MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2026, 22(4), em2805
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/18127

Designing a model-eliciting activity (MEA) to foster understanding of eigenvalues and eigenvector

Alexander Betancur-Sánchez [1,2*], Solange Roa-Fuentes [1]

[1] Universidad Industrial de Santander, Bucaramanga, COLOMBIA
[2] Bucaramanga Education Secretariat, Bucaramanga, COLOMBIA

Received 11 November 2025 ▪ Accepted 10 February 2026

Abstract
This article describes the design of a model-eliciting activity (MEA) that integrates GeoGebra drawing within the models and modeling perspective (MMP) and the action, process, object, and schema (APOS) theory to foster understanding of eigenvalues and eigenvectors. Insights from two pilot tests with linear algebra students led to refinement of the design to ensure alignment with the six principles of models and modeling theory, prior to its implementation with a group of university students in an introductory linear algebra course. Student work related to MEA generated during the implementation is analyzed using elements of genetic decomposition and modeling cycles. The reflections highlight the richness of the MEA, its potential to generate new situations and questions for further exploration of eigenvalues and eigenvectors, and its relevance for developing a more profound understanding of other fundamental concepts in linear algebra.

Keywords: MEA, APOS, MMP, linear algebra, eigenvalues, eigenvectors

INTRODUCTION

Recent research in mathematics and engineering education has increasingly focused on the development of students' mathematical thinking, particularly in the final years of secondary school and in the early years of university, through modeling situations (Hamilton, 2008; Huffman & Mentzer, 2021). In particular, several studies on the teaching of linear algebra have adopted the models and modeling perspective (MMP) (Lesh & Doerr, 2003) to introduce concepts such as systems of linear equations, linear dependence and independence, and eigenvalues and eigenvectors (Possani et al., 2010; Salgado & Trigueros, 2015; Wawro et al., 2019).

One of the approaches explored in this paper is the design and development of model-eliciting activities (MEAs). MEAs are carefully constructed situations that serve as research tools to document and assess student thinking while also encouraging students to invent, refine, and generalize mathematically robust constructions. In MEAs, concepts are tools for describing and explaining situations through the development of models; situations that are intended to respond to "real-life" simulations (Hamilton, 2008; Lesh & Doerr, 2003; Montero Moguel & Vargas Alejo, 2022; Sevinc & Lesh, 2018). For the MMP, the models:

> They are conceptual systems (consisting of elements, relationships, and rules governing interactions) that are expressed using external notational systems and used to construct, describe, or explain the behaviors of other systems–potentially allowing these systems to be intelligently manipulated or predicted (Lesh & Doerr, 2003, p. 10).

Eigenvalues and eigenvectors are fundamental concepts in linear algebra, particularly in science and engineering programs (Altieri & Schirmer, 2019; Karakok, 2019). These concepts are essential inputs to graduate courses in mathematics, physics, engineering, and technology (McDonald et al., 2024). Some studies suggest that university courses often incorporate extensive use of algorithms, exercises, and, in some cases, abstraction. While this approach may be beneficial for the professor, it can be an unfamiliar concept for students, as it often limits opportunities to reflect on and understand the concepts, their meanings, relationships, and applications (Betancur Sánchez et al., 2021; Salgado

✉ albetsan@correo.uis.edu.co **(*Correspondence)** ✉ doraroaf@uis.edu.co

Contribution to the literature

- This study contributes to MEA designs by applying the MMP framework within dynamic geometry environments, such as GeoGebra applets.
- A key contribution lies in documenting the MEA design process from the MMP perspective, guided by GD in accordance with APOS theory.
- This study contributes to the research literature on linear algebra didactics, specifically in the teaching and learning of eigenvalues and eigenvectors.

& Trigueros, 2015; Thomas & Stewart, 2011). Recently, studies on eigenvalues and eigenvectors have focused on designing learning situations that use modeling and computational technology to explore, validate, and model these concepts (Beltrán-Meneu et al., 2016; Possani et al., 2010; Salgado & Trigueros, 2015; Stewart et al., 2019; Trigueros, 2018).

In addition, the research has made complementary use of theoretical elements in mathematics education, such as the action, process, object, and scheme (APOS) theory and the MMP, to address in greater depth emerging research problems, particularly in linear algebra (Altieri & Schirmer, 2019; Salgado & Trigueros, 2015; Sevinc & Lesh, 2018).

This study contributes to illustrating the design process of a MEA that incorporates dynamic geometry (Febriani et al., 2024), using GeoGebra applets, and is based on genetic decomposition (GD) of eigenvalues and eigenvectors (Betancur Sánchez et al., 2022) within the framework of APOS theory. The article demonstrates the emergence of learning eigenvalues and eigenvectors as a dynamic process by modeling cycles using the MEA. By focusing the lens of APOS theory on modeling cycles, this study examines the mental structures and mechanisms involved. Specifically, this article aims to present evidence of the schemas students mobilize when engaging in MEA and to examine how the interaction between the linear transformation (LT) schema and the basis schema might facilitate the learning and understanding of eigenvalues and eigenvectors.

For the study reported here, two guiding questions were: Which MEA fosters learning about eigenvalues and eigenvectors among science and engineering students? What mental structures do students develop when they construct, share, manipulate, modify, and reuse mathematically meaningful systems (i.e., build models) through MEAs involving eigenvalues and eigenvectors?

The first question guided the study in designing the MEA based on the six principles of Lesh et al. (2000) and the GD on eigenvalues and eigenvectors proposed by Betancur Sánchez et al. (2022). The second question motivated placing the GD lens on the emerging modeling cycles of the MEA, in particular on the interaction of schemes from the APOS theory lens.

THEORETICAL ELEMENTS

This study draws on the MMP and APOS theories. It first presents fundamental considerations regarding MMP and principles for MEA design. APOS theory is then articulated, specifying the GD of the concepts of eigenvalues and eigenvectors, and the discussion concludes by showing how these theoretical elements are integrated.

Models and Modeling Perspective

The MMP conceives learning mathematics as a process in which students develop conceptual systems (models) that are continually adjusted, extended, and refined through interaction with activities, peers, and teachers (Lesh & Doerr, 2003). To structure experiences in which students can express, test, and refine their mathematical thinking, we have employed intentionally designed situations, such as "real-life" simulations, in which students' conceptual understanding can be directly documented and assessed (Lesh et al., 2000). Accordingly, the design of MEAs emphasizes the students' production process when solving problems that require developing, testing, revising, and communicating mathematical models. As noted by Lesh and Doerr (2003), MEAs allow observation of how students build and refine their mathematical ideas, enabling identification of the "conceptual tools that they can share, manipulate, modify, and reuse (models) to build, describe, explain, manipulate, predict or control mathematically significant systems" (Lesh & Doerr, 2003, p. 10).

Engagement with MEAs supports the development of mathematization processes, including quantifying, dimensioning, coordinating, categorizing, and algebraizing (Lesh & Doerr, 2003). Through these activities, students establish modeling cycles that foster skills in generating tests, refining models, and providing explanations, thereby enhancing the process of building mathematical models within MEAs.

To design MEAs that effectively reveal students' thinking, Lesh et al. (2000) propose six design principles:

1. *Reality principle (or meaning):* It aims to have students make sense of the problem using their personal knowledge and experiences. The proposed situation should require students to

apply mathematical knowledge so that the solution is not trivial (Lesh et al., 2000).

2. *Principle of model building:* the situation posed should require more than simply answering a question; it is expected to demand explanations, predictions, and justifications to develop a model, thereby motivating the transformation into a mathematical language (Lesh et al., 2000).
3. *Principle of self-assessment:* This principle provides criteria for assessing the usefulness of possible responses to advance model development. The goal is for students to exercise control over the situation's objectives by recognizing when a change in the model is necessary (Salgado & Trigueros, 2015).
4. *Documentation principle:* The given situation must require students to explicitly reveal their thinking, including the objectives and solutions considered. That is, students must contribute to learning and documentation by showing the constructs that enable their reasoning, while simultaneously encouraging self-reflection (Lesh et al., 2000).
5. *Principle of simplicity:* Activities must be both challenging and accessible to students, so that, once solved, they can serve as essential prototypes for learning, and the model and reasoning can be reused to solve other situations (Lesh & Doerr, 2003).
6. *Principle of generalization:* A guiding question for this principle is: Is the model developed from the activity valid only for the situation? The goal is to guide and encourage students to develop general ways of thinking rather than focusing on specific contexts. The development of the model "must become a new mathematical object that students can apply" (Salgado & Trigueros, 2015, p. 26).

APOS Theory

GD is considered the core of the theory. It functions as a cognitive model that describes the mental constructions and mechanisms required for a student to understand a mathematical concept (Arnon et al., 2014). In relation to the concept of eigenvalues and eigenvectors, Betancur Sánchez et al. (2021) propose a preliminary cognitive model, and Betancur Sánchez et al. (2022) report its subsequent refinement. For the development of this study, the validated GD of eigenvalues and eigenvectors serves as a conceptual foundation for the design of the MEA, referred to as the "design of a new clock" (Betancur Sánchez et al., 2022).

The following is a summary of GD, highlighting key elements necessary for understanding eigenvalues and eigenvectors.

The cognitive model is based on a description of fundamental mental structures that enable the construction of new concepts. In this context, an LT scheme is essential (González Rojas & Roa Fuentes, 2017; Roa-Fuentes & Oktaç, 2012) along with a Vector Space Scheme at an inter-level (Parraguez & Oktaç, 2010). The Inter level aims to generate interactions between the elements of each scheme and, through the concept of a vector space basis, to promote interaction between the LT scheme and the vector space scheme. A conception of the process of factoring a polynomial over a field, denoted as K, is also necessary, as well as tools to determine its existence and calculate its roots.

The inter level of the LT scheme allows students to recognize a linear operator as a LT defined on a (finite) vector space V, that is, $T\colon V \to V$ it is a linear operator on a field K Let $\lambda_0 \in K$ and $v_0 \in V$; the construction of the concept of eigenvalue and eigenvector begins by coordinating the processes of scalar multiplication and LT, from which the expression is obtained $T(v_0) = \lambda_0 v_0$. At this stage, it is recognized that the null vector is not an eigenvector.

From the internalization of an action structure that allows the student to find solutions to the vector equation $T(v) = \lambda_0 v$ (using the three interpretations of a LT (González Rojas & Roa Fuentes, 2017); the student can now determine the values λ_0 such that $T(v) = \lambda_0 v$. Thus, two processes must be coordinated to generate a new process, which corresponds to determining all the characteristics of the system associated with the expression obtained in the second process.

Process 3 is coordinated with the null space process, and process 4 emerges, allowing recognition that the solution set of $T(v) = \lambda_0 v$ and $Nul(T - \lambda_0 I)$ are equal and contain the respective eigenvectors associated with λ_0 that is, the equivalence between $T(v) = \lambda_0 v$ and $(T - \lambda_0 I)\, v = 0$ (different matrix representations of T) is stablished.

The eigenvalue and eigenvector process is finally structured through the coordination of process 4 with the determinant process. In this construction, the student establishes the biconditional relationship between the existence of an eigenvalue and its corresponding eigenvector. Through reflection on the totality of the eigenvalue and eigenvector process and the encapsulation mechanism, the eigenvalue and eigenvector objects are obtained. Within this structure, students can consider an ordered pair (λ, v), where the first element is an eigenvalue and the second an associated eigenvector, according to the linear operator T, to perform new actions, such as diagonalizing the operator. Betancur Sánchez et al. (2022) provide a detailed description of these mechanisms, which emphasizes the complexity involved in developing an understanding of this concept.

In APOS theory, the set of actions, processes, objects, and other underlying structures that are coherently related forms a schema of a concept K (Arnon et al.,

Table 1. Characteristics of the population participating in each stage of the study

	Pilot 1 (P1)	Pilot 2 (P2)
General characteristics of the population	Students enrolled in linear algebra in science and engineering programs, who had previously studied linear transformations defined in $\mathbb{R}^2$ and $\mathbb{R}^3$	
Number of students	20	9
Implementation time	2 hours	3 hours
Data collection instruments	Worksheets, class video recordings, and the researcher's log	
Student coding	$P2E_1, P2E_2, \ldots$	$P2E_1, P2E_2, \ldots$

2014). Learning is dynamic, and as relationships and transformations, conscious or unconscious, are built between actions, processes, objects, and other schemas, it can evolve (Betancur Sánchez et al., 2021). The intra-, inter-, and trans-triad allows us to characterize the relationships built between the structures of the schema, whether through assimilation or accommodation, and to account for its development (Trigueros, 2018). When faced with a problem-solving or modeling situation, a student may need to coordinate different schemas (Arnon et al., 2014).

Complementary Use of MMP and APOS

The complementary relationship between MMP theory and APOS theory has been explored in several studies (Possani et al., 2010; Salgado & Trigueros, 2015). In general, within APOS, tasks are designed to create environments that foster the construction of mathematical knowledge, while MMP focuses on mobilizing and applying this knowledge in realistic situations. In this sense, MEAs promote the introduction and development of new mathematical concepts for students; in this research, the concepts of eigenvalue and eigenvector. This approach allows tasks designed considering GD to foster the construction of new, useful conceptual tools that support subsequent work with the model (Salgado & Trigueros, 2015; Trigueros & Oktaç, 2019).

Initially, the GD described in Betancur et al. (2022) is considered to ensure that the MEA design aligns with the mental constructs of the GD and the theoretical orientations of the six design principles of the MEA (Lesh et al., 2000). Subsequently, two pilot tests were conducted to verify compliance with these principles. Based on this validation, the MEA was implemented to teach the concepts of eigenvalues and eigenvectors in a linear algebra course for science and engineering students, using the GD as a reference. Finally, the students' mental constructs, mobilized during the intervention, were analyzed using the GD. In particular, the interaction between the LT scheme and the base scheme was examined. Thus, in each modeling cycle related to the MEA, the APOS theory allows for an in-depth study of the mental processes that generate learning and consolidate the understanding of the concepts of eigenvalues and eigenvectors.

RESEARCH METHOD

This qualitative study is divided into three main stages:

(1) design of the MEA,

(2) its implementation, and

(3) the recognition of mental constructs and mechanisms related to the concepts of eigenvalue and eigenvector.

MEA Design

The GD, as reported in detail by Betancur Sánchez et al. (2022), served as the basis for designing the MEA in line with the six principles proposed by Lesh et al. (2000) and two pilot tests. **Table 1** presents the population characterization for pilot test 1 and pilot 2.

Pilot test 1 primarily aimed to examine whether MEA supported the prior constructions described in the GD for eigenvalues and eigenvectors, as well as to evaluate its form and content in relation to the six MEA design principles.

Pilot test 2 primarily sought to determine whether the adjustments introduced after pilot test 1 improved compliance with the six design principles proposed by Lesh et al. (2000) and to proceed with implementing the MEA in the classroom.

The first researcher was the teacher in charge of the groups in which the two pilot tests were conducted. This ensured that students who had not received prior instruction in eigenvalues and eigenvectors during the intervention semester were included.

MEA_0. Design of a new clock–Pilot 1

The clock design was the situation proposed to students to capture their attention and invite them to act as creators of a new design. The initial conditions and the guiding questions are presented below. The use of GeoGebra in MEA_0 was incorporated to explore design possibilities and engage students in an analytical approach, as suggested by Febriani et al. (2004).

MEA prompt: Imagine you are an artist who has been asked to design an elliptical clock based on a circular wall clock with a radius of 1 decimeter. Considering the center of the circular clock as the origin of a coordinate plane, the design requirements are as follows:

a. If it is 12:00 on the circular clock, in the new design, the location of the end of the minute hand should be one unit up and one unit to the right with respect to its 12:00 position.
b. If it is 3:15, in the new design, the location of the end of the minute hand should be one unit up from its 3:15 position.

From the above: Describe how any point on the circular clock must be transformed to construct the new design.

Suppose the circular clock shows 6:20. Consider the minute hand as a vector. Describe specifically how the position of the minute hand changes to achieve the new design. At what other position would the minute hand undergo a similar transformation? What would be the approximate length of the minute hand for this case?

Suppose the minute hand can be represented as a vector $\boldsymbol{m} = [\boldsymbol{a}, \boldsymbol{b}]$ and some modifications are made to the design requirements so that an E model can describe it. What characteristics must the vector meet such that the transformation under the E model is like that found in item b?

Pilot test 1 was implemented in a university computer lab. The research asked the 20 participating students to form groups of two or three. Initially, the professor presented the MEA_0 document and indicated where the GeoGebra file could be accessed (see, https://www.geogebra.org/m/bsqacavh). The researcher circulated among the groups to discuss students 'developing ideas. When necessary, feedback and broader discussions were facilitated with the entire class. The 2-hour session was video-recorded, and the students' worksheets, along with the researcher's field notes, were collected. Once the intervention concluded, the research team reviewed the video recording and worksheets multiple times. Similar productions and key fragments of dialogue between students and teacher-researchers were identified, transcribed, and analyzed. Selected extracts are presented below, according to the primary purpose of pilot test 1. Students were labeled following the format $P1E_1, \ldots, P1E_{20}$, where P# refers to the pilot, E_k to the students and the subscript (k) indicates the order.

In both group discussions and whole-class conversations, it was observed that the students were able to translate the conditions of MEA_0 in mathematical language involving vectors and LTs. In the following transcript, the student $P1E_1$ provides insight into the construction model associated to MEA_0.

> $P1E_1$: […] basically, we need to find a linear transformation using the "clues" they gave us. […] so, if we use a vector to represent this (referring to conditions a). and b). of the MEA_0), it is as if it went from$[0, 1]$ to $[1, 2]$ […] I did the same with the other one, from $[1, 0]$ *to* $[1, 1]$. I want to see how a is transformed $[x, y]$ by the linear transformation […] I am trying to identify which? matrix can help me.

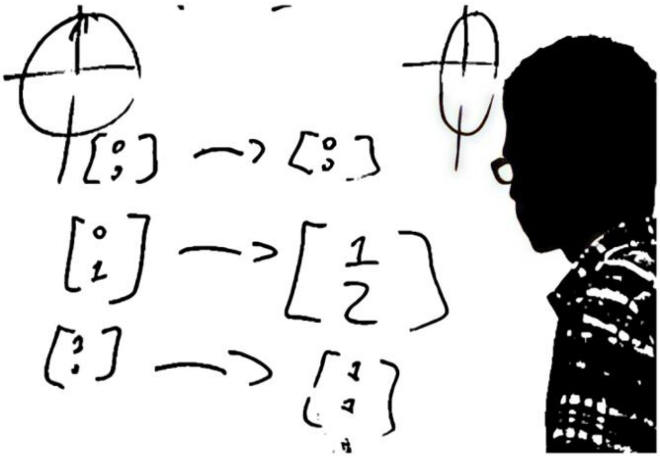

Figure 1. Interpretation of student $P1E_1$ on MEA_0 (Source: Authors' own elaboration)

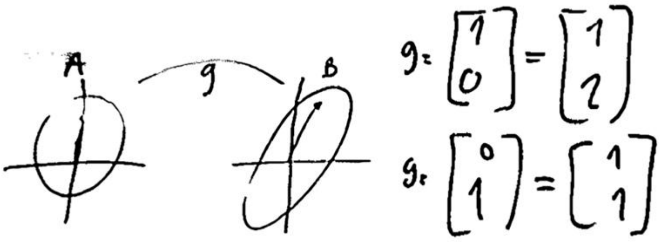

Figure 2. Interpretation of student $P1E_4$ and $P1E_5$ on MEA_0 (Source: Authors' own elaboration)

Figure 1 shows the work written by student $P1E_1$ on the board; this is considered a first approach to developing the model. During the intervention, $P1E_1$ drew on prior constructions related to LTs and bases, although the latter was not made explicitly. In the following dialogue, the students $P1E_4$ and $P1E_5$ articulate these constructions more explicitly.

> $P1E_5$: We are initially must-see what matrix g generates our linear transformation and forms the new design […]
>
> $P1E_4$: We take the canonical base and find the images […] (see **Figure 2**)

Students $P1E_1, P1E_4$ and $P1E_5$ conceptualized an LT as a matrix. By explicitly referring to the use of the basis and the LT, we consider that the students were mobilizing their LT scheme and vector space scheme. Although the problem did not directly refer to these concepts, students were able to draw on their prior knowledge to address the questions in MEA_0. The evidence from pilot test 1 highlights three aspects to consider in the design of the MEA:

(1) the activity supports the mobilization of prior constructions described in GD,

(2) it is necessary to include a condition referring to positions where the minute hands do not overlap; this implies adjusting the GeoGebra applet interface, and

(3) the wording of item 3 in MEA_0 should be improved for greater clarity.

Regarding the GeoGebra applet interface, two graphical representations are required: the first illustrates the design of the circular clock and its deformation into an ellipse; while the second displays both the circular and elliptical outlines together with the vectors. This dual representation promotes the construction of relationships among the associated mathematical elements, such as collinearity, change of base, and the use of additional GeoGebra tools.

MEA_1. Design of a new clock-Pilot 2

Considering the aspects mentioned for adjustments of pilot test 1, MEA_1 was proposed. According to Febriani et al. (2004), the combination of MEA with GeoGebra fosters more engaging, in-depth, and meaningful mathematical learning experiences, helping students to explore mathematical concepts interactively and writing more realistic contexts. Furthermore, the use of the GeoGebra applet in the MEA_1 was intended to facilitate the connection between visual and symbolic representations, particularly in relation to the effects on the clock design of changing conditions or a change of base. Additional guiding questions regarding the use of the GeoGebra applet were added to guide the intervention as presented below.

MEA prompt: Imagine you are an artist who has been asked to design an elliptical clock based on a circular wall clock with a radius of 1 decimeter. Considering the center of the circular clock as the origin of a coordinate plane, the design requirements are as follows:

a. If it is 12:00, in the new design, the location of the top end of the minute hand should be one unit above and one unit to the right of its position on the circular clock.

b. If it is 3:15, in the new design, the location of the top end of the minute hand should be one unit above its position on the circular clock.

Based on these conditions: How is any point on the circular clock transformed to construct the elliptical design? Propose a model for the new clock design.

1. Consider the minute hand of a circular clock as a vector m, and for an elliptical clock, the vector $T(m)$. What are the vectors m, and $T(m)$ when the clock indicates 10 minutes? What geometric relationship exists between these vectors? Does this relationship hold for other vectors?
2. Explore the GeoGebra applet [select the "show clockface" checkbox and move the point. If they exist, which vector(s) in the circular design is the transformation invariant in the new design? What characteristics do these vector(s) have? Justify your answer P?
3. Let $m = [a, b]$ be a vector representing the position of the minute hand on the circular clock and let E denote another model of an elliptical clock. Sketch and describe the characteristics and conditions required for the minute-hand transformation to have invariant positions under E.

Pilot test 2 was implemented in a computer lab with a different group of students than those in pilot test 1. The intervention lasted three hours. As in pilot test 1, a video recording of the students' group work was made, the research provided overall feedback, and the students' written production was collected. The teacher indicated the moments when the GeoGebra applet should be used (https://www.geogebra.org/m/q85bec4d). After the intervention, the research team carefully reviewed the collected information to identify evidence of compliance with the six design principles of an MEA and the GeoGebra applet's relevance. The participating students were labeled as $P2E_1, \ldots, P2E_9$.

Below are some highlights from the research team's analysis of pilot test 2, specifically regarding the mobilization of GD's mental constructs, the role of the condition related to the minute hand's position, and the work with the GeoGebra applet.

Instructions to students: In your group, explain to your two classmates how you understand the problem. A fragment of this discussion is presented below (student $P2E_6$)

> $P2E_6$: They tell me that at twelve o'clock on the circular clock (locate the point), which the vector corresponds $to[1, 0]$. Now 3:15 would be this point (he locates it), $[0, 1]$. At 3:15 they tell me that the end of the minute hand should be one unit up […] I have the images of the unit vectors that would form the basis of my clock (he writes the respective coordinates, which are the images of the vectors of the canonical basis). So, from these vectors (he points to the ones in the basis), having the images of the transformation, we can now know the transformation matrix.

In the previous dialogue, the group $P2E_6$ draws on their prior experience and knowledge of clocks and identify mathematical elements. As mentioned in pilot test 1, the MEA favors the mobilization between the LT scheme and the vector space scheme. With the request for the clock design, which involves positioning the minute hand at the 10-minute, the students are encouraged to explore and reason with the model they have constructed. In the dialogue, the researcher carries out the following procedure together with student $\mathrm{P2E}_4$ (**Figure 3**).

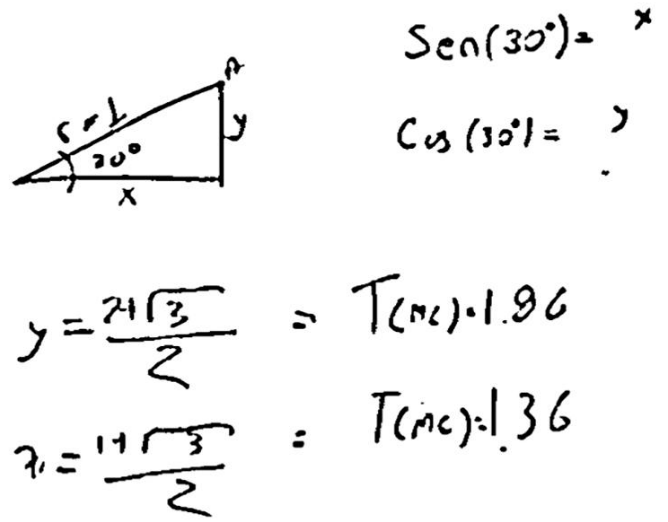

Figure 3. Procedures of $P2E_4$ for determining the position of the minute hand in the new clock design at the 10-minute mark (Source: Authors' own elaboration)

The analysis of the students' worksheets, along with transcripts of the video recordings, shows evidence that MEA_1 allows students to record their progress and reasoning, either verbally or in writing. The question about the relationship of the minute hands at the 10-minute mark supports the principles of documentation and self-assessment. The use of the GeoGebra applet, as evidenced in pilot test 2, facilitates the validation and construction of new clock designs while allowing students to recognize the mathematical concepts identified before using the applet: vectors, collinearity, basis, and LT. Furthermore, students benefited from the use of GeoGebra tools such as drag and drop and matrix mapping. Following the research team's analysis of pilot test 2, the MEA was revised to align with the company's request The questions were adapted to better engage students, increasing realism, and open the possibility of considering different models that met the established conditions. Finally, it was decided not to include the GeoGebra applet, allowing students to develop various approaches in mathematical language and stimulate their imagination. In this regard, the MEA indicated that the company would contact them if necessary, and this communication could include the GeoGebra applet.

From the evidence related to pilot test 1 and pilot test 2, compliance with the six principles for the design of the MEA for eigenvalues and eigenvectors is documented.

1. *Principle of simplicity:* The situation is accessible to students in the sense that the elements involved are understandable and familiar. It allows for the inclusion of progressively more complex mathematical concepts. For instance, students can draw clock hands, associate them with vectors $\mathbb{R}^2$, and then include concepts such as LT and basis.
2. *Reality principle:* Students make sense of the problem by using graphs or geometric representations and can also recognize the need to apply mathematical knowledge to advance the situation. Students are familiar with the parts of a clock and how it works, they can transfer this knowledge to the proposed problem. This modeling problem does not involve a trivial answer; on the contrary, it requires the application of advanced mathematical concepts.
3. *Model-building principle:* The situation requires transforming a real-life context into mathematical language. Students can consider the LT (linear operator) as a model, act upon it to describe the problem, and establish conditions for the existence of positions where the minute hands of both designs overlap (i.e., vectors lie on the same straight line—the GeoGebra applet can be helpful). In this way, students can work with eigenvalues and eigenvectors using LT representations.
4. *Generalization principle:* Students can consider different LTs as models for elliptical clock designs (the GeoGebra applet can be helpful). They can think of vectors in the vector space as having deformations that can be described by a scalar multiple. Discussions between students and the teacher can focus on finding ways to determine the eigenvectors (vectors) and eigenvalues (scalars) of a given linear operator, as well as the conditions for their existence.
5. *Self-assessment principle:* The conditions proposed in the MEA statement can be used both in construction and in assessment and verification. For example, the condition that the minute hands cannot overlap at the 10-minute mark allows students to evaluate whether the model they constructed satisfies this condition. This also makes it possible to consider refining the model, where the GeoGebra applet can be helpful.
6. *Documentation principle:* Students can verbally express their reasoning, conjectures, and the model that describes the new clock design. They are also required to provide illustrations, descriptions, and conditions; in other words, students are expected to express their thinking in their written work explicitly.

Given the evidence on compliance with the principles proposed by Lesh et al. (2000) and the mobilization of the necessary constructs identified in the GD, the MEA was used to implement the teaching and learning of eigenvalues and eigenvectors. The final version of the MEA used for the study is presented below.

MEA. Design of a New Clock

After conducting two pilot tests and making the necessary adjustments, the research team proposed the final version of the MEA in accordance with the first guiding question of the ongoing research. The MEA presented below is the one used in the implementation, and from which the data analysis emerges.

MEA prompt

Imagine you are part of the design team of a company interested in creating new clocks. You are asked to design a new clock with an elliptical outline, starting from a circular one with a radius of 1decimeter. Considering the center of the circular clock as the origin of a coordinate plane, the requirements for the new design are as follows:

a. If it is 12:00, the location of the tip of the minute hand in the new design should be one unit above and one unit to the right of the minute-hand tip on the circular clock.
b. If it is 3:15, the location of the tip of the minute hand on the new design should be one unit above the tip of the minute hand on the circular clock.
c. If the circular clock indicates 10 minutes, the minute hands of the two designs cannot be superimposed.

The company will contact you when it deems this necessary. The work you must do is:

1. Find a model for the new design that describes how any point on the circular clock is transformed. If overlap exists, explain where the minute hands coincide and describe their relationship.
2. Propose alternative models of elliptical clocks that permit positions where the minute hands overlap. Provide illustrations and describe the characteristics and conditions that produce such overlaps.

Application of the MEA

The MEA is applied to a 30-student linear algebra I class at a public university in Colombia. The students were enrolled in science and engineering programs and had not received formal instruction on eigenvalues and eigenvectors prior to the MEA. The teaching implementation followed the activities, classroom discussion, and exercises (ACE) teaching cycle, as defined by APOS theory, and took place in a computer classroom. The opening for teaching eigenvalues and eigenvectors was the MEA presentation. Students were initially organized into groups of three. The researcher presented the MEA and provided an initial overview. Subsequently, discussions are held between groups and/or in general. Additionally, specific tasks related to the MEA were assigned, and the ACE cycle was repeated throughout the implementation (Arnon et al., 2014).

The MEA "designing a new clock" was conducted through two 2-hour sessions. In the first session, each group produced an initial approach to the MEA without using the GeoGebra applet. The professor and researcher circulated among groups to facilitate progress and, when appropriate, invited the two groups to share their work or provided general feedback to the whole class. In the second session, students used the GeoGebra applet to explore different clock designs as required by the MEA.

During the intervention, video recordings of group work, discussions, and whole class feedback were made. The GeoGebra applet screen was captured using PowerPoint, and students' worksheets were collected at the end of each session. The researcher also recorded field notes at the close of each session to document observations and highlights for later analysis.

Recognition of Mental Constructs and Mechanisms

At the end of the MEA implementation, the researcher reviewed the video recordings of each session and each student group's written output. During a discussion session between the teacher and the researcher, similar across-group outputs were identified, and those that offered the most detailed explanations and documentation were selected for transcription. Excerpts from the written output that supported the participants' dialogues were included in the transcripts, along with the researcher's field notes. Through source triangulation and discussion within the research team, key moments in the students' model construction were identified: cycles in which students drafted explanations and clock designs. In the modeling cycles, the analysis focused on identifying the mobilization of mental mechanisms and constructs associated with eigenvalues and eigenvectors, as described in the GD, particularly the interaction between the LT scheme and the basis scheme. This analysis shows how such constructions enable progress in understanding and addressing the MEA students' requirements.

RESULTS

The analysis is organized according to the modeling cycles identified in the student work related to the MEA "designing a new clock." Each modeling cycle was assigned a representative phrase that reflected the students' progress in approaching the MEA and in developing their constructs and mental mechanisms. In accordance with section 2.3, special attention will be paid to the interaction between the LT scheme and the base scheme. Therefore, specific sections are not assigned to each structure in APOS. These are

(1) recognition of conditions expressed as vectors,

(2) initial approximation to the model,

(3) need for model accuracy and validation, and

(4) recognition and establishment of conditions.

The labels assigned to students in the transcripts and to reference their written, verbal, or digital production in the GeoGebra applet environment were $IE_1, \ldots, IE_{30}$.

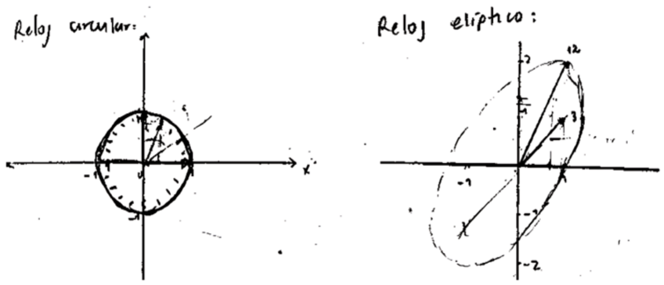

Figure 4. Graphic representation of the clocks designs by IE_4 (Source: Authors' own elaboration)

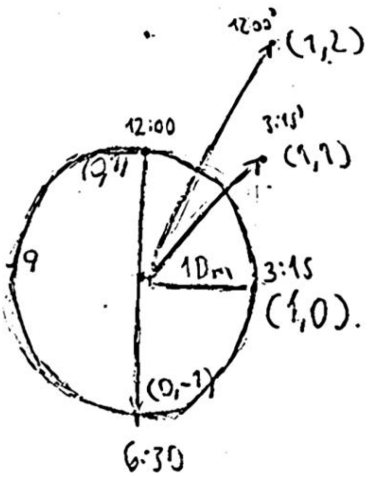

Figure 5. Interpretation by IE_1 of the MEA conditions in terms of vectors that are transformed (Source: Authors' own elaboration)

Recognition of Conditions as Vectors in $\mathbb{R}^2$

During the first 30 minutes, students in each group analyzed the proposed situation. The researcher approached the different groups, listened and observed the students' reasoning, and interacted with them and the teacher.

The researcher identifies how students interpret and represent the situation: they produce a drawing of the circular clock and a drawing for the new design (see **Figure 4**). Student IE_4 and their group create sketches on the Cartesian plane with the aim of visually identifying the hands of the new clock and their position at 12:00 and 3:15 (without explicitly referring to numerical vectors in $\mathbb{R}^2$). When the students identify the position of the minute hands with coordinates on the Cartesian plane, they shift their attention from the change in the location of the arrows as representations of the minute hands to the change, or transformation, of the numerical coordinates. **Figure 5** shows one of student$IE_1's$ worksheets, created with their group. The paragraph written by IE_1 describes their reasoning and the plan they consider viable for finding the model for the new clock design.

> We start from the circular clock with a radius of 1 decimeter, graph the given hours, and begin by locating the coordinates of the minute hand position on the new clock based on the given conditions. We note that we used the canonical vectors of $\mathbb{R}^2$ and transformed them, so we can use these coordinates as the columns of what will be our transformation matrix.

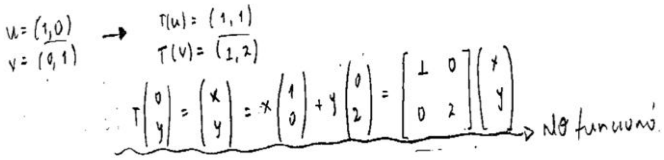

Figure 6. Production of IE_{12} to find the linear transformation of the new model (Source: Authors' own elaboration)

By writing down the coordinates of the minute hands, student IE_1 identified that the coordinates for the circular clock correspond to the canonical vectors of $\mathbb{R}^2$, which are transformed into other vectors for the new design. The researcher records in his field notes, "Students who cannot recognize condition a and condition b in the MEA as numerical vectors fail to establish a plan or advance in their work to find the model." The groups of students who transform the given information into vectors appeared to activate both their basis scheme and LT scheme. That is, once they identified the canonical basis, they associated each vector in the basis with its image and showed evidence of recognizing that this change could be described or generated by an LT.

In **Figure 6**, student IE_{12} and their group recognized conditions a. and b. for the new clock design as vectors and their respective images through an LT. In other words, they considered that there exists a transformation T, such that $T(u) = [1, 1]$ and $T[v] = [1, 2]$ (see **Figure 6**). In attempts to determine the LT, IE_{12} proposes that $T\begin{bmatrix}0\\y\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$ and attempted to express a linear combination between vectors before considering an equality with the matrix product by a vector. This procedure shows an effort to describe the family of vectors that on the vertical axis of the rectangular system for $\mathbb{R}^2$. However, the impossibility of analytically finding the transformation seems to correspond to the fact that IE_1.

IE_{12} was unable to coordinate the Basis Process $\mathbb{R}^2$ of and the LT process. This prevented recognizing the associated matrix with the LT or using basis and linearity properties to determine an algebraic expression for it. Nevertheless, the group of students working with IE_{12} managed to recognize that their model did not satisfy the company's request (they wrote "it didn't work" at the end). Based on this, the professor and the researcher encourage other students in the group to review their reasoning and validate whether the model satisfied the conditions established by the company.

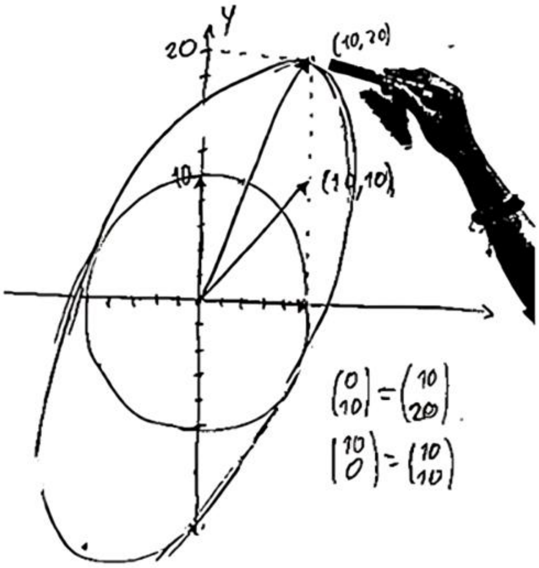

Figure 7. Production of IE_7 regarding the conditions of the modeling problem (Source: Authors' own elaboration)

> Taking the position of the minute hand of the circle as vectors that are transformed to give the new elliptical clock design.

Approach to the Model

The discussions generated in each work group were consolidated into a general discussion that included all the students, the teacher, and the researcher. Student IE_7 was invited to the board to present his group's productions and reflections. **Figure 7** shows the drawing produced by IE_7, which interprets all the information about the situation in mathematical terms: circle, ellipse, numerical vectors, basis, LT, and associated matrix.

> IE_7: […] we thought of a circle of one decimeter, but we expressed it in centimeters because it was too small […] at twelve it said that it should be one unit upwards, that is, 20 centimeters. Then it moved another unit to the right […] then it reached 10 in x and that perpendicular distance was projected and at 20, so there was the coordinate […] the other data they gave us was like a smaller radius of the ellipse.

> IE_7:. We drew an ellipse that touched the circle (referring to the work in his group) […] Another thing we did was to assign coordinates or vectors to these (indicating the positions of the minute hand at 12:00 and 3:15) This point we must be[10,0]and the other [10,0]. We identified that this was like linear transformations. We see that [10,0] and [0,10] are like a basis […] we related these vectors with their transformation. For [0,10] the transformation was[10,20] and for $[10, 0]$ it was [10,10]. But that is as if it were a canonical one; in the end, it can be placed as $T = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$ the linear transformation of the elliptical clock.

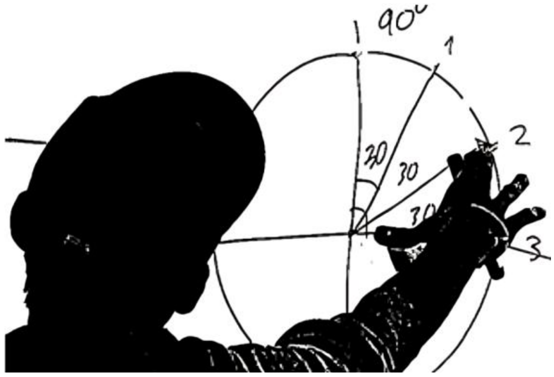

Figure 8. Reasoning ofIE_8 to relate angles to minute hand positions (Source: Authors' own elaboration)

The student IE_7 shows details about the procedures and reflections developed in his group. The students not only recognize the given data as vectors in $\mathbb{R}^2$, but can identify that $[10, 0]$ and $[0, 10]$ as a basis for $\mathbb{R}^2$. Moreover, they express themselves in terms of the canonical basis to directly determine the matrix associated with the LT. This provides evidence that they have constructed the concept of an associated matrix and established the relationship between the transformation and its matrix representation through the basis. Therefore, we argue that the interaction between the basis scheme and the LT scheme plays a fundamental role for IE_7 and his group in the construction of a preliminary model for the new clock design.

In the sketch proposed by IE_7, the ellipse touches the circle tangentially at two points (see **Figure 7**). As a result of the IE_7 presentation, and because some groups disagree with this interpretation, the condition involving the 10-minute position was discussed in class. Subsequently, another student from the group went to the board and presented an alternative idea they have been developing, also related to construction of model.

> IE_8: […] for the 10-minute we were relating it with angles […] If we divide this quadrant (First quadrant) of 90° into sectors of 30°, then we will have one hour, two hours and three hours […] in the first we will have 5 minutes, and in the second 10 minutes.

> IE_{13} : I don't understand the angles (referring to $IE_8's\ explanation$, see **Figure 8**).

> IE_8: […] what we were looking at was that in this part (indicates 3:15 on the circular clock), so that in the new design it would be 3:15, it would be practically like here (draws a straight line indicating 20 minutes, see **Figure 9**) like 5 continuous minutes downwards […]

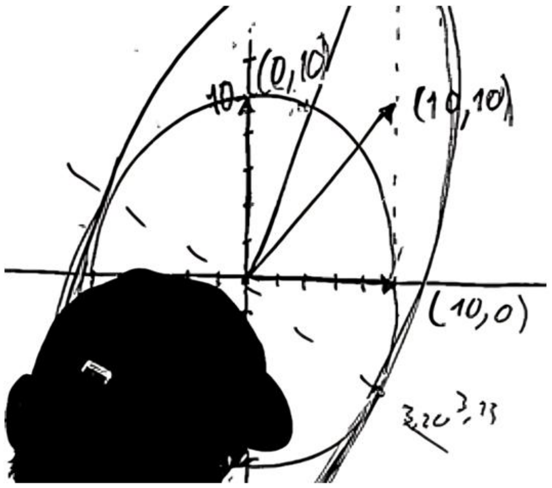

Figure 9. Comparison between the two clock designs when they mark 3:15 by IE_8 (Source: Authors' own elaboration)

> IE_2: […] I was wondering if you could use some trigonometric identity to relate the sides when dividing that quarter of a circle into 30° angles. […] Another thing I think is that the ellipse should be more flattened. The circle should be slightly on the outside.

> IE_{11}: […] when I entered the vector, it gave me a vector that is outside of that ellipse [1,1]

> IE_8: I would think this is just a sketch (points to the outline of the ellipse).

> IE_2: We find the transformations of all the ends of the circle (and draw the ellipse that passes through the ends of those vectors ($[1,0]$; $[0,1]$; $[-1,0]$; $[0,-1]$)

The student IE_8 considers condition c of the MEA. The analysis of this item allows students to reflect on the model that they have constructed with the aim of refining it and determining if it satisfies the condition established for the minute hand when it indicates 10 minutes. The need to determine the exact vector at the 10-minute mark and its corresponding transformation in the new design leads IE_8 to relate minutes to degrees on the Cartesian plane. The intervention of IE_{11}, who question whether the vector $[1,1]$, belongs to the circle (radius 1 and centered at the origin), raises an issue of the model validity. Although the student does not formally recognize that [1, 1] does not lie on the unit circle the question prompts classmates to reconsider the accuracy of their model. Meanwhile, IE_2 shows broader progress by identifying not only the images of $[1,0]$ and $[0,1]$ but also those of $[-1,0]$ and $[0,-1]$. By plotting the images of these four vectors and sketching the ellipse that passes through them, IE_2 demonstrates that the relationship between the circle and the ellipse proposed by IE_8 is not correct.

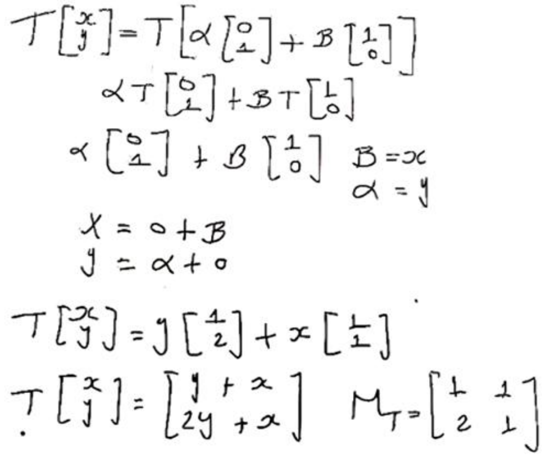

Figure 10. Reasoning of IE_{12} which involves the linearity conditions of a linear transformation and the concept of basis, presented on board (Source: Authors' own elaboration)

At this point, the students acknowledge the progress achieved in developing the model while also recognizing the aspects that require refinement. Their reasoning has already involved the use of mathematical concepts from linear algebra, trigonometry, among others. The first session concludes with an open question regarding the validity of the model and the necessity for its improvement. The teacher encourages the students to continue reflecting on their model and to prepare their analysis and comments for presentation in the following session.

Need For Model Accuracy and Validity

At the beginning of the second session of the MEA implementation, a general discussion takes place in which students present the work they have developed during the first session. The following dialogue emerges from a question posed by researcher to the student IE_{12}: "How can we describe how each position of the minute hand on a circular clock transforms into the minute hand on an elliptical clock?"

> IE_{12}: […] what we did was look at the canonical vectors of $\mathbb{R}^2$ […] in the linear transformation, we can look at the basis vectors. In this case, the vectors are the canonical basis of $\mathbb{R}^2$, which allows me to understand how to apply it $[x, y]$ to anyone.

The response of student IE_{12} demonstrates his understanding of the basis process, as he recognizes that any vector can be expressed as a linear combination of the vectors in a basis of $\mathbb{R}^2$. In particular, the basis he considers corresponds to an inverted order of the canonical basis vectors of $\mathbb{R}^2$. Moreover, by coordinating the basis process with the LT process, this student is able to find a general form for the model, as well as determine the associated matrix with respect to the canonical basis, as evidenced by the procedures outlined in **Figure 10**.

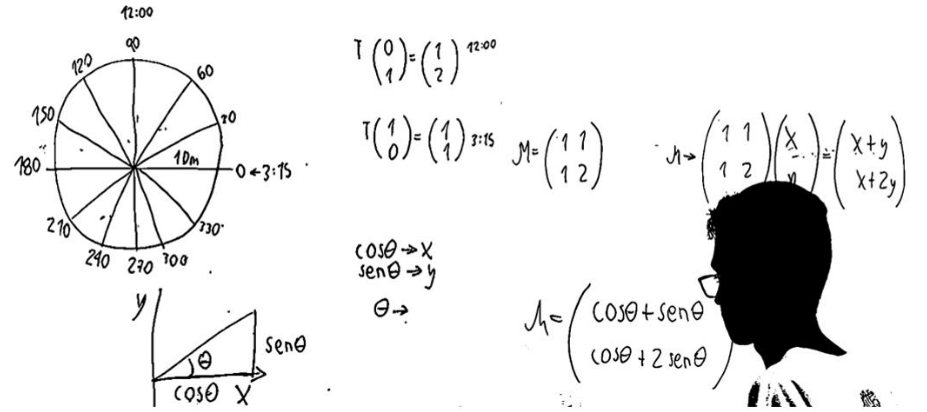

Figure 11. Reasoning of IE_1 to refine the design model of the new clock (Source: Authors' own elaboration)

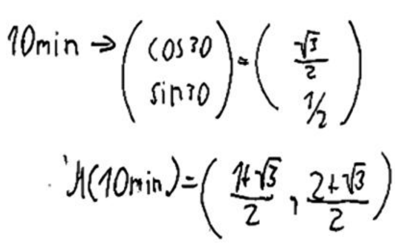

Figure 12. Coordinate vector for the 10-minute position in the new clock design as determined by IE_1 (Source: Authors' own elaboration)

To continue, the researcher asks the students about the relationship proposed in the previous session between angles and the positions of minute hand. Building on this prompt, student IE_1 decides to share his reflections and progress:

> IE_1: […] I can place an angle at each position of the minute hand on this circular clock […] we have to see what the coordinates of each point would be according to that little triangle […] I can relate my coordinate in x with $cos(\theta)$ and y with $sin(\theta)$ and then I can relate the value of θ to the circle. If we want to know how any point [x, y] or position is transformed in terms of θ , then we can multiply it by the matrix, and now we have it on terms of θ (**Figure 11**).
>
> IE_{20}: So, we can use this to look at the 10 minutes?
>
> IE_1: […] Yes. In the ten minutes, you see that it would be like thirty degrees, do you see that? […] then you just insert the value here and that's it.
>
> I: So, what would be the coordinates on the circular clock and in the new design?
>
> IE_1: well, it would be doing the math (asks classmates calculate the result, see **Figure 12**).
>
> IE_7: They are only there every five minutes
>
> IE_1: […] We simply have the angles of thirty by thirty […] so basically what I did was divide each piece into five. Do you understand? […] so when dividing it, we will get for each minute […] it would give me an equivalence of 6°.

The refinement of the model presented by IE_1 allows his classmates to systematically associate each position of the minute hand on the circular clock with an angle and use the proposed transformation model to obtain the corresponding position in the new elliptical design. In this way, the model gains both precision and relevance. At this stage, the student focuses on determining whether there exist positions in which the minute hands overlap. Specifically, after recognizing the transformation that describes the passage from the circular to the elliptical design, they concentrate on studying, through the properties of the LT, whether a vector and its corresponding image are collinear at certain position of the minute hand. After making certain approximations and verifying that, for the 10-minute position, the minute-hand vectors of both designs are collinear, the researcher informs the class that the company owner has sent a simulation of the design for further comparison and analysis, based on the reasoning they have developed. At this point, the students proceed to explore GeoGebra applet (https://www.geogebra.org/m/q85bec4d), where the new design is displayed and can be contrasted with the models they themselves proposed.

Recognizing and Establishing Conditions

Exploring GeoGebra allows students to validate and expand their ideas about the proposed model. For example, some students observed that "the ellipse is more flattened than the ellipse IE_8 had drawn." This investigation also enables them to identify multiple positions where the vectors overlap: "There are four positions, two we have already found, and the other two correspond to opposite directions. Here (points to 5 minutes), there is one, and the opposite of this would be this (points to 35 minutes). These two seem to stretch equally when deformed for the new design." Subsequently, the teacher and the researcher approached different students and asked them to

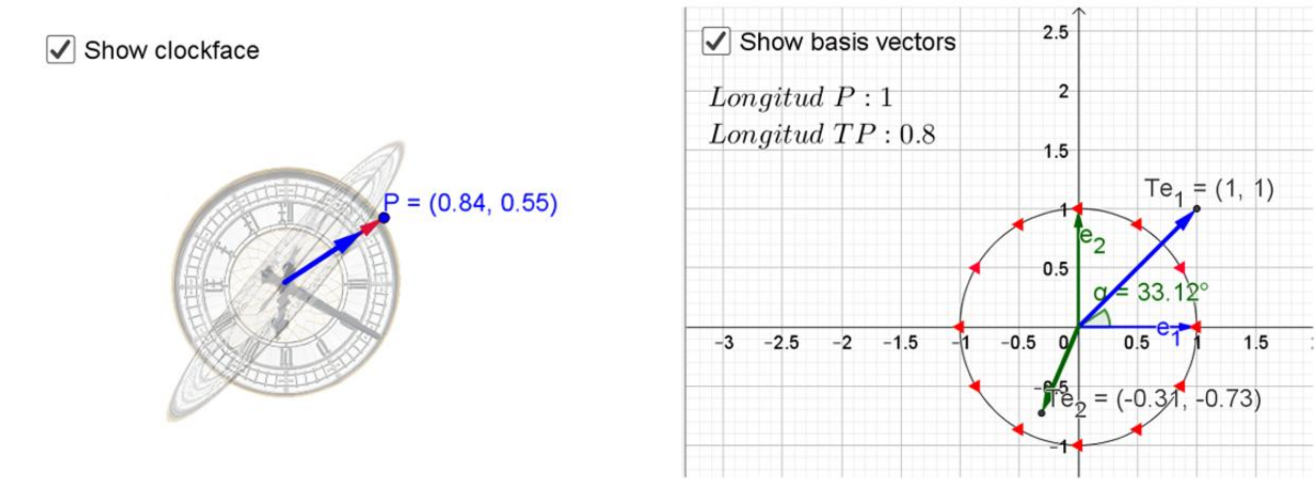

Figure 13. Graphical exploration views in GeoGebra on the MEA (the authors' own elaboration)

describe how the transformation occurs at these four positions of the minute hand. The students referred to these cases as: "they are on the same line," and "through a reduction or enlargement."

These excerpts from student productions illustrate how working in the GeoGebra environment promotes self-assessment and provides feedback on the documentation students had previously generated. Additionally, it allows students to expand their analysis of geometric cases and relationships at the specific positions where the minute hands of both clock designs overlap. In the following discussion, students reflect on how transformations of the canonical basis can define new clock designs.

> IE_{22}: […] we can change the dimensions, sorry, the size of the new clock by moving the vectors (referring to the vectors Te_1 and Te_2 , images of the canonical vectors $\mathbb{R}^2$, see **Figure 13**).

> IE_1: what changes is the transformation.

> IE_{10}: […] the images of the base vectors are being modified […] and this causes the transformation to be modified and changes the design of the clock.

Exploring and analyzing the two graphical views in GeoGebra (see **Figure 13**) allows students to continue reflecting on the concept of basis and LT, particularly in $\mathbb{R}^2$. The right-hand window of **Figure 13** shows the basis vectors and their respective images. Students modify the transformations of the canonical basis vectors in $\mathbb{R}^2$ and, by varying the point P (in the left-hand view) along the circumference, they determine that there are positions where the vectors overlap, that is, they become collinear.

In working with GeoGebra environment, the teacher and researcher guide students to recognize cases where

(1) some vectors in $\mathbb{R}^2$, when transformed, become collinear,

(2) some vectors never become collinear, and

(3) all vectors in $\mathbb{R}^2$, when transformed, lie on the same straight line that contains them.

This instructional guidance was informed by the GD framework used to teach eigenvalues and eigenvectors. Below are examples of student productions, along with the illustrations they reference in their dialogue.

> IE_{16}: […] Here the shape of the clock is almost like a vertical ellipse […] and we can see that the minute hands are superimposed […] in this case the vectors are on the same line (see part a in **Figure 14**) […]. Still, if we place the image of $e_2$over here (remember the vector in the fourth quadrant) the clock hands no longer overlap […] no matter which direction the vectors go, you cannot put them on the same line (see part b in **Figure 14**).

The students' work with the GeoGebra applet fosters a reflective environment in which they validate, reconstruct, or generate new ideas about models that meet the company's conditions. They can identify and explain the positions at which the vectors associated with the minute hands of circular and elliptical clocks are collinear. Furthermore, they formulate new questions and explore relationships between mathematical concepts. At the end of the second MEA class session, some students presented to the professor and researcher additional clock designs that had emerged from group work without the use of GeoGebra.

> First find it by reflecting $T_m\begin{bmatrix}1\\2\end{bmatrix}$ on the axis y and $T_m\begin{bmatrix}1\\1\end{bmatrix}$ on the axis x giving a model similar to the previous one, preserving its characteristics and conditions where$\begin{pmatrix}-x\\y\end{pmatrix}$ is the reflection matrix on the axis y and $\begin{pmatrix}x\\-y\end{pmatrix}$ the reflection matrix on the axis x.

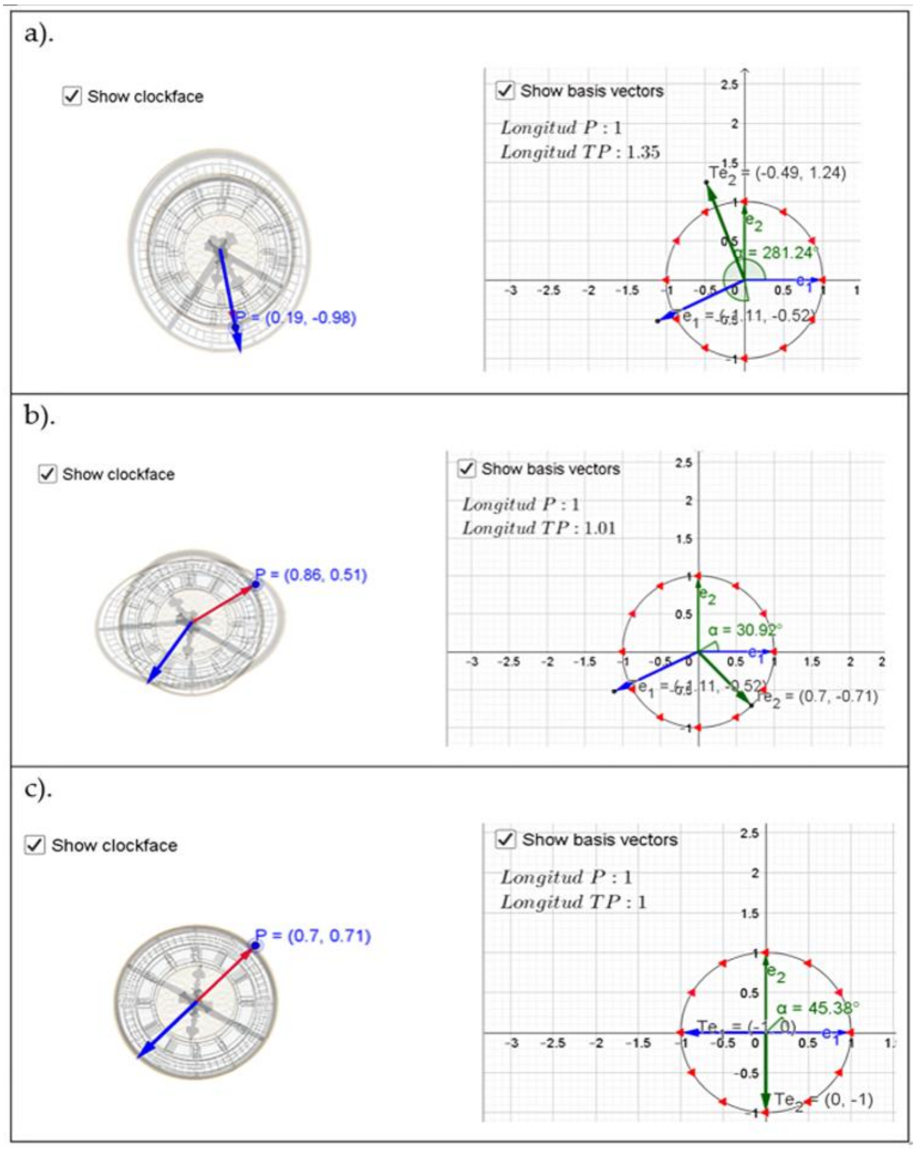

Figure 14. Other clock designs proposed by IE_{16} in GeoGebra and their relationship with eigenvalues and eigenvectors (Source: Authors' own elaboration)

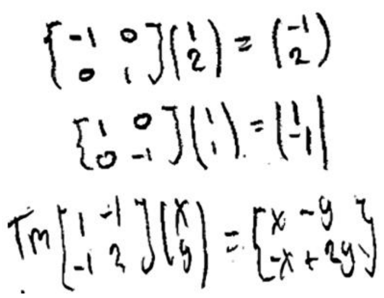

Figure 15. Strategy of IE_{20} to construct other clock models in which there are positions for the minute hands where they overlap (Source: Authors' own elaboration)

As illustrated in **Figure 15**, the student's IE_{20} proposed building an alternative model by applying reflections of the vectors with respect to the x and y axes. This approach opened the discussion on whether the linear operator of the original clock design and the one proposed by IE_{20}share the same eigenvalues, And if so, whether they have the same associated eigenvalues. In this sense, the MEA not only provides a context for introducing eigenvalues and eigenvectors but also encourages to investigate further the relationships and properties of linear operators in $\mathbb{R}^2$ and their matrix representations to specific bases of the vector space.

> When the transformation affects only one axis, the vectors coincide at the points on the $x-$ axis and the $y-axis$, in the same way, $\binom{nx}{y}$ for the x −axis and $\binom{x}{ny}$ for the $y-axis$.

Students were able to generate families of models in which the condition of overlapping minute hands is satisfied. In **Figure 16**, the student IE_{21}considered a family of designs in which "the transformation affects only one axis" Specifically, in one case, the vector $[1, 0]$ is transformed into $[n, 0]$ while $[0, 1]$ remains invariant; in the other case, $[1, 0]$ remains unchanged while $[0, 1]$ is

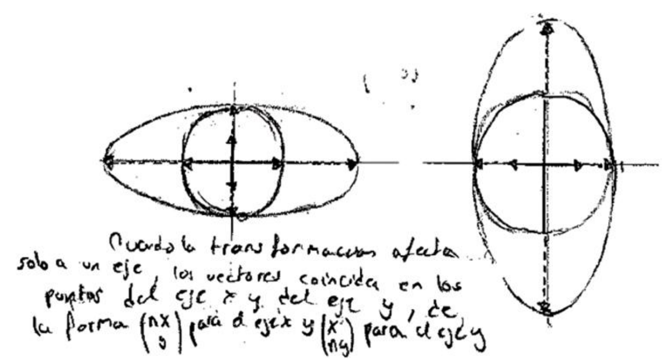

Figure 16. Family of clock models where there are overlapping minute hand positions in the circular and elliptical designs proposed by IE_{21} (Source: Authors' own elaboration)

transformed in to $[0, n]$. IE_{21} demonstrates recognition of the functional form of the two families of transformations:

(1) $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} nx \\ y \end{bmatrix}$ and

(2) $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ ny \end{bmatrix}$.

Furthermore, the student indicates in the drawings that the vectors lying on the $x-$axis and $y-$axis correspond to invariant vectors under these transformations, that is, they are eigenvectors associated with their respective eigenvalues.

DISCUSSION

Some research involving the use of MEAs (Baker & Galanti, 2017; Dede et al., 2017) shows great potential for fostering learning in a modeling context. By making complementary use of the MMP perspective and APOS theory in this research, attention was focused not only on the design but also, from a cognitive perspective, on what might be happening in the student's mind regarding learning and understanding. From the first modeling cycle, as shown in the results for the MEA "design of a new clock," the interaction between the LT scheme and the basis scheme was evident, consistent with the reference GD used in the research (Betancur et al., 2022). This is not surprising, as mentioned in the theoretical framework and confirmed by Trigueros (2018): modeling situations mobilize the evolution and/or interaction between conceptual Schemes in mathematics.

The three situations presented in the GeoGebra applet were analyzed and discussed with the students to promote recognition of the basis for characterizing the linear operator and its invariants, such as eigenvalues and eigenvectors. The teacher and the researcher used these cases to guide reflection on how an LT acts whiting the vector space domain. That is, some transformations have a "special" effect on certain vectors, whereas others act uniformly on all vectors (Betancur Sánchez et al., 2021; Parraguez González et al., 2022). In relation to the GD proposed by Betancur et al. (2022), the three situations support the coordination between the LT process and the scalar multiplication process, while simultaneously activating the LT scheme through the recognition of cases in which a linear operator exerts a "special" effect. During implementation, the teacher and the researcher guided the students in the developing procedures to identify vectors $v \epsilon \mathbb{R}^2$ and scalars $\lambda \epsilon \mathbb{R}$, such that $T(v) = \lambda v$. In this way, the instruction was aligned with GD to promote the construction of the associated mental structures. These results constitute a contribution of the present study in relation to other studies on eigenvalues and eigenvectors, however, they are aligned with that proposed by Trigueros and Oktaç (2019).

Given that students often encounter difficulties when other vector bases are involved (Mendoza-Sandoval et al., 2021; Plaxco et al., 2018; Zandieh et al., 2017) it is important to consider that the eigenvalue is a structural invariant of the LT. Eigenvalues are intrinsic properties of an LT and remain invariant under a change of basis. Eigenvectors, as elements of the vector space, are also independent of the basis; however, their coordinate representations depend on the chosen basis. This highlights the need to foster interaction between the LT scheme and the basis scheme, which in this research was achieved through the designed MEA.

CONCLUSIONS

The research process enables us to answer the two guiding questions posed at the end of the introduction. Regarding the question, "What thought-revealing activity (MEA) promotes learning about eigenvalues and eigenvectors in science and engineering students?", the empirical evidence resulting from the research shows that the MEA "designing a new clock" promotes the learning of these concepts, as demonstrated in the results section. The MEA design report showed that, through implementation in two pilot tests, the MEA was adjusted, refined, and made more relevant to promote students' mathematical activity related to eigenvalues and eigenvectors. Each implementation contributed to adjusting the design conditions of the elliptical clock to ensure compliance with the six principles of Lesh et al. (2000). Furthermore, it ensured the mobilization of prior mental constructs established through GD and the clear and timely presentation of information and instructions for the pedagogical intervention.

Furthermore, regarding the question, what mental structures do students develop when they construct, share, manipulate, modify, and reuse mathematically meaningful systems (i.e., build models) in a mathematically significant learning environment (MEA) involving eigenvalues and eigenvectors? The results of the study show that the MEA "designing a new clock" is an effective scenario for motivating student interest and mobilizing their mental constructs related to the LT scheme and basis scheme to proceed with the construction of eigenvalues and eigenvectors. For instructional development, the MEA became a generator of connections with other concepts, such as collinearity, the transformation matrix with respect to a basis, and the recognition of features associated with linear operators, as shown in **Figure 14**. Through guided exploration and reflection on the GeoGebra applet, students found examples where eigenvectors do not exist, where eigenvectors exist in a nontrivial subspace, and where every vector in $\mathbb{R}^2$ is an eigenvector. In each case, it was a valuable opportunity to recognize the eigenvalue associated with each corresponding eigenspace.

The four modeling cycles presented in the analysis reveal the moments and stages in the refinement of the company's proposed model for the design of the new elliptical clock. By using GD from the perspective of APOS theory (Arnon et al., 2014) in each modeling cycle, new mental constructs can be observed, whether through process coordination, process encapsulation, or the dynamization of schemas. For the MEA considered in the study, evidence is presented that the interaction between the LT schema and the vector space schema contributes to the construction of the eigenvalue and eigenvector concepts, in accordance with Betancur et al. (2022). The complementary use of the PMM perspective and APOS theory ensures a pedagogical intervention design in a learning environment conducive to the MEA approach to a new concept. Furthermore, in each modeling cycle, model refinement deepens the mental constructs necessary for learning and understanding the concept from the perspective of GeoGebra design, a differentiating aspect in the research of Possani et al. (2010) and Salgado and Trigueros (2015).

Limitation

One limitation of the study is the number of students in whom the learning activity "design of a new clock" was implemented. Future studies could implement it across various science and engineering courses and investigate the role of GeoGebra applets in learning about eigenvalues and eigenvectors. Another limitation of the study is the integration with specific tasks to foster the other mental constructs necessary for understanding the concepts.

Contributions

Methodologically, this study contributes to MEA designs (Baker & Galanti, 2017; Dede et al., 2017) by applying the MMP framework within dynamic geometry environments, such as GeoGebra applets (Febriani et al., 2024). A key contribution lies in documenting the MEA design process from the MMP perspective, guided by GD in accordance with APOS theory. While previous studies, such as those by Possani et al. (2010) and Salgado and Trigueros (2015), have applied MMP and APOS complementarily, the MEA design process itself has not been systematically documented. As mentioned by Trigueros and Oktaç (2019), task design in mathematics education is widespread and serves multiple purposes, yet making the design variables and strategies visible is essential for advancing research and supporting the educational community.

Finally, this study contributes to the research literature on linear algebra didactics, specifically in the teaching and learning of eigenvalues and eigenvectors. The MEA "design of a new clock" represents a novel and effective approach to introducing these concepts, integrating theory, visualization, and student reasoning within a structured pedagogical framework (Salgado & Trigueros, 2015).

Author contributions: AB-S & DSR-F: conceptualization; **AB-S:** data curation and writing–original draft; & **DSR-F:** formal analysis and writing–review & editing. Both authors agreed with the results and conclusions.

Funding: This study was partially supported by the project 4247 and the mobility program of the vice-rectorate for research and extension of the Universidad Industrial de Santander (VIE-UIS).

Acknowledgments: The authors would like to thank the participants for their willingness to contribute.

Ethical statement: The authors stated that the study was approved by the institutional ethics committee of University of Santander on November 1, 2024. Written informed consents were obtained from the participants.

AI statement: The authors stated that no AI was used in the preparation of this article; however, the ChatGPT translator was used to assist in translating the manuscript from the original language into English.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Altieri, M., & Schirmer, E. (2019). Learning the concept of eigenvalues and eigenvectors: A comparative analysis of achieved concept construction in linear algebra using APOS theory among students from different educational backgrounds. *ZDM Mathematics Education, 51*, 1125-1140. https://doi.org/10.1007/s11858-019-01074-4

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. Springer. https://doi.org/10.1007/978-1-4614-7966-6

Baker, C. K., & Galanti, T. M. (2017). Integrating STEM in elementary classrooms using model-eliciting activities: Responsive professional development for mathematics coaches and teachers. *International Journal of STEM Education, 4*, Article 10. https://doi.org/10.1186/s40594-017-0066-3

Beltrán-Meneu, M. J., Murillo-Arcila, M., & Albarracín, L. (2016). Emphasizing visualization and physical applications in the study of eigenvectors and eigenvalues. *Teaching Mathematics and Its Applications, 36*(3), 123-135. https://doi.org/10.1093/teamat/hrw018

Betancur Sánchez, A., Roa Fuentes, S., & Ballesteros, S. J. (2021). A preliminary genetic decomposition of the concept of eigenvalue and eigenvector: The analysis of text books as a substrate in the construction of cognitive models. *Revista Latinoamericana de Investigación en Matemática Educativa, 24*(3), 245-276. https://doi.org/10.12802/relime.21.2431

Betancur Sánchez, A., Roa, S., & Parraguez, M. (2022). Mental constructs associated with eigenvalues and eigenvectors: Refining a cognitive model. *Avances de Investigación en Educación Matemática, 22*, 23-46. https://doi.org/10.35763/aiem22.4005

Dede, A. T., Hidiroğlu, Ç. N., & Guzel, E. B. (2017). Examining of model eliciting activities developed by mathematics student teachers. *Journal on Mathematics Education, 8*(2), 223-242. https://doi.org/10.22342/jme.8.2.3997.223-242

Febriani, P. A., Mandailina, V., Abdillah, A., Syaharuddin, S., & Mehmood, S. (2024). Utilizing GeoGebra-assisted model-eliciting activities (MEAs) in mathematics instruction enhances students' comprehension of concepts and improves their problem-solving abilities. *Jurnal Inovasi Pembelajaran, 10*(1), 19-30. https://doi.org/10.22219/jinop.v10i1.25819

González Rojas, D. E., & Roa Fuentes, S. (2017). Linear transformation scheme: Construction of abstract objects from the internalisation of concrete actions. *Enseñanza de las Ciencias. Revista de Investigación y Experiencias Didácticas, 35*(2), 89-107. https://doi.org/10.5565/rev/ensciencias.2150

Hamilton, E. (2008). Model-eliciting activities (MEAs) as a bridge between engineering education research and mathematics education research. *AEE Journal 1*(2).

Huffman, T. J., & Mentzer, N. (2021). The impact of modeling-eliciting activities on high school student design performance. *International Journal of Technology and Design Education, 31*(2), 255-280. https://doi.org/10.1007/s10798-019-09557-x

Karakok, G. (2019). Making connections among representations of eigenvector: What sort of a beast is it? *ZDM Mathematics Education, 51*, 1141-1152. https://doi.org/10.1007/s11858-019-01061-9

Lesh, R., & Doerr, H. (Eds.). (2003). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*. Lawrence Erlbaum Associates. https://doi.org/10.4324/9781410607713

Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. Kelly, & R. Lesh (Eds.), *Research design in mathematics and science education* (pp. 591-646). Lawrence Erlbaum Associates.

Mendoza-Sandoval, E., Rodríguez-Vásquez, F. M., Roa-Fuentes, S., & Romero-Valencia, J. (2021). Study on the cognitive construction of the change of basis matrix in terms of the APOS theory. *Avances de Investigación en Educación Matemática, 20*, 65-87. https://doi.org/10.35763/aiem20.4011

Montero Moguel, L. E., & Vargas Alejo, V. (2022). Modeling cycles and covariational reasoning when carrying out a model eliciting activity. *Educación Matemática, 34*(1), 214-248. https://doi.org/10.24844/EM3401.08

Parraguez González, M., Roa-Fuentes, S., Jiménez Alarcón, R., & Betancur Sánchez, A. (2022). Mental structures and mechanisms that from a geometric perspective model and articulate the learning of eigenvalue and eigenvector in R2. *Revista Latinoamericana de Investigación en Matemática Educativa, 25*(1), 63-92. https://doi.org/10.12802/relime.22.2513

Parraguez, M., & Oktaç, A. (2010). Construction of the vector space concept from the viewpoint of APOS theory. *Linear Algebra and Its Applications, 432*(8), 2112-2124. https://doi.org/10.1016/j.laa.2009.06.034

Plaxco, D., Zandieh, M., & Wawro, M. (2018). Stretch directions and stretch factors: A sequence intended to support guided reinvention of eigenvector and eigenvalue. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh (Eds.), *Challenges and strategies in teaching linear algebra* (pp. 175-192). Springer. https://doi.org/10.1007/978-3-319-66811-6_8

Possani, E., Trigueros, M., Preciado, J. G., & Lozano, M. D. (2010). Use of models in the teaching of linear algebra. *Linear Algebra and Its Applications, 432*(8), 2125-2140. https://doi.org/10.1016/j.laa.2009.05.004

Roa-Fuentes, S., & Oktaç, A. (2012). Validation of a genetic decomposition of linear transformation: An analysis refined by the implementation of the research cycle related to APOS theory. *Relime, 15*(2).

Salgado, H., & Trigueros, M. (2015). Teaching eigenvalues and eigenvectors using models and APOS theory. *The Journal of Mathematical Behavior, 39*, 100-120. https://doi.org/10.1016/j.jmathb.2015.06.005

Sevinc, S., & Lesh, R. (2018). Training mathematics teachers for realistic math problems: A case of modeling-based teacher education courses. *ZDM Mathematics Education, 50*, 301-314. https://doi.org/10.1007/s11858-017-0898-9

Stewart, S., Epstein, J., & Troup, J. (2019). Leading students towards the formal world of mathematical thinking: A mathematician's reflections on teaching eigentheory. *International Journal of Mathematical Education in Science and Technology, 50*(7), 1011-1023. https://doi.org/10.1080/0020739X.2019.1657598

Trigueros, M. (2018). Learning linear algebra using models and conceptual activities. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh (Eds.), *Challenges and strategies in teaching linear algebra* (pp. 29-50). Springer. https://doi.org/10.1007/978-3-319-66811-6_2

Trigueros, M., & Oktaç, A. (2019). Task design in APOS theory. *Avances de Investigación En Educación Matemática, 15*, 43-55. https://doi.org/10.35763/aiem.v0i15.256

Wawro, M., Watson, K., & Zandieh, M. (2019). Student understanding of linear combinations of eigenvectors. *ZDM Mathematics Education, 51*, 1111-1123. https://doi.org/10.1007/s11858-018-01022-8

Zandieh, M., Wawro, M., & Rasmussen, C. (2017). An example of inquiry in linear algebra: The roles of symbolizing and brokering. *PRIMUS, 27*(1), 96-124. https://doi.org/10.1080/10511970.2016.1199618

https://www.ejmste.com