector-field contexts; for example, understanding surface integrals in Gauss's law as the accumulation of infinitesimal flux contributions clarifies the difference between vector fields and scalar accumulations (Li & Singh, 2017). At the same time, misreadings of dx as a purely procedural marker rather than an infinitesimal with units further undermine transfer (Oehrtman, 2009).

Difficulties with vectors are equally persistent. Many students, despite prior exposure, struggle with basic addition, subtraction, and decomposition, both graphically and algebraically (Heckler & Scaife, 2015; Knight, 1995; Liu & Kottegoda, 2019). These gaps distort the reading of core laws (e.g., treating mv^2/r as a standalone formula rather than as an instance of Newton's second law, $\vec{F} = m\vec{a}$) and invite magnitude-only reasoning that ignores direction (Flores et al., 2004, Shaffer & McDermott, 2005). Standard operations are also fragile: the dot product is misinterpreted as vector-valued; the cross product depends strongly on context familiarity (Deprez et al., 2019; Zavala & Barniol, 2013). Even after targeted instruction, a considerable proportion of students demonstrate limited improvement on validated diagnostic instruments (Barniol & Zavala, 2016; Lippiello et al., 2022).

Beyond topic-specific difficulties, a unifying challenge across derivatives, integrals, and vectors is the coordination of varying quantities. Covariational reasoning, i.e., understanding how changes in one quantity correspond to changes in another, is foundational both for calculus and for the modelling practices of physics (Carlson et al., 2002). The CoRP framework (Olsho et al., 2023) provides a structured lens on this interplay by articulating three intertwined dimensions: "proceptual" (procedural and conceptual) Understanding, describing how mathematical operations and representations are blended with physical meaning; physics mental actions (PMAs), which characterize how learners coordinate varying quantities (e.g., identifying related quantities, describing trends, analyzing small chunks of change); and Expert Behaviors, which capture strategic modelling moves typical of physicists (e.g., identifying physically meaningful points, engaging in local analysis, invoking compiled models). Difficulties in these covariational processes underpin many of the challenges discussed above, particularly when students must interpret derivatives as rates, integrals as accumulations, or vector operations as modelling choices grounded in physical variation.

The Role of Teacher Professional Development and Teacher Communities

Bringing these research perspectives and findings into classroom practice requires close collaboration between researchers and teachers. Traditional training initiatives, often short and content-heavy, rarely lead to sustained instructional change (Gilbert, 2010). Effective professional development must instead engage teachers in active, coherent, and sustained processes that foster reflection and collaboration (Desimone, 2009).

TLCs (Vangrieken et al., 2017), which emphasize situated participation, the co-construction of shared practices, and time for reflection, are a possible response to this need. Action research complements the learning community model by positioning teachers as active agents of inquiry rather than passive recipients of innovation (Laudonia et al., 2018). Through cycles of planning, acting, observing, and reflecting, teachers investigate and transform their own practice, especially when such inquiry is embedded in collaborative settings. Integrating action research within TLCs has proven effective in fostering professional dialogue, shared experimentation, and evidence-informed pedagogical innovation (European examples include the LINPILCARE initiative; van de Sande et al., 2017).

Based on these premises, our group has developed a model for physics teacher professional development (*Collabora*; Carli & Pantano, 2021, 2023), which poses teacher communities and action research as its pillars, sustained by the structural features of effective professional learning—content focus, active engagement, coherence, and sustained duration. The model is then nurtured by, and grounded in, Physics Education Research findings. Professional development activities designed according to this model entail regular meetings across one academic year and engage participant teachers in co-designing and implementing research-based activities or teaching sequences in the classroom.

Confidence Assessment

The last element of our theoretical framework is the inclusion of confidence assessment in the designed materials and evaluation instruments. This choice reflects a growing recognition that effective learning depends not only on conceptual understanding but also on metacognitive awareness. Prior research (Foster, 2016, 2022; Lippiello et al., 2025b) demonstrated that the relationship between confidence and performance—termed *calibration*—offers insight into students' self-evaluation and epistemic engagement. Miscalibration, in the form of over-confidence or under-confidence, can hinder learning by distorting students' sense of competence.

Embedding confidence prompts in everyday work thus serves a dual function: pedagogically, it encourages reflection on one's reasoning; analytically, it provides data to assess metacognitive alignment between perceived and actual understanding. This perspective emphasizes the affective and self-regulatory dimensions of learning at the interface between disciplines.

The *Let's Interplay!* Initiative

The *Let's Interplay!* initiative, based on our teacher professional development model, was developed at our university during the 2024-2025 academic year as part of a broader research program on the integration of mathematics and physics in secondary education. It involved teachers from different parts of the region where our university is located. *Let's Interplay!* represents the evolution of previous projects (Lippiello et al., 2025a, 2025b) extending its focus from the structural role of mathematics in physics to a broader, explicitly interdisciplinary framework.

Jointly coordinated by the departments of physics and astronomy, mathematics, and statistical sciences, the initiative pursued three main goals:

(1) to enhance the teaching and learning of physics and mathematics by emphasizing their interconnected nature;
(2) to promote better calibration between students' confidence and actual performance; and
(3) to foster a sustained TLC supporting collaborative and reflective professional development.

The TLC involved 20 in-service secondary school teachers, 13 of whom completed a classroom implementation in either the fourth or fifth year of upper secondary school (students aged 17-19). All participating teachers were qualified to teach both mathematics and physics at the upper-secondary level; however, due to school-level organizational structures, they did not always teach both subjects to the same class. In most of the 13 implementations, mathematics and physics were therefore taught by two different teachers, while only three participating teachers (each working in a different school) happened to teach both subjects to their own class.

The TLC included teachers from multiple schools, with some coming from the same institution. This composition enabled comparison not only across different school contexts but also within the same organizational environment, offering teachers opportunities for mutual support during implementation. This contextual detail is relevant for understanding the implementation choices and the ways in which teachers engaged with the interdisciplinary materials, as well as the constraints and affordances they encountered in their specific school settings. The community operated under a guided action research model adapted from the *Collabora* framework. Teachers meet monthly across one academic year. During these meetings, they worked with research-based instructional materials developed by the research team. Each session combined three components:

- theoretical input, presenting research findings related to the meeting's focus topic;
- collaborative engagement with the teaching materials in small groups; and
- reflective discussion, including teachers' reflections, classroom experiences, and student data from previous research.

As part of the training program, teachers were encouraged to implement the activities in their classrooms, with the flexibility to adjust the schedule and to choose the most appropriate instructional context.

Six three-hour meetings were held over the academic year. The first three revisited the core mathematical-physical topics (vectors, derivatives, and integrals) with a strong interdisciplinary orientation. The fourth meeting addressed covariational reasoning, while the fifth introduced statistical reasoning through examples such as regression analysis and probability distributions applied to physical contexts. After this series, there was a two-month interval with no scheduled meetings, during which teachers continued or completed a classroom implementation. The final meeting was then dedicated to sharing results and reflecting collectively on the implementation. Between meetings, participants maintained informal contact via email and online communication channels.

Although the teaching materials were initially drafted by the research team, the process of classroom implementation was characterized by close collaboration, with teachers contributing actively to their refinement and to the interpretation of classroom experiences. Their contribution unfolded across multiple stages. First, during the TLC meetings, teachers provided immediate feedback on clarity, workload, and feasibility, leading to targeted adjustments before classroom use (e.g., suggested modifications to timing, grouping strategies, or the wording of specific prompts). Second, during implementation, teachers exercised autonomy in orchestrating the activities—deciding how to structure group work and whole-class discussions, and how to deliver feedback to students. Their reports on how these choices played out in practice (such as the time students needed for particular tasks, points of hesitation, or moments that generated productive discussion) fed directly into the subsequent design cycle. Finally, after implementation, teachers' reflections during the end-of-year focus group offered more systematic insights into students' engagement and difficulties. These observations informed the refinement of the materials, particularly concerning pacing, scaffolding, and opportunities for verbalization and metacognitive reflection.

In this sense, teachers' classroom-based expertise functioned as an essential component of the iterative refinement process, ensuring that the instructional materials remained responsive to real classroom conditions and supported the effective uptake of the intended innovations. The initiative thus exemplified a form of boundary practice (Akkerman & Bakker, 2011), in which teachers and researchers collaboratively negotiated the integration of disciplinary knowledge and pedagogical expertise. Through this collaborative structure, *Let's Interplay!* functioned both as a professional development pathway and as a research environment.

METHODS

Building on the collaborative structure of the *Let's Interplay!* initiative, the study adopted a design-based research approach (Guisasola et al., 2017), implemented through our teacher community. The classroom interventions carried out as part of the program therefore constituted the context of the research. Some of the materials were also revised on the basis of a previous implementation carried out in 2022/23 (Lippiello et al., 2025a, 2025b), and the overall design process was informed by ongoing discussion with the participating teachers. For the present analysis, we consider only the fifth year (grade 13) implementation, involving 203 students from 12 classes and their 13 physics and mathematics teachers.

Data collection followed a multi-methods approach, combining pre-post testing of students and teachers' perspectives, collected through a focus group.

As a baseline assessment, the physics inventory of quantitative literacy (PIQL) (White Brahmia et al., 2021) was administered before the intervention. The PIQL assesses students' quantitative reasoning in physics, with particular emphasis on covariational reasoning making it a suitable baseline measure for an intervention focused on introductory calculus. The Italian translation of the PIQL, validated as part of the research project (Lippiello, 2025), was used.

As a post-test, students completed the test of calculus and vectors in mathematics and physics (TCV-MP), a multiple-choice instrument developed in previous research by our group (Carli et al., 2020; Lippiello et al., 2022). The test was recently revised to strengthen coherence within and across the three main conceptual areas it targets (Lippiello, 2025):

- The integrals section was expanded to assess students' understanding across the three complementary conceptions of the integral (antiderivative, area under the curve, and Riemann-sum accumulation);
- The derivatives section was refined to emphasize the construction of rate of change and to include derivatives with respect to spatial as well as temporal variables;
- The vectors section was extended with new items addressing dot and cross products and their interpretation in physical contexts.

Together, the PIQL and revised TCV-MP enabled an integrated assessment of students' mathematical-physical reasoning before and after the intervention. The TCV-MP could not be used as a pre-test because many of its target concepts (e.g., derivatives and integrals) had not yet been taught at the start of the school year, which would have yielded strong floor effects. The PIQL was therefore chosen as a baseline measure because it assesses prerequisite quantitative reasoning skills that all students could reasonably attempt before instruction.

Both assessment instruments included confidence ratings attached to each item, following the methodology described in Lippiello (2025). Students indicated how confident they were in each response using a four-point Likert scale (1 = "not at all confident" to 4 = "very confident"). These data were used to analyze students' confidence calibration, defined as the alignment between students' perceived confidence and actual performance, to explore the metacognitive dimension of learning.

The instructional worksheets used in the intervention were those presented during the community meetings and revised through practitioner review. In total, six activities were designed, two for each target concept (derivatives, integrals, and vectors). Teachers were encouraged to implement all six worksheets, although actual implementation ranged from two to six, as discussed in the results section.

A teacher focus group conducted during the final session of the training course investigated teachers' perceptions of the materials, their enactment of interdisciplinary practices, the role of the teacher community, and perceived constraints and enabling conditions. The focus group was audio-recorded and thematically analyzed to identify recurring patterns in teachers' reflections. These qualitative data were used to contextualize and interpret the quantitative findings reported in the results section.

The research design was quasi-experimental and did not include random assignment or a control group. The intervention was not conceived as the testing of a fixed instructional package under controlled conditions, but as a teacher-mediated implementation within a learning community. Teachers were supported in appropriating and adapting the materials to their own classroom contexts, in line with the aim of fostering sustainable change in practice. This choice necessarily introduced implementation variability, including differences in the number of worksheets enacted and in local constraints, and shifted part of the control from the research team to the teachers. Accordingly, the results should be interpreted as evidence from an ecologically valid implementation, rather than as causal estimates from a controlled experiment. Teacher-level variation was addressed descriptively through documentation of classroom enactment and through focus group data, which were used to contextualize the quantitative findings.

Design and Analysis of the Educational Materials

The grade 13 component of the intervention focused on vectors, derivatives, and integrals, the three domains assessed in the TCV-MP. Two worksheets were designed for each topic. The activities are summarized in **Table 1**.

The worksheets were conceived as *boundary objects*, didactic mediators connecting theoretical research frameworks with classroom practice.

Relying on Uhden et al.'s (2012) modeling framework the worksheets were designed to lead students through successive phases of mathematization (constructing formal models from physical phenomena) and interpretation (retranslating mathematical results into physical meaning), making these epistemic transitions explicit.

This modelling process is also interpreted through the lens of interdisciplinarity. The worksheets create structured opportunities for students to cross disciplinary boundaries and engaging in "boundary crossing mechanisms" such as identification, coordination, and transformation between mathematical and physical reasoning. From the FRA perspective, mathematics and physics are treated as overlapping epistemic cultures that share certain scientific practices while differing in aims, methods, and validation criteria.

To support the cognitive and representational transitions involved in this boundary crossing, the design draws on the CoRP framework, which articulates how learners coordinate varying quantities and construct relationships between them. The worksheets progressively engage students in PMAs, beginning with the identification of related quantities (PMA1) and advancing through recognizing and describing trends (PMA2), constructing value pairs or tables (PMA3), applying discrete scaling reasoning (PMA4), and ultimately analyzing small chunks of change (PMA5). This structure ensures that mathematical relationships, such as derivatives and integrals, emerge naturally from reasoning about change, rather than from rote symbolic manipulation.

Finally, the design of each worksheet was informed by topic-specific research findings and by meaningful physics contextualization aligned with the curriculum of the final two years of high school, thereby facilitating their integration into regular instruction.

The vectors worksheets proposed isomorphic questions in three contexts: pure mathematics, mechanics, and electromagnetism (force, torque, Lorentz force, magnetic flux). This design draws on research findings (e.g., Deprez et al., 2019; Zavala & Barniol, 2013) showing that understanding of vector operations depends strongly on context and representational form. The worksheets systematically vary representations (verbal, algebraic, and graphical) while prompting explicit comparison between dot and cross products and their scalar versus vectorial outcomes.

Table 1. Summary of the activities

Worksheet – topic	Description
Vectors I – Dot product	The worksheet develops the dot product across multiple contexts (work and magnetic flux in physics, as well as purely mathematical settings) while strengthening students' representational fluency (from verbal definitions to symbolic formalism and graphical interpretation). Particular attention to limiting cases (zero and negative outcomes) is leveraged to foreground structural understanding rather than mere procedural manipulation.
Vectors II – Cross product	It develops the cross product across diverse contexts (Lorentz force, magnetic force on a current-carrying wire, torque, and purely mathematical applications) emphasizing transitions among multiple representations (verbal, formal, and graphical), with vectors expressed in Cartesian coordinates, through unit vectors, and in polar form. The activity also invites explicit comparison with the dot product to highlight conceptual and operational contrasts between the two vector operations.
Derivatives I – Language	It converts verbal statements about "rates of change" into formal mathematical expressions and subsequently promotes their interpretation across diverse contexts, both physical and non-physical. In doing so, the activity explicitly targets the intertwined processes of mathematization and interpretation described in Uhden et al.'s (2012) model, fostering students' ability to move fluidly between conceptual understanding and formal representation.
Derivatives II – Reasoning chains	It begins with screening questions to assess students' understanding of the derivative as the slope of the tangent line to a curve, then progresses to tasks requiring the construction of "reasoning chains", sequences of statements (including correct, incorrect, and irrelevant ones) that students must select and logically order. This structure is designed to foster slow, analytical reasoning rather than intuitive or cue-driven responses. The activity is implemented across multiple contexts (kinematics, electric potential, potential energy, and magnetic flux) to strengthen transfer and conceptual coherence.
Integrals I – Special relativity	It guides students in deriving relativistic kinetic energy starting from Newton's second law and the definition of work, employing the integral as a continuous sum, as an antiderivative, and through the application of the FTC. The resulting expression is then compared with the classical case. The activity further includes the analysis of the function $v^2(E)$, exploration of limiting cases, and an experimental verification based on Bertozzi's (1964) results, thus integrating mathematical formalism, physical interpretation, and empirical validation.
Integrals II – Targeted exercises	A sequence of scaffolded problems was developed, including variable-resistivity tasks adapted from Hu and Rebello (2013), displacement from a time-dependent velocity $v(t)$, PIQL-inspired items on the relationship between rate and area (Zimmermann et al., 2025), and real-world graph interpretation tasks involving acceleration and velocity. Each activity requires students to verbalize reasoning, provide justifications, and report confidence levels in their answers, thereby supporting the conceptualization of the integral as an accumulation process rather than a purely procedural operation.

Derivatives were treated both as rates of change and as slopes of tangent lines, tracing the conceptual shift from discrete to infinitesimal reasoning. Drawing on Jones (2017) and Zimmerman et al. (2025), the tasks emphasize the coordination of quantities, units, and physical meaning, moving beyond purely procedural computation.

Integrals were explored through their multiple conceptualizations—as antiderivatives, areas under a curve, and Riemann sums—following Jones (2015a). The activities explicitly link these views to the accumulation of physical quantities, such as work or energy, thereby bridging symbolic formalism with physical interpretation.

Across all topics, the worksheets seek to integrate theoretical coherence, representational diversity, and epistemic authenticity, positioning mathematics as a language for modelling, interpreting, and ultimately understanding the physical world.

We now examine in greater detail two worksheets that illustrate the design approach outlined above and provide complementary examples of how the design principles discussed above were enacted in practice.

Example of Worksheet Analysis: Derivatives II – Reasoning Chains

Among the activities designed for grade 13, the worksheet derivatives – reasoning chains exemplifies how the materials aimed to promote deep conceptual reasoning and foster meaningful integration between mathematics and physics. The activity focuses on the derivative as a rate of change and on its interpretation across multiple physical contexts (kinematics, potential energy, electric potential, and magnetic flux) each involving analogous mathematical structures but distinct physical meanings.

The design was inspired by the work of Speirs et al. (2023), who employed a paired-question methodology to investigate students' difficulties in applying the same conceptual structures and reasoning strategies across different physics contexts. In Speirs et al.'s (2023) work, students first answer a screening question to assess their conceptual readiness, followed by a target question that required similar reasoning in a new context where salient surface features could elicit intuitive but incorrect responses.

In adapting this methodology to an instructional rather than purely diagnostic purpose, we reinterpreted it as a single, integrated learning activity designed to elicit and support analytical, concept-driven reasoning. The screening-target structure was retained but transformed: instead of multiple-choice items, students responded to open-ended screening questions requiring explicit justification, allowing teachers to identify conceptual gaps and making students' reasoning visible. These questions were first posed in a purely mathematical context—encouraging verbalization and formalization of the derivative as the slope of the tangent line—before being linked explicitly to the physics meaning of rate of change.

The core of the worksheet introduced a reasoning-chain format, in which students received a set of statements (some correct, others incorrect or irrelevant) and were asked to arrange them into a coherent chain of inferences that justified a given conclusion. This design, while echoing Speirs et al.'s (2023) approach, was substantially expanded: the number of statements was increased and differentiated across contexts (kinematics, electric potential, potential energy, and magnetic flux), and distinct graphical representations were created to reduce cueing effects and promote transfer. **Figure 2** provides an example of this approach. In each case, students were guided to integrate information across verbal, algebraic, and graphical representations, explicitly engaging the mathematization and interpretation phases of Uhden et al.'s (2012) modelling cycle.

The reasoning-chain format further supported metacognitive engagement by requiring students to reflect on the validity and logical order of statements. At the same time, it promoted representational fluency and coordination between mathematical and physical meaning, thus reinforcing the structural role of mathematics in physics. More specifically, the activity was designed to elicit progressively more sophisticated PMAs. Early prompts targeted PMA1, guiding students to identify related quantities (e.g., recognizing that velocity depends on time or that electric potential varies with position). Subsequent tasks required PMA2, describing the trend of change between variables, for instance, articulating whether velocity increases or decreases and how the rate of change itself evolves. Finally, later prompts fostered PMA5 by encouraging infinitesimal reasoning through the analysis of how very small variations in one variable affect another, including the interpretation of local slope as an instantaneous rate of change, or the connection between the gradient of potential and the electric field, depending on the context. In this way, the worksheet scaffolded students' movement from qualitative to quantitative and eventually to differential reasoning.

Finally, the concluding reflection prompts asked students to compare contexts and articulate how the same mathematical construct, the derivative, acquires different interpretations across disciplinary boundaries. This invited the kind of interdisciplinary reflection described by Akkerman and Bakker (2011), helping students navigate and negotiate meaning between mathematical formalism and physical phenomena.

Example of Worksheet Analysis: Integrals I – Special Relativity

The worksheet (reproduced in full in **Appendix A**) focused on integrals and special relativity.

The task begins within the domain of physics, drawing from Newton's second law, and progressively transitions into mathematics through the explicit construction of the derivative via its limit definition. This first shift exemplifies the structural-technical duality of mathematics in physics: the derivative emerges not as an isolated algorithm but as a tool for modelling the relationship between measurable quantities, in this case force and momentum. The focus then moves to the definition of work, which requires a further epistemic transition—from discrete to continuous reasoning—introducing the integral as a continuous sum. In this section, students use both the definite integral and the antiderivative, aligning with the multiple conceptualizations identified by Jones (2015a), and engage with the FTC as a bridge between mathematical formalism and physical interpretation, consistent with White Brahmia and Thompson (2025).

The subsequent phase of *interpretation*, central in Uhden et al.'s (2012) model, invites students to assign physical meaning to the mathematically derived expression through the work-kinetic energy theorem. By analyzing limiting cases (e.g., $v = 0$) and interpreting the constant of integration, students are guided to recognize how physical quantities acquire meaning through boundary conditions and experimental validation. This step explicitly connects mathematical reasoning to the epistemic values of physics, such as empirical testability and coherence with established laws.

The worksheet further extends to covariational reasoning by asking students to invert the functional dependence between energy and velocity ($E[v] \to v[E]$), explore asymptotic behavior ($E \to +\infty$), and produce a qualitative graph of $v^2(E)$ in both classical and relativistic regimes. This promotes reflection on how the same mathematical operation—functional inversion—

D. Magnetic flux context

Question: Observe the magnetic flux–time graph for two loops in Figure 6.9. Indicate, if it exists, the moment at which the induced electromotive forces in the two loops have the same magnitude.

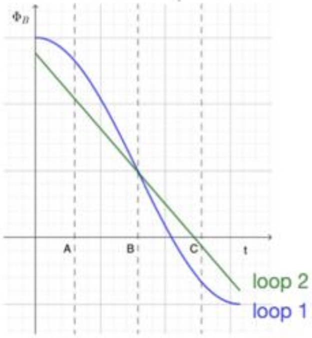

Figure 6.9

To justify your answer, you may use the following list of statements. Construct a chain of reasoning.

a) The magnitude of the instantaneous electromotive force (emf) is given by the derivative of the magnetic flux with respect to time.

b) The change in magnetic flux is given by the area under the emf–time graph.

c) The magnitude of the emf is given by the area under the magnetic flux–time graph.

d) The magnitude of the emf is given by the height of the graph.

e) The average electromotive force, in magnitude, is the ratio between the change in magnetic flux and the time interval.

f) The emf, in magnitude, is the ratio between the magnetic flux and the corresponding time.

g) The curves intersect at point B.

h) The derivative of a function at a point x_0 is $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ if it exists and is finite.

i) The slopes of the tangent lines to the graphs are equal at point A.

j) The magnitudes of the electromotive forces are equal at point A.

k) The magnitudes of the electromotive forces are equal at point B.

l) The magnitudes of the electromotive forces are equal at point C.

m) The magnitudes of the electromotive forces are never equal.

n) The derivative of a function at a point is the slope of the tangent line to the graph of the function at that point.

o) Faraday–Neumann–Lenz's law applies when the magnetic flux through the surface bounded by an electric circuit varies over time.

p) Faraday–Neumann–Lenz's law applies when the magnetic flux through the surface bounded by an electric circuit varies in space.

Chain of reasoning (add words like 'and', 'or', 'because', 'but', 'therefore', ...): ...

Figure 2. Excerpt from the worksheet derivatives II – reasoning chains, illustrating the activity in the context of magnetic flux (the authors' own elaboration)

can yield distinct physical insights depending on the model's domain of validity. Here, modelling acts as a boundary object, enabling students to traverse and integrate the conceptual spaces of mathematics and physics.

Finally, the worksheet concludes with an experimental component referencing Bertozzi's (1964) validation of relativistic kinetic energy. This section situates the integral-derived result within the broader family resemblance of scientific practices (Erduran & Dagher, 2014): modelling, prediction, and empirical verification. By closing the modelling cycle with a comparison between theoretical and experimental graphs, the activity embodies the *transformation* mechanism of interdisciplinarity, where shared conceptual tools—such as the integral—become genuine hybrid objects mediating understanding across disciplinary boundaries.

Confidence Assessment Practice

At the end of each worksheet, students completed a short questionnaire designed to promote reflection on both learning and confidence. The form included two explicit confidence items on a five-point Likert scale (0 = not at all, 4 = very much):

(1) I felt confident while tackling the proposed questions.

(2) I felt confident after working through the questions and receiving feedback.

Students also responded to two open-ended prompts identifying factors that increased or reduced their confidence (e.g., performing calculations, engaging with real-world contexts, working individually or collaboratively).

Table 2. Grade 13 (N = 203): Descriptive statistics for the pre-test (PIQL)

	Test-score (/10)	Confidence-score (/4)
Median	4.500	2.700
Mean	4.429	2.641
Standard deviation	1.528	0.498
Minimum	1.500	1.000
Maximum	8.000	3.650

Table 3. Grade 13 (N = 203): Descriptive statistics for the post-test (revised TCV-MP)

	Test-score (/10)	Confidence-score (/4)
Median	5.250	2.525
Mean	5.422	2.566
Standard deviation	1.903	0.522
Minimum	1.250	1.200
Maximum	10.000	3.950

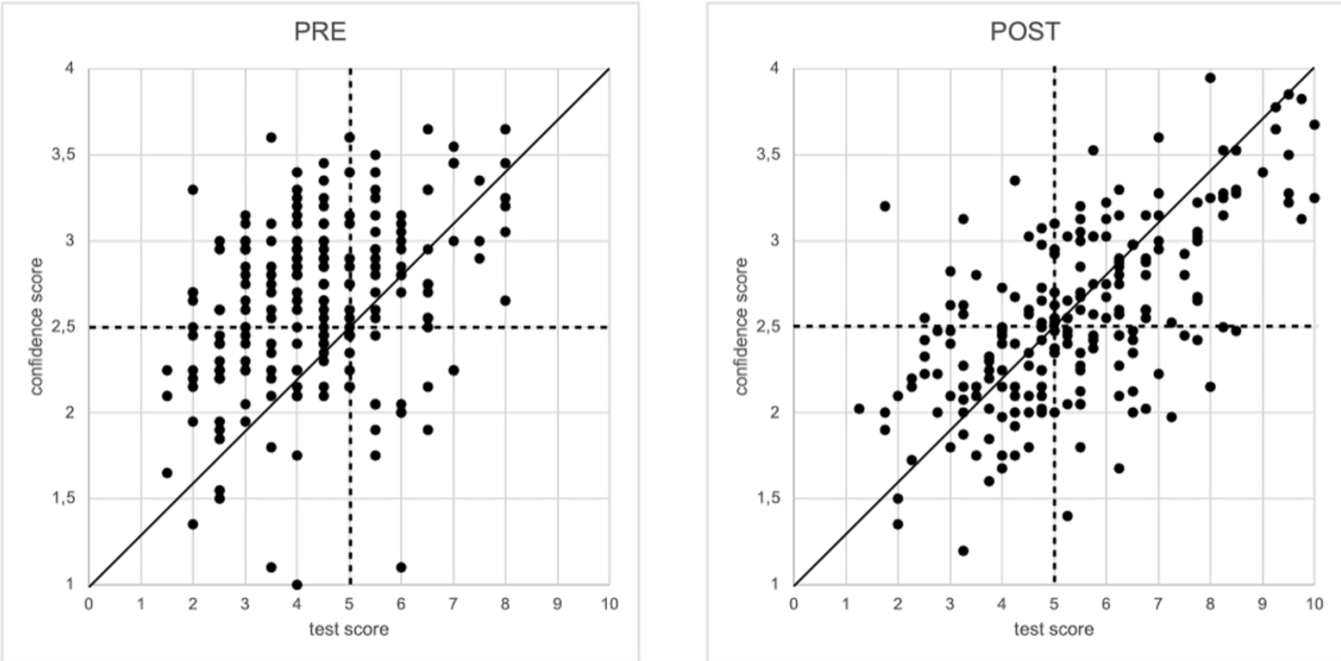

Figure 3. Confidence against performance scores, representing students' confidence calibration for the N = 203 students involved in classroom implementation (left: pre-test & right: post-test) (the authors' own elaboration)

Additional Likert-scale items probed perceived learning, engagement, and disciplinary balance (e.g., The activity helped me better understand the topic; I feel I worked on the key concepts and methods of mathematics/physics). In the worksheet on integrals II, the same confidence question used in the post-test (How confident are you in your answer?, 1-4) was added after each item to allow students to practice confidence assessment at the task level.

RESULTS AND DISCUSSION

Students' baseline understanding of quantitative reasoning was assessed using the PIQL (White Brahmia et al., 2021), administered as a pre-test before the implementation of the activities. The PIQL results (Table 2) showed that students entered the intervention with a partially developed, but fragile command of covariational reasoning: average scores were moderate-to-low (mean = 4.4/10), and confidence ratings suggested frequent overestimation of competence, as shown in Figure 3 (left).

As a post-test, students completed the revised TCV-MP. For analysis, TCV-MP scores were scaled to a 0-10 range to facilitate comparison with PIQL scores. Table 3 shows the results of student performance and confidence. The post-test results suggest better calibration between confidence and competence, as shown in Figure 3 (right). The correlation between confidence and test scores increased from $r = 0.34$ at pre-test to $r = 0.63$ at post-test ($p < 0.001$), indicating that students became more accurate in judging their own understanding. The proportion of overconfident students decreased substantially, with responses aligning more closely to the bisector line. This finding supports the idea that explicitly addressing confidence and metacognitive calibration in instruction can help students develop more realistic self-assessment, a key ability for regulating learning strategies.

Because these results aggregate students who participated in varying amounts of completed activities, a more detailed analysis is needed.

To better examine the impact of the activities, we analyzed students' post-test performance (TCV-MP) as a function of the number of completed worksheets, controlling for initial preparation (PIQL scores, scaled 0-10).

Table 4. Distribution of students and classes participating in the implementation by number of completed activities

Number of activities	Number of students	Number of classes
2	14	1
3	98	5
4	14	1
5	49	3
6	28	2

Table 4 shows the number of students (and classes) who implemented between two and six activities, allowing for a comparative analysis across levels of engagement.

We first examined the association between the number of activities completed and post-test performance using an ANCOVA, with the number of activities as a categorical factor and pre-test score as a covariate. This model was chosen to account for baseline differences in students' initial preparation. The analysis indicated a significant association between the number of activities and post-test scores ($F = 3.92$, $p = 0.004$; partial $\eta^2 = 0.074$), while pre-test performance was also highly significant ($F = 41.75$, $p < 0.001$), confirming its predictive value. However, Levene's test indicated a violation of the homogeneity of variance assumptions ($p = 0.003$). For this reason, the ANCOVA result was interpreted cautiously and complemented with additional checks.

A one-way ANOVA was then conducted to examine whether post-test scores differed across activity groups without adjusting pre-test performance. This analysis confirmed a significant difference among groups ($F = 4.71$, $p = 0.001$). Since this model does not control baseline differences and may still be affected by distributional assumptions, a Kruskal-Wallis test was also performed as a non-parametric robustness check ($H = 20.65$, $df = 4$, $p < 0.001$), indicating that post-test scores differed across at least some activity groups. Taken together, these preliminary analyses suggested that the number of activities completed was associated with post-test performance, but they did not identify a causal effect nor fully account for individual baseline variability.

To better account for individual variability, a multiple linear regression model was then fitted, including pre-test score and the number of activities as predictors. The number of activities was treated categorically, with the two-activity group as the reference category, in order to avoid imposing a linear dose-response assumption. Model diagnostics were satisfactory: residuals were normally distributed, variance was homoscedastic, and no problematic multicollinearity was detected. The model explained 25% of the variance ($R^2 = 0.25$). Pre-test score remained a strong predictor, confirming the importance of controlling baseline performance.

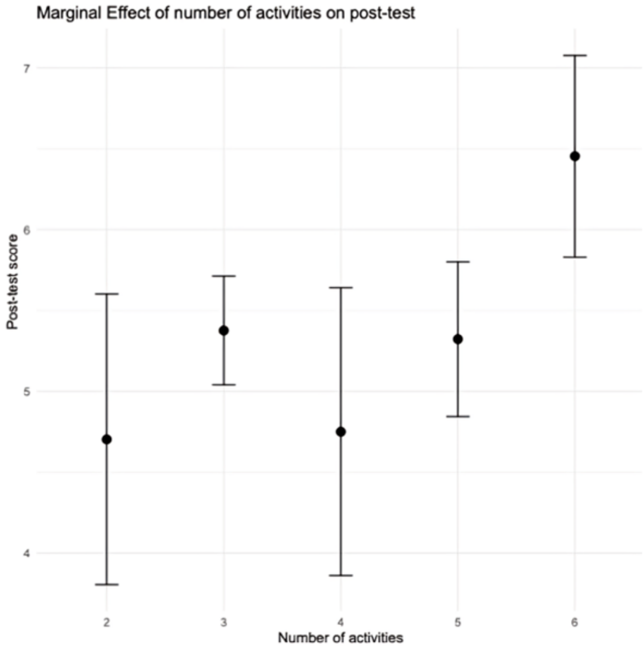

Figure 4. Estimated marginal means of post-test scores (adjusted for pre-test performance) by number of activities completed (the x-axis indicates the number of implemented activities [2 to 6], while the y-axis represents the estimated post-test score; the dots correspond to the adjusted group means, and vertical lines denote 95% confidence intervals) (the authors' own elaboration)

The categorical regression model suggested a tentative non-linear pattern. Students who completed three to five activities performed similarly in the post-test to those who completed two, whereas only those who completed all six activities showed a statistically significant improvement relative to the two-activity group ($B = +1.75$, $p = 0.0019$). A comparison with a model treating the number of activities as a continuous predictor indicated a better fit for the categorical specification (ANOVA $p = 0.035$, AIC = 792.9 vs. 795.8), suggesting that the association may not be simply incremental. This pattern is also visible in the marginal means plot (**Figure 4**), where only the six-activity group's adjusted mean exhibits a confidence interval that does not overlap with those of the other groups. However, given the unequal group sizes, the non-random assignment of students to activity levels, and the teacher-mediated nature of the implementation, this pattern should be considered exploratory and indicative of a possible threshold effect, rather than confirmatory evidence of such an effect.

In sum, the analyses provide exploratory evidence that completion of the full sequence of interdisciplinary activities was associated with higher post-test performance, after controlling for pre-test score. The observed pattern is suggestive of a possible threshold effect, in which substantial gains are more clearly visible only when students engage with the complete set of activities. However, this interpretation remains tentative. The study did not include random assignment or a control group, and the number of activities completed was influenced by contextual and teacher-level factors. Therefore, the results should not be interpreted as demonstrating a causal effect of completing all six activities.

The findings support the design rationale of the intervention: conceptual restructuring and improved calibration appear to require sustained engagement across topics and consistent practice with the processes of mathematization and interpretation, rather than isolated exposure to single activities. This interpretation also resonates with the teacher community setting, which is grounded in the assumption that long-term engagement of teachers and their classrooms is a necessary condition for the development of productive teaching "habits" (Etkina et al., 2017).

While these results are encouraging, they must be interpreted with caution given the moderate effect size, unequal group sizes, and potential class-level influences (e.g., teacher mediation or contextual factors).

To better understand and complement the quantitative findings, we examined teachers' focus group accounts of how the worksheets functioned in classroom practice. This qualitative evidence helps contextualize the observed pattern and provides insight into the ways in which the materials were appropriated and enacted by teachers.

One teacher emphasized the cumulative role of the full set of worksheets, noting that they did not merely help students *"take stock of the situation,"* but also see connections,

> "as if they were revisiting together, in a spiral, things already done and then taken up again" (T6).

Another explained that the worksheets *"served as post-explanation reinforcement"*, allowing students to revisit previously introduced topics through practical activities (T4). These accounts are consistent with the tentative quantitative pattern suggesting that higher post-test performance was associated mainly with completion of the full sequence, rather than with isolated exposure to individual activities.

Teachers' reflections also help clarify the kind of learning processes that the materials may have supported. Several comments pointed to the role of the worksheets in making the mathematics-physics relationship more explicit and in guiding students from intuitive reasoning toward formalization. For example, one teacher reported using the worksheets to work on the distinction between formula, definition, and formalization, stating that the materials helped students understand

> "what it means to study in depth with reference to specific language" (T7).

Another teacher highlighted the importance of the worksheet structure, describing it as a way to move progressively *"into the concept"*, reaching formalism only at the end, after starting from a more intuitive idea (T3).

Future controlled studies are needed to further investigate the pattern suggested by the quantitative analysis and to better disentangle the underlying mechanisms. As for this study, the qualitative findings can help contextualize the results and support a cautious interpretation that the activities may have had a positive impact on students' learning. However, the worksheets also contributed to other dimensions of the intervention, particularly in relation to teacher professional development, as discussed below. For this reason, we examined teachers' accounts to understand how the materials were appropriated within the teacher community and how they supported classroom enactment.

Evidence from the teachers' questionnaires and focus group converges on two main levers of change:

(1) ready-to-use, research-anchored materials that concretize the mathematics-physics relationship, and

(2) a community structure that sustains collaborative sense-making and supports the transfer into practice.

Teachers consistently value having access to classroom-ready worksheets and to the accompanying research evidence. Their comments are consistent with an interpretation of the materials as practical "boundary objects" that made interdisciplinarity actionable. As one of the teachers (T3) put it, *"Having concrete things to do, not just talk"*, while another teacher (T8) emphasized how sharing the worksheets within the school facilitated collaboration:

> "I opened up, and by sharing the materials, my colleague understood. The materials helped to share and clarify the path we wanted to take".

Several teachers reported that the materials unlocked changes in assessment and task design:

> "What I saw in the worksheets helped me structure assessments differently …" (T11).

For others, the materials changed teachers' beliefs of what is feasible:

> "What previously seemed like something fantastic and practically impossible turned out to be feasible—discovering that there's research behind it that yields results is not only reassuring but also makes it doable" (T10).

The worksheet design was recognized as applicable also beyond the immediate target grades and topic, enabling interdisciplinary work even in earlier classes:

> "Even second-year students [grade 10] can work with different representations if you present them." (T11).

Finally, teachers highlighted the value of ungraded, discussion-oriented tasks for engagement and metacognition:

> "Not having a grade on the worksheets was fundamental … It gave students the freedom to express themselves, even when making mistakes." (T5).

Alongside the materials, teachers pointed to the collaborative routines of the TLC, especially co-planning, sharing resources, and reflecting on student evidence, as essential supports. Collaboration between mathematics and physics teachers within the same school has amplified the impact:

> "This year I was more fortunate because one of my colleagues attended the Let's Interplay! course, and we shared everything." (T8).

The community reduced professional isolation and normalized common challenges:

> "… to see that the class problems are the problems of everyone, in all their forms, faced together." (T4).

Teachers also emphasized how the program design strengthened the research-practice connection:

> "For me, the most important thing is the exchange with the university…" (T3).

Participation in the TLC also appeared to support teachers in moving from simply valuing interdisciplinarity to enacting it with greater structure:

> "Having a structure, even in the worksheets, is something I'm thinking about often: how to design an exercise if we need to move from one subject to the other, starting from the questions themselves, and having an approach that goes deeper into the concept until reaching the formalism as the final step, starting maybe from a more intuitive idea. So yes, it's giving me structure." (T3).

Teachers also identified practical constraints: timetable pressure, syllabus pacing, textbook misalignment, and the need for aligned colleagues:

> "Managing time is incredibly challenging in fifth year …" (T4)

> "Sometimes it's the textbooks themselves that don't help …" (T9)

> "If you have a colleague who shares your vision, it's feasible. Otherwise, you need to teach both disciplines …" (T1).

Despite these challenges, teachers described concrete curricular adjustments, like introducing analytic geometry earlier to support physics:

> "I also brought forward analytic geometry in the fourth year—I'm really glad I did." (T4),

and some assessment activities co-designed with the research team.

Looking forward, participants wished for continuity of the community, earlier and longitudinal integration, and the co-design of tools targeted to a wider number of topics and competences:

> "If we could start earlier … Little by little." (T6)

> "Wherever difficulties are widespread, it would be useful to have accompanying materials that can also be used in earlier years." (T8).

Overall, the qualitative data clarify the mechanism behind the quantitative pattern. Teachers' accounts suggest that the materials can be especially valuable when used over time, as part of a coherent way of working that allows students to revisit concepts, make connections across topics, and progressively deepen their reasoning. In this sense, research-based materials operationalize interdisciplinarity in class, while the TLC provides the social infrastructure to adapt, share, and sustain those practices.

CONCLUSIONS

We conclude by providing an answer to the RQs guiding this work.

Design of Theory-Driven Instructional Activities

The *Let's Interplay!* initiative shows how multiple theoretical frameworks can be effectively integrated to inform the design of classroom materials that bridge mathematics and physics. The structural view of mathematics in physics (Uhden et al., 2012) provided the conceptual foundation for treating mathematical reasoning not as a procedural tool but as a meaning-making structure that shapes how physical phenomena are represented and interpreted. This principle was enacted through the worksheets' design, which systematically guided students through the dual processes of mathematization and interpretation, making explicit the transitions between symbolic manipulation and physical reasoning.

The combination of the FRA (Erduran & Dagher, 2014) and the boundary-crossing framework (Akkerman & Bakker, 2011) offered a powerful lens to operationalize interdisciplinarity. The worksheets acted as boundary objects, shared artefacts allowing students and teachers to move between the epistemic cultures of mathematics and physics while maintaining each discipline's identity. Each activity foregrounded overlapping cognitive-epistemic practices (e.g., modelling, quantification, representation) while also drawing attention to the distinctive validation norms of each field.

The CoRP framework (Olsho et al., 2023) further anchored the design in cognitive theory, structuring reasoning progressions from qualitative to quantitative and differential levels through explicit engagement with covariational reasoning.

The resulting materials exemplify a design-based synthesis: theory-driven yet grounded in classroom feasibility. Activities such as derivatives II – reasoning chains and integrals I – special relativity concretely embody this synthesis, translating abstract theoretical constructs into accessible student experiences. In this sense, the project contributes an actionable model of how interdisciplinary frameworks can inform instructional design, advancing both conceptual understanding and epistemic agency in upper-secondary physics.

Student Outcomes and Teacher Enactment of the Interdisciplinary Worksheets

Quantitative results provide tentative evidence that students engaged productively with reasoning tasks at the interface between mathematics and physics after the intervention. The post-test results on TCV-MP indicate that students were able to engage with calculus- and vector-based reasoning tasks in physics contexts, while the baseline PIQL provided information on their initial quantitative reasoning resources. Moreover, the strong post-intervention correlation between confidence and performance ($r = 0.63$, $p < 0.001$) suggests a more aligned relationship between students' perceived and demonstrated understanding. This result is consistent with enhanced calibration, although it should be interpreted cautiously given the design of the assessment.

This improvement is noteworthy because calibration was not only monitored but explicitly practiced through embedded confidence prompts and post-activity reflections. These tasks were designed to support self-assessment and regulation of cognitive effort—competences often underdeveloped in traditional physics instruction. The pattern aligns with our previous findings (Lippiello et al., 2025b) showing that making confidence an explicit learning target can foster awareness of the limits and reliability of one's reasoning.

The regression analyses by number of completed activities suggested an exploratory, non-linear pattern: higher adjusted post-test scores were observed mainly among students who completed the full sequence of six activities. Given the unequal group sizes, the absence of random assignment, and possible class- and teacher-level influences, this finding should not be interpreted as confirmatory evidence of a threshold effect. Rather, it offers a tentative indication that sustained engagement with the full sequence may be relevant for supporting students' reasoning across successive phases of mathematization and interpretation.

Qualitative data from teachers' focus group and questionnaires help contextualize and enrich the quantitative findings. Teachers described the worksheets as mediators of interdisciplinary integration: boundary objects that made connections between mathematics and physics tangible, and that provided shared reference points for discussion and reflection. They valued the materials' concreteness and adaptability, noting that the structure of the tasks helped them design lessons and assessments with a clearer interdisciplinary trajectory.

Participation in the TLC further amplified this effect. Co-planning, sharing resources, and discussing student evidence fostered professional reflection and cross-disciplinary dialogue, reducing isolation and encouraging shared ownership of the innovation. Interdisciplinarity emerged less as an abstract aspiration and more as a feasible, structured process supported by common tools, language, and community work.

However, several constraints remained, such as time pressure, curricular rigidity, textbook alignment, and dependence on collegial support. Despite such challenges, teachers' testimonies indicate that the worksheets served as catalysts for curricular coherence and cross-subject planning, extending beyond the immediate target grades.

Limitations and Future Directions

While promising, these findings should be interpreted in light of several limitations. The uneven distribution of students across activity levels limits causal inference, and some of the observed differences may reflect class-level or teacher-level effects. Moreover, confidence measures, though informative, rely on self-report and may be sensitive to affective factors beyond metacognitive accuracy. Future research aims to further validate the tentative threshold pattern and to disentangle the respective contributions of instructional design, teacher mediation, and student engagement. Ongoing extensions of the project will involve additional cohorts, grade levels, and disciplinary topics to test the robustness and transferability of the proposed framework. Finally, refining the confidence-assessment protocol—potentially by integrating trace-based data or think-aloud methodologies—may provide deeper insights into how calibration and metacognitive awareness develop through structured reflection.

In conclusion, this study illustrates how theoretical frameworks on the structural role of mathematics and interdisciplinarity can be transformed into practical instructional tools that meaningfully connect physics and mathematics learning. The *Let's Interplay!* worksheets served as boundary objects that operationalized complex theoretical constructs, such as mathematization, covariational reasoning, and metacognitive calibration, into accessible classroom practice.

The findings suggest that these materials played a dual role. For students, they provided structured opportunities to engage with mathematics as a resource for physical reasoning and to reflect on their own confidence and understanding. For teachers, they functioned as boundary objects that supported professional reflection, interdisciplinary dialogue, and the appropriation of research-informed practices. Although the quantitative results remain exploratory and should be interpreted cautiously, their integration with teachers' accounts indicates that the worksheets contributed to making the mathematics-physics relationship more visible, discussable, and teachable.

By embedding theory into design, classroom enactment, and assessment, the initiative offers a replicable model for bridging research and practice in STEM education. Its contribution lies in showing how research-informed materials, when developed and discussed within a TLC, can support sustainable forms of interdisciplinary teaching.

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Ethical statement: The study did not require formal approval from an institutional ethics committee under Italian regulations. All data were collected in accordance with national data-protection laws and with the institutional guidelines of the participating schools. Participation by teachers and students was entirely voluntary and based on informed consent. No identifiable personal data were collected, and all results are reported in anonymized and aggregated form.

AI statement: During the preparation of this manuscript, ChatGPT (OpenAI) was used for language editing, including checking translations, improving grammar, and refining the clarity and style of the English text. It was not used to generate scientific content, interpret results, or formulate conclusions. The authors take full responsibility for the content of this article.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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APPENDIX A: WORKSHEET INTEGRALS I – SPECIAL RELATIVITY

Classical Physics

Newton's second law can be expressed as $\vec{F} = \frac{\Delta\vec{p}}{\Delta t}$, with $\vec{p} = m\vec{v}$ the momentum. What happens if $\Delta t \to 0$? What is the relationship between force and momentum? Calculate the expression for the force, remembering that velocity is a function of time:

Relativistic Physics

Newton's second law can still be expressed as $\vec{F} = \frac{\Delta\vec{p}}{\Delta t}$. However, the expression for momentum becomes $\vec{p} = \gamma m_0\vec{v}$.

What happens if $\Delta t \to 0$?

What is the relationship between force and momentum?

Calculate the expression for force, recalling that velocity is a function of time, and the Lorentz factor is a function of velocity, which is in turn a function of time, therefore, also depends on time:

Let's verify the result obtained: $\vec{F} = m_0 \frac{d\gamma}{dt}\vec{v} + m_0\gamma\frac{d\vec{v}}{dt}$. What is the definition of work? But what if the force is not constant—how can we proceed?

Hint (complete): "Small amount of work" = … product between the force and a "…"

Translating into symbols and considering $d\vec{s}$ the component of displacement parallel to the force: $dW = \vec{F} \cdot d\vec{s}$. Since $d\vec{s}$ is small, we can assume that the velocity remains approximately constant during that small displacement. Therefore, $d\vec{s} = \vec{v}\, dt$. Replace the force with the expression previously derived and substitute the expression for accordingly. Then simplify the expression: $dW = \vec{F} \cdot d\vec{s} = \cdots$

Is work a scalar or a vector quantity? To obtain the total work, we sum all the small contributions dW. Which mathematical operator allows you to calculate a sum of infinitesimal contributions, a continuous sum? $W = \cdots dW$ (Which symbol do you insert in place of the dots?)

Before moving on, let's do a quick check!

Precisely, the work done by the force from point A to point B is a definite integral $W = \int_A^B dW$. However, to compute it, we must go through the indefinite integral $\int dW$.

What is the relationship between the definite and indefinite integral?

Which theorem highlights this connection?

So, by substituting: $\int dW = \int(m_0 v^2\, d\gamma + m_0\gamma v\, dv) = \int m_0 v^2\, d\gamma + \int m_0\gamma v\, dv$. To calculate these integrals, we proceed one at a time:

1) $\int m_0 v^2\, d\gamma$; 2) $\int m_0\gamma v\, dv$

To integrate with respect to γ express v^2 explicitly as a function of γ: ...

Verify that you have obtained the expression: $v^2 = c^2\left(1 - \frac{1}{\gamma^2}\right)$. Now you can compute the first integral. Pay close attention to the fact that you are integrating with respect to γ, while m_0 and c are constants. $\int m_0 v^2\, d\gamma = \cdots$

2) To integrate with respect to v, express γ explicitly as a function of v: …

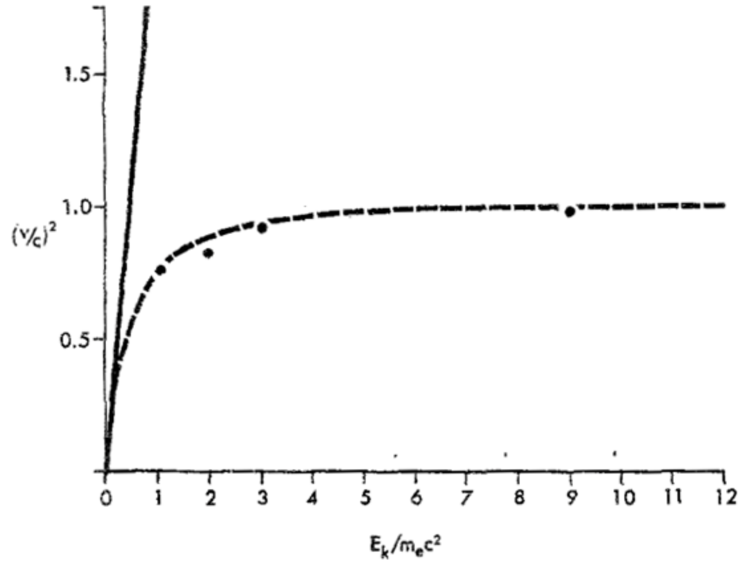
Verify that you have obtained the expression: $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$. Now you can compute the first integral. Pay close attention to the fact that you are integrating with respect to v, while m_0 and c are constants. $\int m_0\gamma v\, dv = \cdots$

So, putting it all together, $W = m_0c^2\gamma + \text{constant}$. Precisely, $W = \int_A^B dW = [m_0c^2\gamma + \text{constant}]_A^B = m_0c^2\gamma_B - m_0c^2\gamma_A$. On which physical quantity does γ depend? So, what changes between state A and state B?

State the work–kinetic energy theorem. So, what could $m_0c^2\gamma + \text{constant}$ represent? For $v = 0$, what is a reasonable value for this quantity to assume? We can now determine the constant ... We therefore obtain that the relativistic kinetic energy is: ... How can we interpret this result? Compare the relativistic and classical kinetic energy expressions by explicitly writing v^2 as a function of energy in both cases. Then, compute the limit of v^2 as energy tends to infinity ($E \to +\infty$) in each case, and sketch a qualitative graph of $v^2(E)$, placing energy on the x-axis and on the y-axis.

Experimental Confirmation

Observe the graph below obtained from Bertozzi's (1964) experiment at MIT and compare it with the results from your two predictions.



Based on the experimental setup shown below, try to explain how the experiment was conducted and how the data were collected.