

Commognition as an approach to studying proof teaching in university mathematics lectures

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Abstract

In this position paper, we propose commognition for the study of proof teaching at university lectures through an integrative literature review. We critically examine studies that focused on proof teaching but did not use the commognitive framework. Through this examination, we gain an understanding of the pedagogical aspects of proof teaching and address gaps in the literature. After we introduce the key principles of the commognitive framework, we review a set of studies that used commognition for the investigation of lecture-type teaching in higher education. In reviewing these studies, we pay special attention to the added value of the use of commognition. We conclude by proposing an approach to the study of proof teaching by using the commognitive framework. This approach offers insights into opportunities provided by lecturers for the de-ritualization of proving routines.

Keywords: commognition, de-ritualization, proving routines, university mathematics, lecture

INTRODUCTION

Teaching through lectures remains the predominant way of teaching in higher education (Viirman, 2021). Through lectures students are provided with opportunities to familiarize themselves with a large variety of mathematical ideas orchestrated by the lecturer (Sfard, 2014). However, lecture-type teaching has been largely criticized especially because it positions students in a passive role while it does not prioritize interaction between lecturers and students during the learning process (Jaworski et al., 2017). Indeed, during lecture-type teaching, students barely contribute to the lecture and the communication is mostly done one-way, from the lecturer to the students (Viirman, 2021).

As a matter of fact, university lectures follow a traditional format within the definition–theorem–proof (DTP) paradigm (Weber, 2004). In this paradigm, the lectures are structured in ways in which proof is seen as a final destination. In general, proof is recognized as the mathematical practice, which offers

“common criteria for accepting and generating new mathematical knowledge and connects the new theorems to the previous ones” (Hemmi, 2010, p. 273).

Through proof, continuity among the generations of mathematicians is achieved because mathematicians familiarize themselves with proofs already available to the mathematics communities and explore mathematics further by bringing new questions that need proof (Hemmi, 2010). However, students as newcomers to the mathematical community, face various difficulties while proving at the university level (Gueudet et al., 2016) that are related to the nature of proofs, the level of rigor, and the transition to advanced mathematical thinking (Selden, 2011).

The last two decades, researchers turned their attention to the communicative aspects of teaching to characterize the communication in lectures by using the commognitive framework (Sfard, 2008). Given that lecturing is more than a monologue from the lecturer, new methodological approaches emerged for the investigation of the teaching in advanced university mathematical courses (Nardi et al., 2014; Sfard, 2014). Commognition can nevertheless offer the analytical tools needed to discuss teaching and highlight lecturing at the micro-level. In this paper, we put an argument forward about the use of commognition for studying proof teaching in higher education. To support this argument, we performed an integrative literature review (Snyder, 2019). With integrative literature reviews developed or

Contribution to the literature

- This study proposes commognition for the study of proof teaching at university lectures, a topic underexplored in university mathematics education through sociocultural perspectives.
- In this position paper, we perform an integrative literature review to examine how commognition is used to study proof teaching. Thus, the overviews of the studies around proof teaching for pedagogical purposes and of the studies that used commognition for the investigation of lecturing highlight the added value of the use of commognition as an approach to study proof teaching in the lectures.
- The study concludes that the study of proof teaching with the use of commognition offers insights into opportunities provided by lecturers for the de-ritualization of proving routines.

emerging topics are discussed. In our case, we focus on the latter, where the purpose is

“to create initial or preliminary conceptualizations and theoretical models, rather than review old models” (Snyder, 2019, p. 336).

We critically synthesize the literature, qualitatively and not systematically, to conclude to our argument and proposal of commognition for the study of proof teaching in the lectures. In the beginning of this paper, to gain a broad view on the pedagogical aspects of proof teaching in the university level we provide a critical overview of this literature. These studies of teaching have used other than commognitive perspectives. Next, we introduce the main principles of the commognitive framework and discuss the learning of mathematics as participation in the mathematical discourse. Moreover, this section is concerned with studies that have used commognition for the investigation of university teaching towards students' learning. We provide a critical overview of the literature and we focus directly on the methodological considerations of the papers and the added value of using commognition. Not many studies that used commognition have focused directly on proof teaching. Then, we argue about using the commognitive framework for the study of teaching toward a de-ritualization of students' proving routines in the lectures. We propose commognition for the exploration of the characteristics of proof teaching in the lecturer discourse. Lastly, we conclude with the final remarks.

PROOF TEACHING AT THE UNIVERSITY FOR PEDAGOGICAL PURPOSES

Researchers in mathematics education have investigated the teaching of proof for students' learning at the university level for the last two decades, through mainly other than sociocultural frameworks (e.g., Fukawa-Connelly, 2012; Weber, 2004). For instance, Weber's (2004) study was one of the first that examined teaching in a lecture by using observations of teaching and interviews with the lecturer. For the analysis, the categorization of lectures as formal or informal was used. Interviews with the lecturer helped with the

identification of the interaction between the formal or informal aspects of teaching while the lecturer's rationale regarding the presented proving skills in the lecture emerged. Thus, three “teaching styles” of the lecturer were discerned (logico-structural, procedural, and semantic). A main result of the study was that in the traditional DTP format of lecturing, there was not a single paradigm for teaching and DTP varied according to the content that was presented. The three teaching styles were the adaptations made on the traditional DTP by the lecturer. This study opened up the road to other researchers to conduct empirical research by investigating the teaching in lectures. The study also highlighted how lecturing can be analyzed toward students' proving skills. For instance, in the last interview the researcher connected the teaching styles with lecturer's rationale behind them. This connection offered an understanding of lecturer's goals regarding specific content and his interpretation of students' learning needs within the course.

Almost ten years later, Lew et al. (2016) investigated whether the lectures of an exemplary lecturer benefited students' learning, giving emphasis on the effectiveness of communication that happened in the lecture. For the analysis of a lecture, the researchers searched for the key points the lecturer presented in the teaching and classified them by using de Villiers' (1990) purposes of proof (verification, explanation, discovery, communication, and systematization). From a proof on the convergence of Cauchy sequences, five points were identified. An example of them, with a discovery purpose, was that

“one can prove that a sequence with an unknown limit is convergent by showing it is Cauchy” (Villiers, 1990, p. 175).

To investigate students' learning from the lectures, the researchers searched for four broad skills (listening, encoding, recording the points of the lecturer for example by taking notes, and reviewing) based on the frameworks of Suritsky and Hughes (1991) and Williams and Eggert (2002). It appeared that students did not recognize the identified points on the proof. One possible explanation was that students paid attention only to what was written on the blackboard. Another

explanation was that the students focused on the algebraic manipulations rather than the colloquialisms the lecturer used. Indeed, the main idea was delivered verbally to the students. The data used for this study was limited and the fact that the lecturer was exemplary did not necessarily establish effective communication among the students and the lecturer.

Issues related to communication and classroom discourses are typically addressed in studies that adopt a sociocultural framework. One such example is Hemmi's (2010) study, which examined lecturers' pedagogical views of proof teaching with data collected through individual interviews. The researcher considered the social practice perspective of proof. She argued that proof can be viewed as an artefact in mathematical practice mediating between the person and the social practice of the community of the university mathematics department of the study. The definition of proof was in accordance to the researcher's consideration of the role of proof within this community and was related to the acceptance, generation, continuity, stability, and precision of mathematical knowledge. A conceptual frame was created from the combination of the social practice perspective (Lave & Wenger, 1991; Wenger, 1998) with the existing literature in teaching and learning of proof, consisting of the pairs induction/deduction, intuition/formality, and invisibility/visibility. From the analysis, the researcher identified three styles (progressive, deductive, and classical) that highlighted lecturers' pedagogical views on proof teaching. The findings of this study highlighted the importance of research on proof teaching within sociocultural perspectives; the social practice perspective supported the identification of the role of proof within a mathematical community.

A few things become apparent from the findings of these studies. On the one hand, the studies provided the language to characterize the teaching styles of the lecturers that took into account students' learning (i.e., progressive, deductive, and classical (Hemmi, 2010), and logico-structural, procedural, and semantic (Weber, 2004)). On the other hand, the existing literature has not only provided the basis and tools for conceptual frames of lecturing but has also provided methodologies and tools for the analysis of the teaching of proof (e.g., Lew et al., 2016). The existing literature reported in this section can nevertheless provide the background for studying the teaching of proof, thus a broad categorization as a starting point, which sociocultural frameworks can use.

What has been missing in the knowledge base outside sociocultural frameworks is the way a community intersects in the lectures and influences the lecturers' engagement with the specific characteristics incorporated in their teaching. For instance, Hemmi's (2010) study, which examined proof teaching from a sociocultural point of view, addressed this gap to a

certain extent. Hemmi (2010) combined extant literature with the social practice perspective for the development of the conceptual frame, which considered the relation between the communities (e.g., the views of the members of a community around proof) and the social practice of proving (social, cultural, historical practices). The main benefit of the findings of this study is that proof took meaning from the mathematical practice and the teaching of proof was related to the participation of the newcomer to the historically established mathematical community while the value of proving has remained stable throughout the years serving similar functions. While Hemmi's (2010) study provided insights into how mathematicians talked about proof, the how and why their talk appeared in the teaching was missing. Commognition can offer micro-level insights into how and why the talk of the lecturer is incorporated into their teaching given that their talk includes their views about the various communities they participate. It is precisely the value of these micro-level insights into teaching proof in higher education that we argue about in this position paper, proposing the adoption of commognition as both a theoretical framework and pedagogical approach. In what follows, we substantiate our argument through a critical synthesis of related contemporary empirical studies that use commognition for the investigation of lecturing.

THINKING AS COMMUNICATION

Since the emergence of commognition (Sfard, 2008) researchers in mathematics education have turned their attention to communication and language (e.g., Kontorovich, 2021; Nardi et al., 2014). Commognition takes its name from the combination of the words, communication and cognition (Sfard, 2008). The inspiration of the commognitive framework came from the work of Vygotsky, Wittgenstein, postmodern philosophers, and the ongoing studies in linguistics (Sfard, 2008). The basic tenet of this framework is that thinking can be conceptualized as communication—the communication of a person with oneself (Sfard, 2001, p. 26). Hence, learning mathematics is conceptualized as participation in a special form of communication.

A discourse indicates an act of communication and mathematics is an

“historically established discourse” (Sfard, 2020, p. 95).

Generally, the discursive approaches are often described as participationist (Kieran et al., 2002; Nardi et al., 2014). While participating in a discourse, one is able to talk about the mathematical objects of the discourse, which appear and have specific meaning within the mathematical discourse (e.g., mathematical limits). Each (mathematical) discourse is distinguishable by community's word use, visual mediators, (endorsed)

narratives, and routines (Sfard, 2008, p. 133-135). *Word use* has to do with the characteristic keywords of the discourse (e.g., circle, function, four). *Visual mediators* are visual objects that are employed for communicational purposes (e.g., graphs, algebraic symbols). *Narratives* are, for example, the proofs, theorems, and definitions. These are sequences of utterances, which describe objects, relations among objects, or relations of processes with objects, and are subject to endorsement or rejection within a discourse. Lastly, *routines* are patterns that appear repeatedly and are characteristics of a certain discourse. The patterns within a discourse are the results of “rule-governed processes” (Sfard, 2008, p. 201).

The rules of a discourse can be object-level or meta-discursive (meta-rules) (Sfard, 2008). Specifically, as Sfard (2008) argued, object-level rules are

“narratives about regularities in the behavior of objects of the discourse” (p. 201)

such as mathematical rules or facts (Viirman, 2021). Meta-discursive rules (meta-rules) are narratives that

“define patterns in the activity of the discursants trying to produce and substantiate object-level narratives” (Sfard, 2008, p. 201).

Meta-rules relate to the participant of the discourse and to what the participant is doing. Because meta-rules relate to a person it is possible for them to evolve over time and vary. The rules that a student accounts as their own are called endorsed (Sfard, 2008, p. 204). Given these, two types of learning appear in commognition, the object-level and the meta-level learning. Object-level learning reflects an increase in the number and complexity of the endorsed narratives and the routines of a discourse with which the learner is already familiar. The result is an

“endogenous expansion of the discourse” (Sfard, 2008, p. 300).

The meta-level learning is associated with change towards the meta-rules of the discourse and reflects the transition to a new (for the learner) discourse. The change is “exogenous” (Sfard, 2008, p. 256); in other words, the previously accepted narratives and rules of the familiar discourse are changing to fit the new discourse.

The interest of the mathematics education community in using commognition to study the teaching of mathematics lecturers is drawn upon evidence that points to the potential of the commognitive approach to supporting teaching and learning. The commognitive framework emerged in the mathematics education field and provided the analytical tools to address critical research questions as for example, how students’ thinking is developed (Kontorovich, 2021; Morgan, 2020). More specifically, in university mathematics

research, the investigation of introductory courses and lecturers’ attempts to encourage students’ participation has drawn the attention of researchers. In the next paragraphs, we review the findings of studies that used commognition for the investigation of lecturing in mathematics. The studies explored the affordances of the framework by focusing on one or more of the four characteristics of a discourse (visual mediators, narratives, routines, and word use), essentially describing aspects of the different discourses of the lecturers.

Güçler (2013) identified and compared the characteristics of the discourses on limits of the students and of the lecturer in an introductory calculus course to highlight the instances where communication failed, suggesting different understandings of limit among the discursants. Focusing on the meta-rules, the framework allowed for an identification of two contexts, the informal definition of limits and the computation of limits, with a shift between them in the lecturer’s discourse. In particular, the lecturer “shifted” the elements of his discourse to affirm the operational and objectified narratives of limit (Güçler, 2013, p. 451). So the shift appeared in lecturer’s meta-rules, word use, and endorsed narratives to support the narratives “limit is a number” and “limit is a process”. Students used the same meta-rules, words and visual mediators for the endorsement of the same narratives of limits with the lecturer. However, these characteristics of students’ discourse were not so coherent as the lecturers’ ones and the students struggled with the transition from the dynamic to the more static aspects of limit. The commognitive framework supported the identification of the occasions where the communication failed, while the researcher connected the teaching of limits and its impact on students’ learning by focusing on the communication between the lecturers and the students. Güçler’s (2013) study focused on all four characteristics of the discourse.

Park (2015) examined three lecturers’ discourses on the derivative with a focus on the transition between the derivative at a point and the derivative as a function on an interval; a topic that students find difficult. Similarly, with Güçler’s (2013) study, Park (2015) focused on all four characteristics of the lecturers’ discourse. The lecturers taught an introductory calculus course and had different years of teaching experience. Through the commognitive analysis of lectures, the researcher identified components of the definition of the derivative (i.e., functions, difference quotient and limit, derivative) and patterns of the connections between these components by discerning the visual representations (symbols, graphs, gestures, algebraic representations) the lecturers used. In this study, the use of the words and visual representations of the lecturers characterized the routines and the narratives of their pedagogical discourse. The findings showed that firstly the lecturers’

discourse focused on the symbolic and algebraic representations of the derivative. Secondly, a limited discussion between lecturer and students was held on how the derivative as a function varies regarding its different representations. Thirdly, the lecturers used a few examples for the justification of its properties and employed several visual mediators in graphs of curves (e.g., secant lines, tangent line) but with no connections among them. The researcher, through the above findings about the lecturers' discourses and the documented students' difficulties in literature, also attempted to connect teaching and students' difficulties with the derivatives.

Viirman (2014, 2015) examined the teaching practices of the same group of lecturers during the teaching of functions in introductory university courses (basic algebra, linear algebra, calculus). Viirman (2014, 2015) in these two studies focused only on the routines of lecturers' discourse, in contrast to the previous studies from Güçler (2013) and Park (2015). The researcher identified the different routines in the mathematical and pedagogical discourses of the lecturers by analyzing the transcripts of the lectures, seeking for

“patterns and characteristics of the use of mathematical and didactical narratives” (Viirman, 2015, p. 1170).

Within the mathematical discourse, a categorization of construction routines (which aim at the creation of new endorsed narratives) and substantiation routines (which focus on the decision to endorse an earlier presented narrative) emerged (Viirman, 2014). The study showed that despite the fact that the lectures seemed similar in that the lecturers mainly talked and wrote on the blackboard, differences in the discursive practices of the lecturers and the way they communicated the mathematics appeared. These differences concerned the frequency in the occurrence of the identified routines in lecturing and their character, which comprised of a set of different types within the routines. For example, the different types of substantiation routines included definition verification, proof, and claim contradiction. As to proof routines, the introductory courses of Viirman's (2014) study did not offer opportunities to the lecturers to use them extensively. However, the investigation of proof routines is important for university mathematics because of the role of proof and proving in university mathematics discourse.

In a follow-up study, Viirman (2015) examined the nature of the lecturers' pedagogical discourse. Typologies of explanation, motivation, and question posing routines emerged from the analysis. In their pedagogical discourse of mathematical objects (e.g., functions), the lecturers used the routines in “different ways and to different extent” (Viirman, 2015, p. 1177), which is in agreement with the findings reported in the earlier study. In Viirman's (2014, 2015) studies, the

identification of routines in the mathematical and pedagogical discourses contributed to a characterization of teaching practices of university lecturers. The investigation of the routines in these two studies provided a fine-grained level of analysis. Other aspects of the discourse such as visual mediators were explored in further research.

The focus of these studies was on the communication between the lecturer and the students, in an effort of the former to introduce the latter to the new (for the students) mathematical discourse, as a more experienced participant of the discourse. The commognitive framework offered the language and the analytical tools to explore the teaching practices within lecturers' discourses and to identify when the communication occurred, how, when it failed and when it created misconceptions for the students. For instance, the framework provided insights into the reasons for the existence of some of the students' difficulties of connecting the visual representations and the words of the lecturers with their routines and narratives (Park, 2015). Moreover, by using commognition one can identify specific communicational patterns (which either help students or create their misconceptions) and communicational gaps, by focusing on the meta-rules of the discursants (Güçler, 2013). The commognitive framework was also useful for the characterization of teaching practices; for example, Viirman (2014, 2015) identified differences and similarities between different lecturers in different courses. In the following paragraphs, we move to a critical overview of the most recent studies that used the commognitive framework.

EXPLORING LECTURING WITH COMMIGNITION IN A MICRO-LEVEL

The studies in the previous section focused on the teaching and learning of mathematical topics (function, limit, and derivative) that can be explored with commognition. The researchers explored the affordances of commognition. More recently, researchers have considered the learning opportunities offered by the lectures to the students and lecturing beyond specific topics. The studies that we discuss next used the framework in a more delicate and complex way, providing a micro-level analysis of lecturing. These studies addressed aspects in lecturing that went beyond observations, highlighting the implicit aspects of teaching. The framework assisted the researchers to go behind the verbal communication and address issues with regard to curriculum (Pinto, 2019), enacted or endorsed meta-rules (Viirman, 2021), and feedback given (Kontorovich, 2021).

Pinto (2019) studied the variability (extent and nature of the diversity) in the formal and informal content of real analysis presented in lecture-format tutorials by two teaching assistants (TAs) who shared the same agenda.

Particularly, the researcher investigated the TAs' mathematical discourse (i.e., narratives in teaching and also the word use and routines within narratives) by focusing on the variability in the opportunities for object-level or meta-level learning of students. The instances, where the enactment of courses differed from the agenda, were first identified in the transcripts of the tutorials and later discussed in interviews centered around teaching decisions. The adaptations in the lesson plan at the object-level learning appeared to provide different opportunities for students' meta-level learning. For example, in the definition of derivative both TAs used discourse on the derivative at an object level by elaborating on the words, the visual mediators and the narratives of definition. However, with regard to students' meta-level learning, one emphasized the narrative that "each and every word counts in mathematics" whereas the other highlighted "how more experienced mathematics students, such as himself, do mathematics". The researcher found that the communication among the TAs in the preparation of the course happened in the object-level learning and not in the meta-level learning as they focused mainly on the mathematical narratives and rarely on the mathematical rationale. This study, part of a larger project, focused on one tutorial session (the first from each TA) while the interviews took place at the end of the semester (four months after the session). The researcher identified as a limitation the fact that the thinking and goals of the TAs at the interview may differ from those after the session. In accordance to this, Weber (2004) argued about a change in the goals of a lecturer during the semester in his study: at the beginning the lecturer focused on formal aspects of proof whereas by the end of the course to more informal aspects.

Similarly, Viirman (2021) analyzed the data of earlier studies (Viirman, 2014, 2015) by focusing on the mathematical discourse of lecturing in an effort to model lecturers' mathematical behavior towards students. The 2021 study differs from the previous ones (Viirman, 2014, 2015) of the researcher who had focused on the characterization of teaching practices. In the more recent study, he focused on the rules of "doing mathematics" having as a unit of analysis lecturers' discursive activity. For the analysis, firstly, the researcher searched for words that denoted the "normative aspects" of mathematical discourse, and for both rhetorical questions and routines related to the mathematical behavior students may recognize. Secondly, he identified and categorized meta-rules in instances that implied statements related to "doing mathematics". Lastly, he used the constant comparative method (Charmaz, 2006) for these instances, seeking the reification of the categories. A number of meta-rules emerged from the analysis within the mathematical discourse that were either endorsed (explicitly displayed in the lecturers' discourse) or enacted by the lecturers

(implied by their discursive activity). All the meta-rules that were identified had as a purpose to produce or substantiate a narrative about a mathematical object. The study revealed that meta-rules existed in the lecturers' discourse and the identification of them facilitated the researcher to model mathematical reasoning and behavior. Such an analysis could be fruitful for studying proof teaching in a lecture, as Viirman (2014) proposed, and studies of this kind could identify and categorize the narrative purposes and the rules related to the practice of proving in lectures. Also, lecturers' awareness of the existence of meta-rules could assist lecturers' reflection on their teaching. The researcher in this study used only observations of the lectures due to his goal to model lecturing. A limitation of such an approach is that the insights into lecturers' intentions towards students' learning were left unexplored as well as the relation of such intentions and identified meta-rules.

Kontorovich (2021) focused on the communication between the lecturer and the students, and between the lecturer and the mathematics education researcher, about the feedback the lecturer gave on students' written proofs. The researcher aimed to examine the affordances of the collaboration of researchers in mathematics education with professional mathematicians (lecturers). He asserted that the use of appropriate organizational frames can raise lecturers' awareness of their teaching practices and help researcher's investigation of teaching at university. The discourse that was examined was the didactical discourse on proof (DDP), which accounted as part of the pedagogical discourse that guided lecturers' teaching. DDP consisted of mathematical components (what accounts as a proof and how to produce it) and pedagogical components (about facilitating the participation of the newcomer into proving). Focusing on the DDP, the researcher analyzed the "comments and the points" (Kontorovich, 2021, p. 218) on lecturer's feedback on proofs to create the organizational frame. Reflection sessions took place afterwards where the researcher provided the lecturer with preliminary narratives that emerged from researcher's analysis of the feedback provided. The reflection sessions helped towards the triangulation of researcher's observations. The affordance of the collaboration of a researcher in mathematics education with a professional mathematician/lecturer was explored through a case study. Narratives were generated from the analysis of lecturer's feedback on students' written proofs, using the organizational frame. The findings of this study showcased the benefits of the organizational frame for the communication and collaboration among researchers and mathematicians towards the generation of narratives. Furthermore, they highlighted the characteristics of lecturer's feedback that were mainly around the idea and representation of proof, with a sensitivity towards students' proofs. Finally, the findings provided valuable insights into the feedback

given on proofs by the lecturer, setting the foundations for studying DDP in lecturing.

In the three studies reviewed in this section, a change in research foci has appeared from the specific context (function, derivative, limit) to the university teaching that may facilitate students' learning. The use of the commognitive framework highlighted a communicational aspect of university teaching even if there is a lack of dialogue in the lectures. Specifically, Viirman's (2014) analysis of meta-rules gave an understanding of the lecturer's teaching practices in lectures and provided more depth than observing solely the "more explicit object-level rules" by also exploring the meta-level rules. Pinto's (2019) work in the proof-oriented course opened up a discussion on the object-level and meta-level learning opportunities for students, offered by the lecturers. These opportunities were mainly implicit even though the lecturers discussed the agenda of the course before teaching. Lastly, Kontorovich (2021) highlighted the communicational aspects of lecturer's feedback and the importance of DDP for a reflection on the given feedback not only for the mathematicians but for all communities related to proof and proving. Thus, studying the teaching of proof by using commognition may add to the discussion and understanding lecturers' attempts to communicate the mathematics in university lectures towards students learning. Next, we focus on the value of using commognition for such an understanding.

TEACHING FOR SUPPORTING STUDENTS' DE-RITUALIZATION OF PROVING ROUTINES

One aspect of teaching that remains underexplored is teaching for students' de-ritualization of proving routines. Our synthesis shows that the majority of the research studies focused on examining the teaching and the meta-rules that appear within it. But the focus has not been yet on the meta-rules for the entrance of the newcomers and their participation in the university mathematical discourse. In this section, we propose a conceptual framework from the methodological standpoint of commognition to address this gap in the knowledge base, as evidenced in our critical synthesis of two lines of work: literature about proof teaching in the university and literature about the use of commognition in mathematics education.

Before we lay out our argument, we first further discuss about the routines of the discourse. Lavie et al. (2019) conceptualized learning as a process of routinization (in commognition, routines are defined as repetitive patterns, and characteristics of a discourse) of a person's actions. The actions in this case were the response to a given situation. This conceptualization had as a starting point that learning was participation of a person in a discourse. The researchers argued that

routines suffice for the communication of the person within the discourse, as they enable story telling about aspects of reality (Lavie et al., 2019). Thus, the routines may function as the unit of analysis in studies of learning. To achieve this, they first operationalized the routines as task-procedure pairs, highlighting in that way the personal and repetitive characteristics of the routines. The procedure is the response to the exact task, but also, it is identical to previous performances of a person for the response to similar tasks. Following that, the researchers proceeded to classifications of routines with first one the distinction between practical and discursive routines. The practical routines occurred in everyday physical actions, like cycling. In contrast, discursive routines happened for and from the sake of communication. The latter routines were discerned in rituals and explorations. Ritual routines appeared in a learner's discourse for social purposes and accounted as process-oriented routines with a focus on the performance and motivation by other, more experienced, participants. These routines were developed for the sake of social approval and for pleasing the expectations of someone else who is already a participant in the discourse, or for avoiding potential punishment. Rituals appeared to be stepping-stones for the participation in a new discourse, thereby enabling a ritualized participation in the discourse. Exploration routines were outcome-oriented routines and concerned the production of a narrative. These routines were developed for the sake of a narrative as they aimed at the generation and endorsement of the intended narrative by a learner. The motivation for the development of an exploration routine was intrinsic. Lavie et al. (2019) recognized that one valuable teaching goal was helping students transform the rituals into exploration routines; thus seeking for an explorative participation in the discourse.

The transformation of a ritual routine into exploration has been defined as the process of de-ritualization (Lavie et al., 2019; Sfard & Lavie, 2005). This process can happen vertically (building on previous results) or horizontally (bringing new approaches to achieve an already known result). The de-ritualization can be "gradual and slow", sometimes not finished during the school years (Lavie et al., 2019, p. 167). In empirical research, the occurrences of pure rituals or pure explorations rarely appeared as the outcome was not always separated from the specific procedure one needed to follow. Indeed, in between pure rituals and pure explorations is "a spectrum of intermediate possibilities" (Heyd-Metzuyanim & Graven, 2019, p. 143). To study the process of learning in terms of de-ritualization, researchers seek for changes in the learners' performance over time. In empirical research, these changes appeared to be characteristics of the routines that indicate a move towards the explorations when the attention of the learner shifts "from the

Table 1. Observed changes to explorative participation (Lavie et al., 2019)

Change	Observed changes in learners' performance
Flexibility	The learner performs a task in more than one ways
Bondedness	Each step of a procedure of the learner "feeds" the next step
Applicability	When the learner can isolate the performance and apply it on different occasions
Performer's agentivity	The learner makes more decisions on his/her own during the learning process
Objectification	The learner increases the level of abstraction of a mathematical object
Substantiability	The learner establishes the criteria to assess outcome of his/her performance based on his/her judgment

performance as such to its outcome" (Lavie et al., 2019, p. 167). In **Table 1**, a list with some changes to explorative participation, i.e., "desirable characteristics of routine" (Lavie et al., 2019, p. 167), is presented that occurred for the study of Lavie et al. (2019) as well as earlier empirical works of the researchers.

Analysis of teaching in lectures towards de-ritualization of proving routines has the potential to extend researchers' current understanding of proof lecturing towards students' learning. The conceptualization of routines as constructed by Lavie et al. (2019) concerned learning and what changes a researcher may observe in the discourse of a learner to identify if a student's shift towards the de-ritualization of routines appears. However, typically lecturers teach large cohorts of university students and usually perform proving routines for the students; students are passive listeners and do not actively engage in the learning process (e.g., Petropoulou et al., 2020). Hence, lecturers may perform a proof with more than one ways (i.e., flexibility), make connections between the different steps of the proof (i.e., bondedness), discuss the application of a proving process in other situations (i.e., applicability), make decisions for the proving processes, evaluate and showcase how one can explore them (i.e., agentivity), increase the level of abstraction of a mathematical object (i.e., objectification), and establish the criteria to judge and reflect on the essence/key ideas of the outcome of the proving process (i.e., substantiability). We propose an approach to studying proof teaching by the identification of such characteristics in the lecturer's discourse of proof teaching in order to gain insights into opportunities provided for the de-ritualization of proving routines in a lecture. Indeed, characteristics of this kind in lecturer's discourse may motivate students to de-ritualize their proving routines, thus supporting students' explorative participation in the mathematical discourse. This methodological approach provides a fine-grained analysis of teaching towards students' learning of proof in lectures.

Ritual and exploration routines grasped the attention of researchers after their introduction from Sfard and Lavie (2005). For instance, they appeared useful for the investigation of how lecturer's assumptions about learning appeared in her discourse and were connected with opportunities offered for ritual or explorative participation of university students (prospective elementary school teachers) (Heyd-Metzuyanim et al.,

2016). The assumptions of the lecturer concerned instances of mathematizing (i.e., failure is embarrassing). The connections between lecturer's assumptions about learning and ritual or explorative participation were made through both qualitative and quantitative analysis of lecturer's discourse. Also, in school mathematics education, Sfard (2017) investigated the teacher's talk and the opportunities given for students' transition from rituals to explorations in class. Moreover, interviews with students took place exploring the operationalization of these opportunities. Later on, a special issue with seven studies was published, where the researchers explored rituals solely but also as a dyad with explorations in learning, teaching, or learning-to-teach contexts (Heyd-Metzuyanim & Graven, 2019). Heyd-Metzuyanim and Graven (2019) synthesized the studies of this issue in three main themes: the logic of rituals and their persistence in class (Coles & Sinclair, 2019; Lavie et al., 2019; McCloskey et al., 2019; Nachlieli & Tabach, 2019), questions about the co-existence of rituals and explorations and whether they are binary or in a continuum (Heyd-Metzuyanim et al., 2016; Nachlieli & Tabach, 2019; Viirman & Nardi, 2019), and the different conceptualizations of rituals and explorations (Coles & Sinclair, 2019; McCloskey et al., 2019; Robertson & Graven, 2019). Most of the studies above focused on school mathematics and not on university mathematics. However, Sfard (2017) suggested that the transition to explorative routines may not take place during the school years. It thus is important to address this investigation in university education and explore the characteristics of teaching that could relate to students' explorative routines.

Lavie et al. (2019) offered a conceptualization of the rituals and studied students' de-ritualization of routines through the observation over time of the characteristic changes in students' routines. The study of rituals and explorations is new and earlier studies (e.g., in the aforementioned special issue) focused on rituals for the entrance to the school mathematical discourse. As discussed earlier, the knowledge base on students' ritual or explorative participation in the mathematical discourse is limited. Such an examination is significant given that it has the potential to showcase how students shift from an observer's position who imitates the experienced participants of the discourse, to a more independent and agentic position within the discourse.

CLOSING REMARKS

In this paper, we built upon literature on ritual and explorative participation and proposed an approach towards gaining a nuanced understanding of teaching proof in lectures for students' learning. We focused specifically on the characteristics of discourse that could promote students' de-ritualization of proving routines. A research focus on de-ritualization in university lectures will provide an understanding of exploration routines as well as opportunities for students' explorative participation in mathematical discourse. A way for researchers to analyze such opportunities for students is through an interpretation of characteristics in the lecturer's discourse of proof teaching, such as flexibility, bondedness, applicability, agentivity, objectification, and substantiability.

We maintain that further research on teaching proof in lectures will provide useful insights not only for researchers in mathematics education but also for lecturers who aim to support students in overcoming mathematical difficulties in introductory mathematics. Researcher's identification of characteristics such as flexibility and bondedness in lecturers' proving routines will foster researcher's understanding of teaching proof. Discussion with the lecturers about such characteristics in their discourse will support lecturers in reflecting on and redesigning their own practices.

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