MODESTUM

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Characterization of teacher profiles in teaching specific geometric topics based on a validated instrument

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Abstract

Mathematics education research highlights a need for instruments to better understand teaching practices specifically concerning geometry education. The research is underpinned by the theory of mathematical working spaces and is based on a quantitative approach. This study presents the results of research organized in stages that demonstrates the process of construction and validation of an instrument aimed at characterizing the teaching of specific geometry topics in secondary education (similarity of figures, homothecy, and Thales' theorem). The instrument was applied to a sample of 63 secondary education mathematics teachers in Chile. The teachers work in public and private educational institutions. Based on confirmatory factor analysis, three empirical dimensions and items for each dimension are established, which allows teacher profiles to be characterized according to how they decide to teach specific geometric topics. The results can be used for decision-making in future research, for teaching training, and for proposing didactic improvements.

Keywords: teacher profiles, teaching geometry, secondary education, questionnaire validation, mathematical working space

INTRODUCTION

The present work is centered on the validation of an instrument that permits the characterization of profiles of secondary education mathematics teachers when teaching specific geometric topics. The work is based on the theory of mathematical working spaces (MWS) (Kuzniak et al., 2022). For this purpose, dimensions and items have been considered to analyze the work of teachers and the organization of their teaching of specific geometry topics. Overall, the study aims to contribute to the community of researchers, teacher trainers, and in-service teachers involved in teaching specific geometric topics of secondary education by proposing an instrument designed to reflexively study teaching practices in a manner that allows for finding patterns, recurring characteristics, and opportunities for improvement.

Research on geometry education is wide-ranging, and its development in the last decade has been increasing (Jones & Tzekaki, 2016; Sinclair et al., 2017; Villa-Ochoa & Suárez-Téllez, 2022); this continuous evolution has been especially apparent in secondary education (Herbst et al., 2018; Weigand et al., 2025). Over time, the development of studies in geometry education has been supported by different theoretical perspectives and methodological approaches. For instance, some studies have explored various aspects of geometry education, such as Aravena-Díaz et al. (2016) who quantitatively assessed students' levels of geometric reasoning using the van Hiele model, while Özdemir et al. (2024), using a quantitative approach, determined the effect of ACE cycle-based learning on seventh-grade students. From a qualitative perspective based on a case study, Espinoza-Vásquez et al. (2025) analyze the classroom teaching of Thales' theorem in secondary education, incorporating mathematics teachers' specialized knowledge and MWS theories. Similarly, Kuzniak and Nechache (2021) examine different forms and paradigms of geometric work demonstrated by pre-service teachers when solving a specific task.

Some researchers place attention on specific cognitive processes, with one example being the development of visualization and reasoning in geometry (Duval, 2005).

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Contribution to the literature

- This study presents an issue that has remained underexplored in the field of mathematics education in the domain of geometry. Specifically, it presents the process of construction, validation, and results of the application of an instrument that characterizes the teaching of specific geometry topics in secondary education.
- A practical contribution of this study is the instrument (MWS-IG questionnaire), which is proposed as a validated theoretical tool for examining teacher practices in teaching specific geometry topics in secondary education, which can be used on a larger scale for quantitative studies. Likewise, the methodological design allows for replicability or future adaptations.
- The authors describe the five most common profiles of secondary school teachers in relation to their geometry teaching practices in specific topics. These profiles can be used in future research to design tasks or teaching proposals that promote specific cognitive and epistemological aspects.

Kuzniak (2018) addresses the teaching of geometry based on reflection on the nature of mathematical work in diverse educational contexts, taking cognitive processes and epistemological aspects into account. In the systematic review by Stylianides et al. (2024), the authors consider proof and demonstration in school and university mathematics, and, in particular, they report on various studies in the domain of geometry. In other research, geometry education is addressed through its use in a dynamic geometry environment (e.g., Flores-Salazar, 2018; García López et al., 2021; Henríquez-Rivas & Kuzniak, 2021; Lagrange & Richard, 2022; Prieto-González & Gutiérrez-Araujo, 2024; Richard et al., 2019; Vízek et al., 2024).

In parallel, some research focuses on analyzing the geometric activity of teachers, with attention to both in-service teachers and pre-service teachers (e.g., Avcu, 2022; Ayvaz et al., 2017; Creager, 2022; Flores Salazar et al., 2025; Sunzuma & Maharaj, 2020). For example, an investigation based on the mathematics teacher's specialized knowledge model describes a secondary school teacher's knowledge of mathematical practice in geometry classes and concludes by indicating the lack of empirical data supporting teacher training (Zakaryan & Sosa, 2021).

Other studies analyze the mathematical work of secondary school teachers in teaching specific topics (Henríquez-Rivas et al., 2021), revealing a preference for algebraic treatments in the teaching of geometry and the need to place greater attention on task design. For their part, Tachie (2020), focusing on teaching Euclidean geometry in schools, demonstrates teachers' lack of mathematical knowledge and states the need for improved teacher training and more research of this type, but on a larger scale (quantitative approach). These studies indicate the importance of the development of research based on empirical evidence, which allows the recognition of fundamental aspects of teachers' geometric work to contribute to decision-making with educational implications relevant to students' contexts.

In relation to the above, the design of instruments is fundamental, and their validation is an important process in relation to their use and utility based on the data obtained (Duke et al., 2020). For example, various studies on mathematics education consider validation processes using expert judgment (e.g., Espinoza Salfate et al., 2023). Others address construct validation by factor analysis (e.g., Magaña Medina et al., 2023). Different types of instruments specifically focused on mathematics teachers have been developed and validated, such as observation guidelines to study teaching practices (e.g., Arteaga-Martínez et al., 2021; Olfos Ayarza et al., 2022) and different types of questionnaires to evaluate teachers' opinions or knowledge (e.g., Pincheira-Hauck & Vásquez-Ortiz, 2018; Seguí & Alsina, 2023). While diverse instruments exist underpinned by theoretical perspectives and specific contexts and purposes, the literature review carried out for this study found a scarcity of instruments that examine teaching practices in the domain of geometry among secondary education teachers.

As noted above, studies on geometry teaching represent a topic of interest in teacher professional development. The problem presented here emphasizes the need to have instruments available to study geometry teaching that allow for analytical explanation of different forms of geometric work by teachers on a larger scale. In this context, the central research questions include the following:

- How is an instrument to examine the teaching practices of geometric topics specific to secondary education comprised?
- How can a validated instrument be utilized to characterize teaching practices for specific geometric topics among secondary education teachers?

This study presents the results of the reliability estimation of an instrument and analyzes the factor loadings of the items it includes. This allows for the confirmation of its theoretical dimensions and, subsequently, a description of the profiles of the participants based on aspects of their teaching practices.

THEORETICAL FRAMEWORK

The theory of MWS is based on the reality of educational actors, both in their teaching practices and in learning in the classroom, in different educational contexts (Kuzniak, 2022). The richness of this theoretical corpus facilitates a deeper study of the mathematical work carried out by researchers, teachers, and students (Kuzniak et al., 2022).

In an MWS, tasks occupy an important role, since they are understood as a means to solve problems (Kuzniak, 2022). The notion of *task* is defined in a broad and open manner (based on Sierpinska, 2004), referring to any type of mathematical exercise, question, or problem, with clearly formulated assumptions and questions, that students can solve in a defined MWS (Nechache, 2017). Various studies show the potential of MWS theory as an analytical and methodological tool for research associated with the study of specific mathematical tasks (e.g., Henríquez-Rivas & Kuzniak, 2021; Kuzniak & Nechache, 2021). For example, Nechache and Gómez-Chacón (2022) present methodological aspects used in MWS to describe mathematical work within one or more mathematical domains.

MWS theory involves two dimensions: on the one hand, there are the epistemological principles of the objects that are studied within a mathematical domain (e.g., geometry, calculus, or probability) (Montoya-Delgadillo & Vivier, 2016), and on the other hand, the human component, which entails considering a cognitive dimension (Kuzniak et al., 2016). These two dimensions, termed *epistemological* and *cognitive* planes, are presented as being linked, with the aim of capturing the mathematical contents of the domain studied and the cognitive activity of the individual when acquiring, developing, or using these contents (Kuzniak, 2022).

The epistemological plane comprises three components: *representamen*, associated with the set of objects based on the interpretations and relations constructed by the individual; *artifacts*, including tools associated with drawing or construction, software, or a symbolic system; and *theoretical referential*, which corresponds to a mathematical theoretical reference system based on definitions, properties, and theorems. The cognitive plane, meanwhile, includes three processes: *visualization*, linked to the deciphering and interpretation of signs and the representation of the objects involved; *construction*, based on actions triggered by the artifacts used and associated usage techniques; and *proof*, understood as any verification reached through processes that produce validation supported by the theoretical referential (Henríquez-Rivas et al., 2021; Kuzniak et al., 2016).

The articulation between the epistemological and cognitive planes occurs through three *geneses*: *semiotic*, *instrumental*, and *discursive*. These geneses allow the

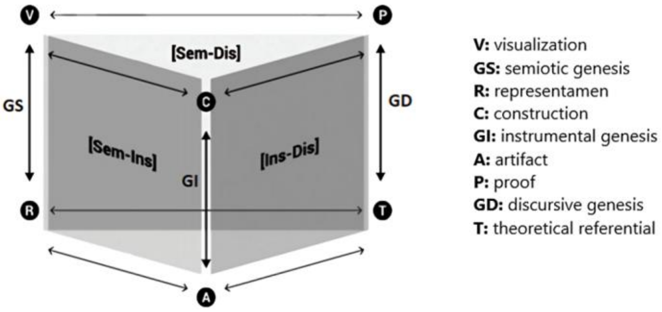

Figure 1. Diagram of MWS (Adapted from Kuzniak, 2022, p. 11)

nature of mathematical work to be coordinated and explained in various educational and institutional contexts (Kuzniak, 2011). Semiotic genesis is based on the registers of semiotic representation that allow tangible objects to act as operational mathematical objects. Instrumental genesis allows artifacts to be operationalized in the construction process carried out by the individual. Finally, discursive genesis of the proof gives meaning to the definitions, properties, or theorems to put them to use for mathematical reasoning.

Furthermore, Kuzniak and Richard (2014) recognize the idea of *vertical planes*, understood as the interactions between two geneses and the components involved. Three vertical planes are identified in these interactions (Kuzniak et al., 2016): *[Sem-Ins]*, associated with the use of artifacts in the construction of objects or in the exploration of semiotic representations; *[Ins-Dis]*, when experimental proof processes are carried out, a construction is validated, or deductive reasoning is used in which instrumented exemplifications are built; and *[Sem-Dis]*, relating the coordination of the process of visualization of represented objects with validation through discursive reasoning.

As demonstrated above, MWS research is based on studying and understanding the dynamics of mathematical work through the relationship between components, geneses, and vertical planes when the individual solves specific tasks (Kuzniak, 2018). These relationships are illustrated in the following diagram (**Figure 1**).

In the domain of geometry, MWS theory has been applied in various studies examining the mathematical work of teachers and future teachers. For instance, Montoya Delgadillo and Vivier (2014) demonstrated *domain changes* in their analysis of tasks designed by teachers to facilitate the transition from geometry to numbers and algebra. Similarly, Gómez Chacón and Kuzniak (2015) analyzed the geometric work of future teachers using technology, proposing MWS study to enhance comprehension in geometry and subsequent work in secondary school classrooms. In the work of Henríquez-Rivas and Montoya-Delgadillo (2016), the synthetic and analytical geometric work of secondary school teachers is analyzed, highlighting the theory for designing tasks and identifying difficulties and errors in mathematical work.

Other works show the theory as an analytical and methodological tool for research in geometry (e.g., Henríquez-Rivas et al., 2021; Kuzniak & Nechache, 2021) and highlight its potential for task design, the description and evaluation of geometric work, the detection of successes, errors, and difficulties, as well as its complementarity with other theoretical perspectives (Espinoza-Vásquez et al., 2025; Flores Salazar et al., 2025).

Lastly, within this theory, three types of MWS are distinguished (Gómez-Chacón et al., 2016): *reference MWS*, related to persons or institutions responsible for the school in terms of official mathematical criteria (Montoya-Delgadillo & Reyes-Avendaño, 2022); *personal MWS*, linked to the reality of students' work when they appropriate and manage problem-solving (Menares-Espinoza & Vivier, 2022); and *idoine MWS*, understood as a space linked to the process of task selection for teaching, which entails the design, adaptation, and implementation of tasks in the classroom in a given context and institution, with the intention of helping students construct their learning in a specific context (Henríquez-Rivas et al., 2022).

The *idoine MWS* encompasses the mathematical work undertaken by a researcher or teacher within a school context. This involves differentiating between the a priori planning, termed the *potential idoine MWS*, and the implementation of teaching, referred to as the *actual idoine MWS*. In this way, the actual *idoine MWS* helps explain what is actually taught, based on the choices made by the teacher (or researcher) to adapt to local constraints, their knowledge, and the resources used for teaching (Gómez-Chacón et al., 2016; Henríquez-Rivas et al., 2022). The present study focuses on the actual *idoine MWS* of teachers when teaching some geometric topics in secondary education. Thus, the richness of this theoretical corpus guides the design and validation of the instrument, which in turn allows us to characterize profiles of mathematics teachers.

METHODOLOGY

General Aspects

This research employed a quantitative approach based on a survey design (Creswell & Creswell, 2023), since the collection of information was carried out using an instrument (MWS-IG questionnaire) that had been previously designed based on elements of the theoretical framework described above. The MWS-IG questionnaire was developed within the framework of a broader research project whose purpose is to identify elements of the geometric work of secondary education mathematics teachers in Chile. This project has specifically centered on the investigation of self-reported teaching practice, particularly as it pertains to a specific domain of geometry encompassing the similarity of figures, homothecy, and Thales' theorem. This choice of topics is based mainly on curricular and epistemological factors, since they are present both in the Chilean school curriculum for ninth grade (students aged approximately 14) (Ministry of Education of Chile [Mineduc], 2015) and in the *Disciplinary Standards* of teacher training (Center for Improvement, Experimentation, and Pedagogical Research [CPEIP], 2021). Moreover, in both cases, their study is proposed in an interconnected manner, since they are addressed from the perspective of the group of transformations in the Euclidean plane.

This research was organized in four stages. In the first stage, theoretical dimensions and the MWS-IG questionnaire items were proposed based on a literature review and subjected to content validation by expert judges (Almanasreh et al., 2019). The validation process entailed the adjustment and improvement of the dimensions and items (for greater details, see Henríquez-Rivas & Vergara-Gómez, 2025).

In the second stage, a construct validation was carried out, considering a sample of 63 teachers and justified based on factorial convergence (Alavi et al., 2024). The selection of the sample was intentional, with the aim of capturing maximum variability and robustly characterizing the metric properties of the instrument. For the specification of the characteristics (metrics) of the instrument, the coefficient Cronbach's alpha was utilized as an estimator of reliability, complemented by McDonald's omega. Based on these estimates, some items were eliminated, followed by a confirmatory factor analysis (Bandalos & Finney, 2019) of the remaining dimensions and items. From the results of the factor loadings, three theoretical dimensions were inductively confirmed in relation to the actual *idoine MWS* of the teachers.

In third stage, the dimensions of the previous stage were deductively reviewed and refined, supported mainly by existing literature on MWS theory. Additionally, the analytical fragmentation of the teachers' mathematical work was inductively explored according to aspects of theoretical interest addressed by each item. Finally, in stage five, the teachers' profiles were inductively characterized, considering the three dimensions and the dichotomization of the responses for the items associated with each of these. The five stages are illustrated in **Figure 2**.

Figure 2 highlights that the operationalization of the *actual idoine MWS* theoretical construct is reflected both in the dimensions and the questionnaire items. These have been attained as a result of the literature review process, validation, and improvements to the MWS-IG questionnaire, as indicated in the four stages that have

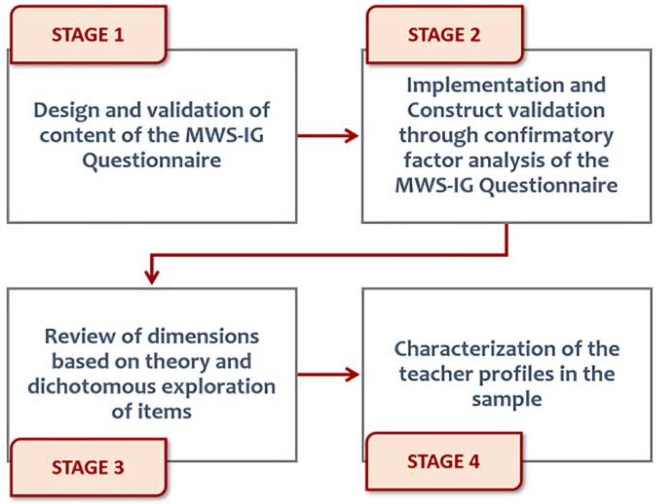

Figure 2. Research flow chart (Source: Authors' own elaboration)

been described, which allows the theoretical coherence of the instrument to be strengthened.

Sampling and Data Collection and Analysis

The population of this study was in-service Chilean mathematics teachers who at the time of the research were teaching at the ninth-grade level, because the teaching of the geometric topics involved is seen at this school level in Chile. An intentional sampling approach was employed to select teachers, prioritizing their current experience in teaching geometry at the secondary level. Initially, more than 100 teachers were contacted via email. The sample deliberately included educators from a range of public and private educational institutions and spanning urban and rural areas in order to ensure a broad spectrum of teaching practices. Additionally, the teachers possessed diverse degrees of professional experience (in terms of years teaching), contributing to a robust sample heterogeneity. This comprehensive diversity was vital for evaluating the instrument's capacity to adapt to and detect nuances across different teaching contexts, consequently reinforcing the external validity of the profiles generated.

The sample was ultimately composed of 63 practicing teachers who shared the previous characteristics and freely and voluntarily answered the questionnaire. The sample size in this study is justified based on the fact that, while confirmatory factor analysis tends to be associated with large sample sizes, the literature indicates that its adequacy depends fundamentally on the relationship between sample size, the number of estimated parameters, and the parsimony of the model. A number of classic and contemporary studies alike indicate that well-specified factor models with moderate to high factor loadings can be estimated in a stable manner with moderate samples when the sample-to-parameter ratio is adequate (Bentler & Chou, 1987; MacCallum et al., 1996). In this study, as stated above, the confirmatory factor analysis was carried out with a sample of 63 teachers, which is acceptable considering that the final model includes only three latent dimensions, each measured by a reduced number of items, limiting the number of free parameters to be estimated.

For data collection and analysis and publication of results, all of the corresponding ethical considerations have been considered, including the signing of informed consent forms by the participating teachers (Creswell & Creswell, 2023). Furthermore, informant anonymity was ensured by anonymizing all personal data through the use of codes. Access to the information linking these codes to individual identities was restricted solely to the principal investigator (first author). This sensitive data was stored in a cloud environment with password-protected, restricted access. The remaining two researchers were granted access exclusively to the blinded data, having previously signed confidentiality agreements.

Regarding the analyses, a deductive-inductive approach was applied (Creswell & Clark, 2018; Goetz & LeCompte, 1988), which refers to the alternation between theoretical interpretation of results and empirical recognition of categories based on data. This was carried out based on MWS theory and in line with the established objectives, which allowed, on the one hand, validation of the designed instrument and evaluation of the relevance of the theoretical dimensions associated with the actual *idoine MWS* of teachers and, on the other hand, characterization of profiles of mathematics teachers in relation to the teaching of specific geometric topics. Likewise, the analytical strategy adopted responds to a confirmatory–exploratory logic, understood as a process of dialogue between the initial theoretical structure and the empirical evidence obtained based on the data. It should be noted that while the dimensions of the instrument are theoretically underpinned by the theory of MWS, the confirmatory factor analysis allowed for said structure to be adjusted based on the observed factor coherence, strengthening the construct validity of the instrument (Bandalos & Finney, 2019).

RESULTS

Below, the results are presented from stage 1 to the final stage, which entails the characterization of the profiles of participating teachers.

Stage 1. Construction and Content Validation of the Instrument by Expert Judgment

For the initial design of the MWS-IG questionnaire, first a review of literature and documents was undertaken focused on the teaching and learning of geometry, from which the following stand out:

Table 1. Organization of the MWS-IG questionnaire according to dimensions

Dimension	Subdimension	Item category	General characteristics	n
Teaching preparation (items 1 to 8)	Organization of teaching	Planning (1), organization of topics (2), learning assessment (3)	Items related to the teaching of homothecy, Thales' theorem, and similarity of figures.	3
	School curriculum	Mathematical skills in the school curriculum (4 to 6), task design (7), school curriculum approach to teaching (8)		5
MWS components and geneses (items 9 to 17)	Components	Cognitive plane: visualization, construction and proof (9 to 11), epistemological plane: representamen, artifact, and theoretical referential (12 to 14)	Items related to geometry teaching, without focusing on specific topics.	6
	Geneses	Semiotic (15), instrumental (16), discursive (17)	Items related to the teaching of homothecy, Thales' theorem, and similarity of figures.	3
Vertical planes (items 18 to 23)	[Sem-Ins] vertical plane	The practice of teaching geometric topics that join two geneses, according to each vertical plane (18 to 23)	Items related to the teaching of homothecy, Thales' theorem, and similarity of figures.	2
	[Ins-Dis] vertical plane			2
	[Sem-Dis] vertical plane			2

Note. n: Number of items

1. Research on geometry education, which includes various theoretical and methodological perspectives, focused on in-service and pre-service teachers (e.g., Zakaryan & Sosa, 2021), along with the search for existing instruments in mathematics education that entail rigorous validation processes, especially those focused on the mathematics teacher (e.g., Arteaga-Martínez et al., 2021).
2. Literature review supported by MWS theory (Kuzniak et al., 2022), with special attention to theoretical and empirical investigations that consider the geometric domain and the teacher as a subject of study (e.g., Henríquez-Rivas et al., 2022; Gómez-Chacón et al., 2016; Morales, 2018; Panqueban et al., 2024),
3. Selected documents that guide teaching work in Chile and initial teacher training, such as the national curriculum (Mineduc, 2015) and the standards of the teaching profession for mathematics pedagogy programs in Chile[1] (CPEIP, 2021), as well as geometry textbooks (e.g., Chuaqui & Riera, 2011).

Designed to investigate the current actual *idoine MWS* (self-reported teaching practice), the MWS-IG questionnaire's initial framework is built upon a situated perspective of teaching knowledge. This framework incorporates perception, interpretation, and decision-making (Depaepe et al., 2020), relying on an assessment of self-reported practice (e.g., Yang et al., 2020). Thus, in its initial stage, the instrument includes 23 multiple choice items for which a single answer must be chosen. Each question presents five forced-choice options (Bartram, 2007), presented in familiar language intended for Chilean teachers. The items comprise questions addressed to the teacher that investigate three dimensions linked to the definition of *idoine MWS* (Henríquez-Rivas et al., 2022):

(1) teaching preparation,

(2) MWS components and geneses, and

(3) MWS vertical planes.

Each of these dimensions presents subdimensions with question types aimed at the geometric domain in general and others focused on teaching specific topics (homothecy, Thales' theorem, and similarity of figures). For the construction of the dimensions and subdimensions, specific theoretical elements of MWS theory are considered. On the one hand, transversal aspects of the theory are considered (including MWS components, geneses, and vertical planes) that allow for characterizing the teacher's mathematical work, and on the other hand, specific theoretical aspects related to teaching (corresponding to the *idoine MWS*). The details of the items according to dimension are shown in **Table 1**.

The call for expert judges was made following the criteria proposed by Skjong and Wentworth (2001): previous experience in the research area, scientific experience, availability and motivation to participate, impartiality, and other qualities inherent to a researcher in the area. Subsequently, 12 experts with doctorate degrees in mathematics education or related areas, and with recognized academic experience in the training of mathematics teachers, were selected.

[1] In Chile, teachers from the preschool to secondary level are educated at university in Pedagogy degree programs (carreras de Pedagogía). These programs are equitable to a bachelor's degree in a specific subject area with a teaching certification and are imparted by universities; they must be accredited by the National Accreditation Commission.

For validation purposes, the experts were fully informed about the objectives of the research, as well as the dimensions and subdimensions organizing the instrument and the items themselves. Using a non-comparative Likert-type scale from 1 to 4 (where 1 is unsuitable and 4 is fully suitable), the experts evaluated each item according to three criteria: clarity, relevance, and importance. Additionally, evaluation of the sufficiency of the items for each subdimension was requested in terms of what each subdimension was intended to measure.

Once the data was collected, the content validity was evaluated item-by-item according to Aiken's (1985) V coefficient, using a significance level of 0.05 for the upper and lower limits of the confidence interval. This was done for the three criteria in each of the 23 items constructed. To carry out the final selection of elements, the fulfillment of two conditions was considered:

(1) a global Aiken's (1985) V coefficient for the three criteria obtained with Euclidean distance greater than 0.8 and

(2) a lower limit of the global confidence interval of the three criteria reached with Euclidean distance greater than 0.6.

Furthermore, to measure the degree of concordance and internal consistency of the variables among the experts, the non-parametric test of Kendall's W index was applied by dimension, because this coefficient is used to determine the degree of concordance among *k* sets of ranges (Legendre, 2005).

According to the judges' evaluation, all items pass the inter-judge consistency test when the three criteria are considered together, which supports the preservation of all items. The average Aiken's (1985) V coefficient is 0.92, considering the 23 items, which indicates a high level of internal consistency. However, in the individual analysis by criterion, the criterion *clarity* initially appeared weaker, with 10 items identified as requiring adjustments in their wording. Regarding the *sufficiency* criterion, the results allow the items to be validated by dimension.

Regarding the concordance among the experts' judgments, according to the values of Kendall's W coefficient for all dimensions, there is a significant degree of agreement, with a p-value of less than 0.05 for each of the three criteria in each of the dimensions.

Finally, in this stage, modifications were made to the items, addressing the balance between elements of the theoretical framework and the teachers' understanding. Indeed, although it is important to ensure that the design of the questions adequately incorporates the theoretical principles of MWS, the use of the theoretical lexicon, with specific terms such as *proof*, *visualization*, and *artifacts*, could likewise affect teachers' understanding and interpretation. Therefore, these terms were revised and adjusted based on clearer lexicons, using the school mathematics curriculum as a frame of reference. The details of the results of the content validation of the instrument and the final version of the MWS-IG questionnaire at this stage are exhibited in Henríquez-Rivas and Vergara-Gómez (2025).

Stage 2. Confirmatory Factor Analysis

Once the instrument has been adjusted (in stage 1), the questionnaire is applied to a sample of 63 in-service teachers, either in-person or virtually according to accessibility. The construct validation process consists of searching for evidence in the data to justify the dimensions that operationalize each construct (Bandalos & Finney, 2019). However, these dimensions are not always precisely supported by the data; therefore, the structures of the groups must be explored to uncover latent constructs or traits (e.g., Ferrando & Lorenzo-Seva, 2018). This process is characterized as a dialogue between the exploratory and confirmatory in factor analysis.

Based on the factorial sedimentation and the ranking of self-values (own values), the data support evidence in favor of the existence of 11 latent traits or factors, which naturally creates a complex context for validation due to the existence of factors for only one variable or question (an obviously counterintuitive situation in factor analysis).

Likewise, from the perspective of assumptions, that is Bartlett's sphericity and the sampling adequacy statistic or Kaiser-Meyer-Olkin (KMO), the following situations are presented: When testing the hypothesis that the variance-covariance matrix is the identity, it is rejected, which means that the data support evidence in favor of the factor analysis or the analysis is justified. However, rejecting this hypothesis does not necessarily mean that the correlations are strong. To assess the overall adequacy of the correlations, the KMO statistic is calculated. This index summarizes the correlations among variables and indicates whether factor analysis is appropriate. Values below 0.5 suggest that the data are not suitable for this type of analysis.

In this context, an exploratory factor analysis is developed, attempting to explain the latent internal structures varying the extraction methods and rotations. The data support evidence in favor of three constructs or factors that correspond to the organization of the initial dimensions. At this stage, the questions linked to each dimension are presented in **Table 2**. As illustrated, in the first factor grouping there is a subscale characterized by items 3 and 5 for one, and 8 and 1 for another, which together characterize scale or dimension 1. Meanwhile, the second factorial grouping considers items 9, 14, and 16, which characterize dimension 2. The third factorial grouping considers the items 19, 21, and 22, which characterize dimension 3.

Table 2. Confirmatory factor analysis results

Item	Factor loadings				Uniqueness
	Factors				
	1	2	3	4	
3	0.979				0.00500
5	0.397				0.83879
8		0.967			0.00500
1		0.445			0.74648
21			0.729		0.40078
19			0.470		0.76464
22			0.394		0.79969
16				0.635	0.57853
9				-0.410	0.73406
14				0.334	0.87533

Note. The "maximum likelihood" extraction method was used in combination with a "varimax" rotation

This explicit latent structure verifies the necessary assumptions for a factor analysis and, therefore, the present factor structure is validated.

Stage 3. Reformulation of Dimensions and Dichotomous Exploration of Items

In this stage, the confirmatory factor analysis demonstrated the need to adjust the conceptual structure initially proposed in light of the empirical data obtained in stage 2. Thus, the three adjusted dimensions are presented as follows:

1. **School curriculum:** This dimension is understood in relation to geometry teaching and its organization for the classroom in a specific context and institution, which implies considering curricular aspects for the planning of teaching and assessment of learning of specific school topics.
2. **Specific cognitive processes:** This dimension is understood as being linked to specific cognitive processes of the mathematical work that the teacher favors or encourages in their geometry teaching, which entails considering processes including visualization, proof, and the use of instruments in activities proposed for teaching geometric topics.
3. **Teaching practice:** This dimension is related to the interaction of aspects of mathematical work that teachers prefer in their practice of teaching geometric topics, which involves considering relationships between two MWS geneses and their components from the epistemological and cognitive levels involved.

In this manner, for the definition of teacher profiles, items have been considered that, according to their factor loadings, are greater than 0.4, because the practical utility of the pattern coefficients begins in the range between |.30| and |.40| (Bandalos & Gerstner, 2016). Therefore, the items considered definitively for each dimension include

(1) *school curriculum*, items 1, 3, and 8,

(2) *specific cognitive processes*, items 9 and 16, and

(3) *teaching practice*, items 19, 21, and 22.

Table 3 displays the items considered for the definition of the profiles according to the category representing them.

Based on theoretical support and empirical data, for the characterization of mathematics teacher profiles regarding the organization of their teaching of geometry and specific (geometric) topics from the school curriculum, the alternatives of each item have been dichotomized according to the geometric work that they favor, which is shown below (**Table 4**).

Stage 4. Characterization of Teacher Profiles

After dichotomizing the responses (**Table 4**), all possible mathematical combinations between the responses per item in each dimension are identified.

Table 3. Items considered for the characterization of teacher profiles

Empirical dimension	No	Item category	Item
School curriculum	1	Planning of teaching	In the planning of your teaching, how many weeks do you allocate to preparing the teaching of homothecy, Thales' theorem, and similarity?
	3	Learning assessment	How do you assess learning of the topics of homothecy, Thales' theorem, and similarity?
	8	Learning approach of the school curriculum	In relation to the learning approaches proposed in the school curriculum, what aspects do you consider for the teaching of homothecy?
Specific cognitive processes	9	Visualization process	Which visualization activities do you privilege in the teaching of geometry?
	16	Use of instruments	Which types of tasks or activities do you encourage the use of instruments for in the teaching of the topics similarity, homothecy, and Thales' theorem?
Teaching practice	19	[Sem-Ins] vertical plane	Which of the following statements best represents your teaching practice for homothecy?
	21	[Ins-Dis] vertical plane	Which of the following statements best represents your teaching practice in relation to the validation of Thales' theorem?
	22	[Sem-Dis] vertical plane	Which of the following statements best represents your teaching practice for homothecy?

Note. No: Item number

Table 4. Dichotomy of responses for profiles for each item according to dimension

Dimension	Item category	Dichotomy of responses
School curriculum	Planning for teaching (PT)	In relation to planning of teaching specific geometric topics, teachers allocate: four weeks or more (1), less than four weeks (0)
	Learning assessment (LA)	In relation to tasks for learning assessment for specific geometric topics, teachers design: non-closed tasks, such as open research tasks, with mixed questions (1), closed tasks with alternatives (0)
	Learning approach of school curriculum (LC)	In relation to the learning approach of the school curriculum that the teacher uses to propose tasks (or learning activities) for specific geometric topics: they consider relationships with other disciplines (1), they do not consider relationships with other disciplines (0)
Specific cognitive processes	Visualization process (VP)	In relation to activities encouraging visualization processes that the teacher includes in geometry teaching: they foster diverse types of visualization in their teaching (1), they do not include the visualization process in their teaching (0)
	Use of instruments (UI)	In relation to activities that foster the use of instruments included by the teacher in the teaching of specific topics: they include the use of some type of instrument in their teaching, for example, to measure, construct, or utilize technology (1), they do not include the use of instruments in their teaching (0)
Teaching practice	[Sem-Ins] vertical plane (SI)	In relation to their teaching practice for specific geometric topics, the teacher: privileges constructions in their teaching, considering the use of some type of artifact, whether material, technological, or symbolic (1), does not privilege constructions in their teaching, but rather the use of conceptual relations (0)
	[Ins-Dis] vertical plane (ID)	In relation to their teaching practice for specific geometric topics, the teacher: includes processes of construction to facilitate some type of proof (1), does not consider processes of construction to facilitate some type of proof (0)
	[Sem-Dis] vertical plane (SD)	In relation to their teaching practice for specific geometric topics, the teacher: includes discursive reasoning to explain visualization processes (1), does not include discursive reasoning to explain visualization processes (0)

Table 5. Sub-profiles by dimension according to dichotomization of the validated items

Dimension 1				Dimension 2			Dimension 3			
PT	LA	LC	Sub	VP	UI	Sub	SI	ID	SD	Sub
1	1	1	P1	1	1	Q1	1	1	1	R1
1	0	0	P2	1	0	Q2	1	0	0	R2
0	1	0	P3	0	1	Q3	0	1	0	R3
0	0	1	P4	0	0	Q4	0	0	1	R4
1	1	0	P5				1	1	0	R5
1	0	1	P6				1	0	1	R6
0	1	1	P7				0	1	1	R7
0	0	0	P8				0	0	0	R8

These combinations determine and define the sub-profiles per dimension.

In the case of the dimension *school curriculum*, eight sub-profiles are obtained (termed P1, …, P8); in the dimension *specific cognitive processes*, four sub-profiles are obtained (Q1, ..., Q4); and in the dimension *teaching practice*, eight sub-profiles can be defined (R1, …, R8), which are presented in **Table 5**. Thus, from the perspective of possible combinations among the sub-profiles, there is a theoretical total of 256 profiles.

Below, the results obtained from the sample of teachers who have answered the MWS-IG questionnaire are presented. The 10 predominant profiles are included based on the distribution of maximum frequencies indicated by the data. This distribution is shown in **Figure 3**, according to the descriptions shared in **Table 4** and **Table 5**.

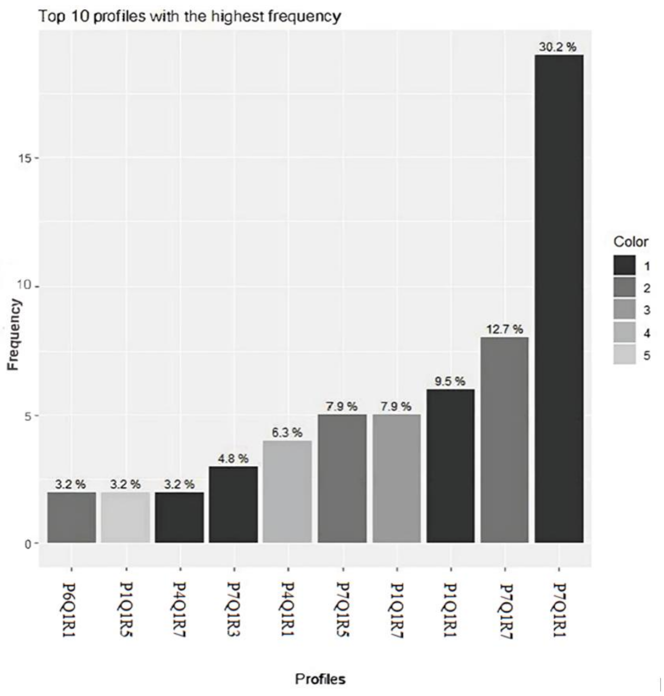

Figure 3. Graph of 10 teacher profiles (Source: Authors' own elaboration)

These profiles were constructed and described based on the three dimensions and representative categories of each item established by the empirical results and the theoretical aspects that support the study.

In the 10 predominant profiles (see **Figure 3**), it can be observed that sub-profile Q1 in dimension 2 is an invariant; that is, a common characteristic among the teachers is that they foster various types of visualization in their teaching of specific geometric topics and include the use of some type of instrument in their teaching, for example, to measure, construct, or utilize technology. To characterize the profiles in greater depth, the five most frequent have been selected, which together encompass more than 60% of the teacher participants.

It should be noted that since dimension 2, *Specific cognitive processes*, is the same across all profiles (teachers foster various types of visualization and utilize some type of instrument for teaching), this dimension will be ignored in the details of the descriptions of the five profiles selected, which are presented below.

- P7Q1R1, 19 teachers. *School curriculum* dimension: less than four weeks are allocated to planning specific geometric topics; when designing evaluation tasks, teachers consider non-closed tasks, for example, open research tasks; the curricular approach considered to propose tasks is based on relationships with other disciplines. *Teaching practice* dimension for specific geometric topics: teachers privilege constructions in their teaching, considering the use of some type of artifact (material, technological, or symbolic), in turn utilizing them to facilitate processes of proof and discursive reasoning to explain processes of visualization of figures. This profile consistently appears as the most common.
- P7Q1R7, 8 teachers. *School curriculum* dimension: as in the previous profile, teachers dedicate less than four weeks to planning specific geometric topics, and when designing evaluation tasks, teachers consider non-closed tasks. In addition, they consider a curricular approach that prioritizes relationships with other disciplines. *Teaching practice* dimension: teachers do not privilege constructions in their teaching with the use of artifacts but rather prioritize conceptual relations. However, when they do develop a construction, it is to facilitate some type of proof process, and discursive reasoning is used to explain visualization processes.
- P1Q1R1, 6 teachers. This profile is similar to the first and most common, with the difference lying in the *school curriculum* dimension, as teachers dedicate four weeks or more for planning specific topics. As with the first profile, when designing evaluation tasks, teachers consider non-closed tasks. In addition, they consider a curricular approach that prioritizes relationships with other disciplines. *Teaching practice* dimension: teachers privilege constructions with the use of artifacts and, in turn, use them to facilitate proof processes and discursive reasoning to explain figure visualization processes.
- P1Q1R7, 5 teachers. In the *school curriculum* dimension: teachers dedicate four weeks or more for planning specific topics, preferring the design of open-ended tasks, and consider a curricular approach that prioritizes relationships with other disciplines. In the *teaching practice* dimension: teachers do not privilege constructions in their teaching, but rather the use of conceptual relations; however, when processes of construction are included, it is to facilitate some type of proof, and discursive reasoning is included to explain visualization processes.
- P7Q1R5, 5 teachers. *School curriculum* dimension: teachers dedicate less than four weeks for planning, prefer the design of open-ended task, and consider a curricular approach that prioritizes relationships with other disciplines. *Teaching practice* dimension: teachers privilege constructions above conceptual relationships in the teaching of topics, include processes of construction to facilitate some type of proof, and do not include discursive reasoning to explain visualization processes.

It should be noted that the remaining participants are distributed among other non-predominant profiles according to the statistical analysis. Across all ten profiles, subdimension Q1 (see **Figure 3**) is a commonality, associating with the promotion of diverse visualization types, often incorporating instruments for teaching specific geometric topics. Notably, the five profiles detailed here additionally share the use of open tasks in evaluation activities and the search for connections with other disciplines. This shared characteristic may be attributable to specific features of the Chilean educational system, including adherence to curricular guidelines concerning learning assessment and interdisciplinary integration. More pronounced differences among the 10 profiles emerge in dimension 3, where all of teachers includes processes of construction to facilitate some type of proof, but not all favor geometric constructions utilizing various artifacts, or yet not all report connecting these constructions with the fostering of discursive reasoning.

DISCUSSION AND CONCLUSIONS

This study seeks to characterize profiles of mathematics teachers based on an instrument designed and validated to examine the teaching of specific geometric topics. From a theoretical perspective, attention is centered on the actual *idoine MWS* of teachers when teaching geometric topics (similarity of figures, homothecy, and Thales' theorem). The instrument permits information to be gathered about the actual *idoine MWS*—related to the way in which teachers organize mathematical work for the classroom in a specific place and context (Henríquez-Rivas et al., 2021)—based on a self-reported approach. The richness of this theoretical framework guides the design and validation of the instrument, which in turn allows the characterization of profiles of mathematics teachers. To achieve this, the reliability estimate of a questionnaire is carried out and the factor loadings of the items that compose it are analyzed, allowing the adjustment of theoretical dimensions and, subsequently, description of the participants' teaching profiles.

Methodologically, this study is quantitative, using an approach that consists of four stages and allows the design and validation of the questionnaire based on the use of diverse sources of information (reports, scientific literature, questionnaire) to obtain teacher profiles on the teaching of geometric topics. In this sense, the study is in line with other research analyzing mathematics teachers' self-reported practices, which point to how these offer pathways for sustainable professional development, as they identify that which is most frequent or favorable for teachers (e.g., Martin et al., 2022). Likewise, other future research can be proposed that delves into specific characteristics, such as continuing with the analysis categories utilized or the in-depth study of *circulations* in MWS (Montoya et al., 2014), in complementarity with the findings of the present study and potentially based on a qualitative or mixed-methods approach, implying other sources of data collection, for example, class observations, and document analysis, among others. This could contribute to the theoretical and methodological corpus of the MWS (particularly *idoine MWS*), as the research based on this theory is mainly from a qualitative approach (Panqueban et al., 2024).

The study results highlight the reformulation of the dimensions, which are initially posed deductively and prior to adjustments based on confirmatory factor analysis. In particular, the dimension *specific cognitive processes* does not consider items associated with discursive reasoning in the proof, according to the results of the factor loadings, which from the point of view of MWS theory is a missing component. Nevertheless, this is consistent with previous qualitative works supported by MWS theory, in which it appears that processes related to proof are rarely addressed or present difficulties in teaching geometry. (e.g., Espinoza-Vásquez et al., 2025). On the other hand, the processes of visualization and use of certain tools are relevant in teaching geometric topics. In future research, this matter could be investigated in order to encourage the activation of a *complete* MWS (Kuzniak et al., 2016), through tasks in which all geneses and components of the ETM are present in the geometric work of teachers and students, as shown in Aguilera Moraga et al. (2025).

Regarding the profiles, a significant number of the participating teachers (56 out of 63) are linked to the 10 predominant profiles found, based on the distribution of maximum frequencies shown in the data. An aspect that deserves attention in the *school curriculum* dimension is that, in all 10 profiles, the teachers state that they consider other disciplines in the teaching of specific topics; for the most part, they also report using diverse tasks in the classroom. This may be due to the growing presence of interdisciplinarity in the school curriculum in relation to homothecy (Mineduc, 2015). In subsequent work, we could research which tasks of an interdisciplinary nature are used and how they are implemented in the classroom.

Another aspect that merits attention is the homogeneity in the dimension *specific cognitive processes*, since all teachers, according to the profiles compiled, respond according to the characterization of sub-profile Q1 (see **Table 5**), that is, they report that they foster various types of visualization in their teaching, in addition to including some type of instrument for the teaching specific topics. Likewise, and the *teaching practice* dimension, 54 teachers declare that they consider the construction of figures to facilitate proof processes, which is linked to the activation of the vertical plane [Ins-Dis]. Additionally, although the visualization process seems to be an important process in teaching, it is not always related to the development of constructions or proofs. For example, the profile P7Q1R5 seems to favor a type of proof with figures, but, seemingly, without words (Richard, 2004), which has been reported previously (e.g., Henríquez-Rivas & Verdugo-Hernández, 2023).

The limitations of the study include the limited scope of the results due to the sample size. In future research, these results could be a basis for research hypotheses in order to study the similarities and stability of results with a larger and more random sample. Moreover, as pointed out by Zhu et al. (2021), in terms of specific teaching and learning processes, it is necessary to delve deeper into the links and mechanisms underlying the self-reported perceptions, the effectiveness, and the practices of mathematics teachers, broadening research to include the observation of practices, which would bolster the characterization of the actual *idoine MWS*. On the other hand, although the study focuses on specific geometric topics, this limitation of the questionnaire was deliberate, as using an instrument of this nature contributes to the specificity of the study from the perspective of MWS theory.

The MWS-IG questionnaire is proposed as a validated theoretical tool that allows for the examination of self-reported practices of mathematics teachers when teaching specific geometry topics. The instrument validation process guarantees the replicability of the study, opening the possibility of applying the instrument on a larger scale in order to recognize and analyze profiles of broader groups of teachers working in other cultural and institutional contexts. The identification of profiles would reveal trends in how mathematics teachers report and perceive their practices for teaching geometry (or specific geometric topics), thereby contributing to the continuous improvement of teacher professional development and pre-service teacher education, sustained by a theoretical and methodological perspective, an important theme in mathematics education today (e.g., Kuzniak & Nechache, 2021; Prieto-González & Gutiérrez-Araujo, 2024). Additionally, in future applications of the instrument, the perspective of *geometric paradigms* could be considered in the description of teacher profiles; this is a construct that emerged from the Theory of MWS during its fledgling stage (Houdement & Kuzniak, 1999).

Finally, these results could be considered in conjunction with other studies in geometry (e.g., Flores Salazar et al., 2025; Kıyıcı & Dikkartın Övez, 2025; Zhang et al., 2025), which contribute to the proposal of improvements in future teaching practices and teacher training, or in the design of tasks to promote forms of mathematical work in relation to specific geometric topics, for example, to encourage discursive reasoning and the presence of theoretical references. Additionally, it could contribute to decision-making at the curricular level, if the instrument were applied to a larger sample of teachers. In this manner, both the data collection instruments, and the results obtained can be utilized as theoretical and methodological input from the perspective of the teacher's *idoine MWS* in the domain of geometry.

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