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Elementary pre-service teachers' horizon knowledge for teaching addition and subtraction: An analysis of video presentations

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Abstract

Horizon mathematics knowledge, i.e., teachers' understanding of how various mathematical topics are interrelated, can help mathematics teachers organize the discrete parts of mathematical content to develop coherent teaching lessons. For example, when teaching whole number addition and subtraction, connections to concepts such as base-10 concepts and the inverse relationship between addition and subtraction could help students solve the addition and subtraction problem better. Thus, teacher training programs have been increasingly promoting this knowledge among pre-service teachers (PSTs) to help them enhance their teaching skills and better prepare them for future teaching. However, little is known about what kind of horizontal knowledge PSTs have developed and how well they utilize it in their teachings. By analyzing video presentations of 43 elementary PSTs, this study examined their horizon mathematics knowledge related to backward and forward conceptual connections of whole number addition and subtraction concepts. The findings revealed that PSTs tend to make connections with previously learned mathematics concepts (i.e., backward conceptual connections) but pay relatively less attention to connecting with other relevant mathematics concepts students will learn in future grades (i.e., forward conceptual connections). In addition, the findings showed that PSTs displayed various types of inaccuracies when connecting base-10 place value and regrouping. These findings offer important insights for teacher training programs to adapt their mathematic method courses to help PSTs improve their horizontal knowledge and proactively address inaccuracies observed in the present study.

Keywords: pre-service teachers, video analysis, mathematics connections, horizon knowledge, backward conceptual connection, forward conceptual connection

INTRODUCTION

Whole number addition and subtraction are required topics in most early-grade elementary mathematics curricula (Li, 2000; Ding & Li, 2010; Nunes et al., 2016; Zhou & Peverly, 2005). Students' competence in whole addition and subtraction developed in their earlier schooling constitutes a critical knowledge base for them to further develop a conceptual understanding of advanced topics, such as multiplication, division, fractions, and algebra, in different stages of their later schooling (Reigosa-Crespo et al., 2013; Wu, 2011). As a result, curriculum standards are developed to include addition and subtraction operations as one of the major content areas (Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000).

Facilitating this goal, teacher training programs are increasingly promoting teachers' mathematics knowledge that can help pre-service teachers (PSTs), an important source of future teachers, better prepare for future teaching in whole number addition and subtraction (Fuson, 2020). For example, recent research suggests that teachers' mathematics horizon knowledge of whole number addition and subtraction (e.g., connecting whole number addition and subtraction to

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Contribution to the literature

- This study contributes to the existing literature by examining preservice teachers' (PSTs) horizon mathematics knowledge in the specific context of whole number addition and subtraction using video analysis.
- This study highlights the tendency of PSTs to establish backward conceptual connections but overlook forward conceptual connections with other relevant mathematics concepts that students will encounter in higher grades.
- This study identified and summarized the different types of inaccuracies that PSTs exhibit when connecting base-10 place value and regrouping, shedding light on potential areas for targeted intervention and improvement in teacher preparation programs.

base-10 concepts, counting, inverse relationship between addition and subtraction, and properties of numbers and operations) provides a lens to improve student learning and comprehension of this topic (Sun et al., 2019). As an important part of mathematics content knowledge, mathematics horizon knowledge reflects teachers' understanding of how various mathematical topics connect with each other (Hill et al., 2008; Zazkis & Mamolo, 2011), including both backward and forward conceptual connections.

In backward conceptual connections, teachers connect current learning topics with students' prior knowledge (Ma, 1999; Wu, 2011). For example, in the case of addition and subtraction, teachers can use base-10 concepts (Fuson, 2020; Thanheiser & Melhuish, 2019) and counting strategies (Fuson, 2020; Sun et al., 2018) to help students understand the conceptions of the whole number system. In comparison, forward conceptual connections focus on making connections between current learning topics and future learning topics (Ma, 1999; Wu, 2011). For example, while teaching addition and subtraction, teachers can connect concepts in this area with examples relating to the inverse relationship between addition and subtraction (Selter et al., 2012) and properties of numbers and operations (e.g., associative property, commutative property) to enhance student's learning and comprehension (National Research Council & Mathematics Learning Study Committee, 2001; Sun et al., 2019). Previous research suggested that making backward and forward conceptual connections to whole number addition and subtraction helps students develop a deep understanding when learning these concepts. For example, making connections to base-10 concepts while teaching whole number addition and subtraction helps students understand the value represented by each place and regrouping (Fuson, 2020; Thanheiser & Melhuish, 2019). In addition, despite that subtraction can be understood by two models: taking away and determining the difference (Usiskin, 2008), many students focus solely on taking away, being too one-sided (Selter et al., 2012). To solve this problem, teachers can connect the inverse relationship between addition and subtraction, emphasizing subtraction as a model for determining the difference (Selter et al., 2012). While both types of horizon knowledge are helpful in improving teaching effectiveness and students' understanding (Ma, 1999; Montes et al., 2013), insights are limited due to an important gap existing in the literature.

Specifically, although the importance of teachers' connection-making in mathematics knowledge (i.e., horizontal knowledge) has been increasingly recognized, most existing studies are mainly focused on examining the connection of whole number addition and subtraction to place value and regrouping (e.g., Matthews & Ding, 2011; Thanheiser, 2009, 2012; Thanheiser & Melhuish 2019). As a result, little is known about what other types of connections are used by PSTs and how well PSTs are using those connections. This research gap limits the ability of teacher training programs to better prepare PSTs and leaves PSTs with missed opportunities to enhance their knowledge while in the program.

To address this gap, this study examines PSTs' horizon mathematics knowledge related to backward and forward conceptual connections of whole number addition and subtraction concepts by analyzing PSTs' teaching videos. Specifically, it investigates the following two questions:

1. Which backward and forward conceptual connections do PSTs make when teaching whole number addition and subtraction?
2. How accurate are these connections demonstrated by PSTs?

RELEVANT LITERATURE

Theoretical Framework

The conception of horizon knowledge proposed by Ball et al. (2008) and substantiated by other scholars (Montes et al., 2013; Zazkis & Mamolo, 2011) provide a theoretical foundation that frames the focus and analysis of this study. This literature suggests that teachers' understanding of how mathematics concepts and procedures they need to teach are related to each other in the curriculum, shaping the coherence of their planning or teaching across a series of lessons (Montes et

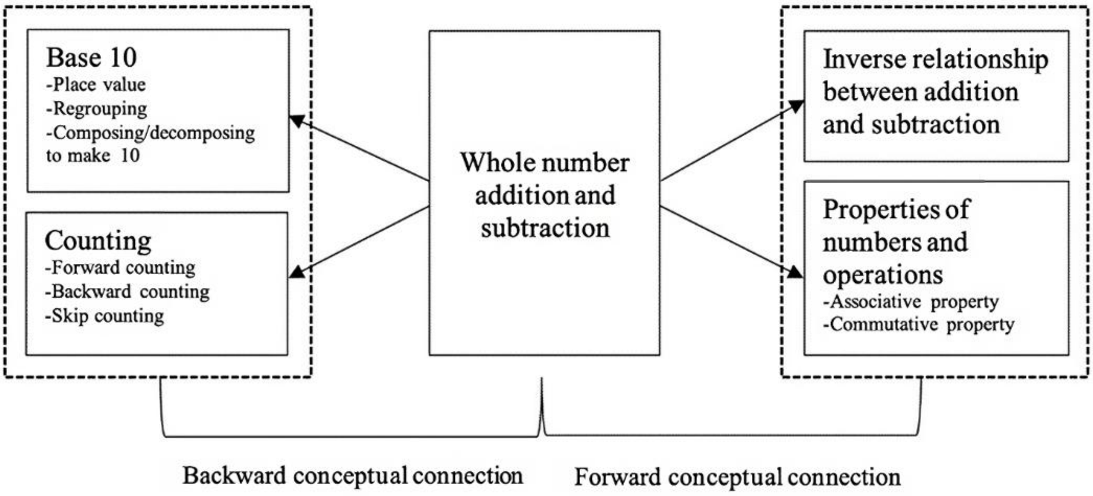

Figure 1. Backward and forward conceptual connections to whole number addition and subtraction (Source: Authors' own elaboration)

al., 2013; Rowland, 2009). Understanding these connections between different concepts and procedures will help teachers develop explanations and examples for students' questions in their instructions (Borko & Livingston, 1989; Leinhardt & Smith, 1985; Rowan et al., 1997), thus affecting the quality of their mathematics teaching (Ball & Bass, 2009).

Horizon knowledge in whole number addition and subtraction

The horizon mathematics knowledge for teaching is one of the six strands of teachers' mathematics knowledge and an important content knowledge strand (Ball et al., 2008). To improve horizon mathematics knowledge, teachers need to develop an appropriate understanding of how mathematics concepts are related to each other (Ball & Bass, 2009).

Teachers' horizon knowledge in whole number addition and subtraction can be observed as two types of mathematics conceptual connections (Ma, 1999; Montes et al., 2013). **Figure 1** illustrates sample connections based on state-level elementary mathematics curriculum standards (Texas Education Agency, 2013). As aforementioned, backward conceptual connections concern teachers' knowledge of how mathematical concepts or ideas students have learned previously are associated with the focal concepts and procedures they are currently learning. For example, as shown in **Figure 1**, when teaching concepts in whole number addition and subtraction, teachers may help students connect addition and subtraction to related topics that students have learned previously, such as base-10 place value and counting (Fuson, 2020; Sun et al., 2018; Thanheiser & Melhuish, 2019).

To illustrate an example of connecting to base-10, suppose students are working on an addition operation, 27+8. In this problem, the 7 of 27 is in the ones place that can only hold 0-9 ones, and the 2 is in the tens place only containing 0-9 tens. First, we add ones place to get 15 ones. Since ones place can only hole 0-9 ones, we need to regroup, which results in 5 ones in the ones place and 1 ten in the tens place. Now, we have 1 ten and 2 tens in the tens place, and when we add them, we get 3 tens. So, we get 3 tens and 5 ones, or 35. In the simple subtraction operation, 31-6, 6 is going to be subtracted from 31 in which 1 in the ones place does not have enough units. Thus, one of the three tens in the tens place needs to be regrouped into ones in the ones place, yielding 11 ones in the ones place. Six ones can now be subtracted from the eleven ones leaving 5 ones that remain in the ones place. Finally, 2 tens in the tens place and 5 ones in the ones place are composed to make the result of 25. These base-10 concepts help students develop a conceptual understanding of the concept and procedure in whole number addition and subtraction (Fuson, 2020; Thanheiser & Melhuish, 2019).

In terms of counting, counting on a number line has been viewed as a basic strategy to solve simple addition and subtraction problems (Bartolini Bussi, 2015; Fuson, 2020; Sun, 2019). Teachers may encourage students to use forward, backward, and skip counting strategies to solve their addition and subtraction problems (Fuson, 2020; Sun, 2019). For example, in the previous addition problem, 27+8, the result can be obtained by starting at 27 and then counting eight positions forward to end with 35. Backward counting in the previous simple subtraction example, 31-6, can be realized by starting with 31 and then counting six positions back on a number line to yield 25. Skip counting can solve a simple addition problem, 15+56, by starting from 15, skip counting by 10 five times to get to 65, and then skip counting by 2 three times to yield 71.

Forward conceptual connections reflect teachers' understanding of how a particular mathematical concept or procedure they are teaching relates to other concepts students will be learning in the future (Ma, 1999; Montes et al., 2013). As shown in **Figure 1**, when teaching whole number addition and subtraction, teachers can help students better understand concepts in addition and subtraction by linking them to other concepts they will learn in the future, including the inverse relationship between addition and subtraction, and the associative and commutative properties of addition (Selter et al., 2012; Sun et al., 2019). Making such connections may help students comprehensively understand addition and subtraction (Selter et al., 2012; Sun et al., 2019). It can also help students prepare for future learning of more complex addition and subtraction problems and develop algebraic thinking (Reigosa-Crespo et al., 2013; Sun et al., 2019; Tent, 2006, Wu, 2011). Below are some examples demonstrating how to connect whole number addition and subtraction to these forward conceptual connections.

Here we use the previous example 31-6 again to illustrate the connection of inverse relationship to whole number addition and subtraction. This way of thinking helps students to view subtraction as determining the difference. To find the difference between 31 and 6, we start with 6 and forward count until we get 31 (6+?=31). We first count four positions to get 10, skip counting by 10 two times to get 30, and another forward counting by one position to get 31. Since 25 positions have been counted (4+10+10+1=25), students will get that 6+25=31. This result indicates that the difference between 31 and 6 is 25, i.e., 31-6=25. While using this "adding up" strategy, we are actually connecting whole number addition and subtraction to the inverse relationship between addition and subtraction. As aforementioned, many students primarily view subtraction solely as a model of taking away (Selter et al., 2012), overlooking the importance of viewing it as determining the difference. This connection helps students to understand subtraction as determining the difference. This connection also helps students to develop algebraic thinking (Sun et al., 2019). For example, as students progress to higher grades, they learn how to solve equations such as "6+χ=31". One common method is to subtract 6 from both sides of the equation, which yields the solution χ=25. By applying this technique and others like it, students can develop a strong foundation in algebra and problem-solving that will serve them well in many areas of math and science (Selter et al., 2012).

The associative property of addition suggests that when we add more than two numbers, the grouping of the addends does not change the sum (Howe & Epp, 2008; Tent, 2006). For example, (3+4)+5=3+(4+5). The commutative property of addition indicates that changing the order of addends does not change the sum, such as 3+6=6+3 (Howe & Epp, 2008; Tent, 2006). Connecting whole number addition and subtraction to the associative and commutative property of addition helps students develop a deep understanding of arithmetic (National Research Council & Mathematics Learning Study Committee, 2001). For example, in the previous addition operation, 27+8, the addends can be decomposed by place value to 20+7+8 in an expanded form notation. Next, since ten ones are needed to regroup to make 10, we can decompose 8 to 5+3. As a result, we will have 20+7, adding 5+3.

Figure 2 illustrates the way to think about it. In this process, we actually apply the commutative property of addition (i.e., (20+7)+(5+3)=(20+7)+(3+5) and the associative property of addition (i.e., 20+[7+3]+5), then we have the ten ones composed and regrouped to yield three tens and five ones, yielding 35. Other than helping

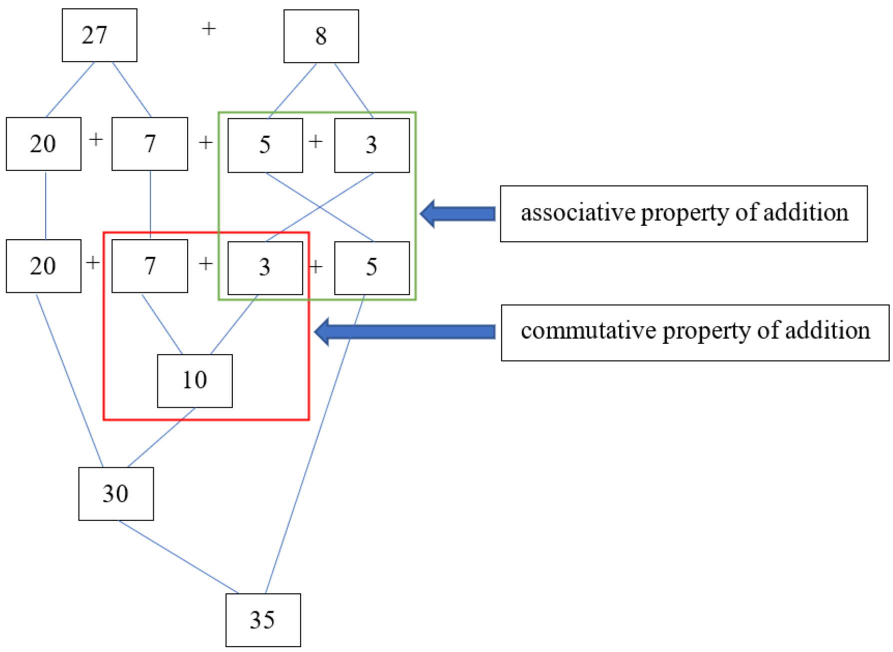

Figure 2. An example of using forward concept connections in teaching whole number addition (Source: Authors' own elaboration)

students develop a deep understanding of arithmetic (National Research Council & Mathematics Learning Study Committee, 2001), making these connections also prepare students to learn and understand the properties of addition, which they will learn in upper grades (Tent, 2006; Wu, 2011).

Empirical Bases

To situate the questions of this study in the relevant empirical literature, we searched the empirical studies relevant to PSTs' horizon mathematics knowledge for teaching addition and subtraction. Our review of the literature yielded several findings.

First, the empirical examination of PSTs' horizon knowledge of teaching addition and subtraction is underdeveloped. Most existing studies are limited to the examination of the connection of whole number addition and subtraction to base-10 place value and regrouping that students might have already learned as summarized in a literature review (Thanheiser et al., 2013). In an early effort to explore PSTs' horizon knowledge, Thanheiser (2009) asked fifteen U.S. elementary PSTs to solve multi-digit addition and subtraction problems involving base-10 regrouping and then interviewed them for explanations of their solutions. The author found that while most participants were able to perform the calculations correctly, they struggled with explaining their solution processes using appropriate base-10 and regrouping concepts. The result was confirmed by a follow-up study with 33 PSTs using survey and interview instruments (Thanheiser, 2012), and another study surveying 133 PSTs (Matthews & Ding, 2011). As a result, recent research suggested that the lack of deep understanding of whole number addition and subtraction limited PSTs' ability to address student understanding. For example, Son (2016) found that 20 out of 80 PSTs have difficulty justifying why certain student-generated whole number subtraction strategies work or not. To help PSTs develop a deep understanding of place value and regrouping, Thanheiser and Melhuish (2019) developed a sequence of tasks using tally system, the Egyptian system (a grouping system), and the Mayan system (a place-value system).

In addition, prior literature indicated that PSTs' inability to explain addition and subtraction involving base-10 place value and regrouping is not limited to the U.S. PSTs. For example, a survey study with 140 PSTs in Turkey on their explanations about the mistakes that students made in addition and subtraction involving base-10 place value came to similar conclusions (Tarim & Artut, 2013). Moreover, Verzosa (2020) examined 230 Philippine PSTs' ability to solve multi-digit subtraction problems and found that their reasonings relied heavily on rules and procedures rather than conceptual understandings. PSTs' difficulties in explaining their solutions to multi-digit addition and subtraction involving place value and regrouping persisted even when they were engaged in well-designed tasks targeting their initial limited conceptions specifically (McClain, 2003; Thanheiser, 2015).

Second, even though PSTs are found to be more fluent in performing whole number operations, including addition and subtraction (Thanheiser et al., 2013), their fluency in calculation begins to break down when multi-step addition and subtraction are involved. Glidden (2008) examined 381 PSTs on their solutions to multi-step operation problems involving addition, subtraction, multiplication, and division using a survey instrument. The author found that although showing confidence in their mathematics ability and favorable perceptions of mathematics, fewer than half of PSTs were able to use the sequence appropriately to solve the multi-step operation problems.

The existing relevant studies contribute significantly to the understanding of PSTs' struggles with using base-10 place value and regrouping concepts in addition and subtraction, and their faulty fluency with calculating multi-step operations involving addition and subtraction. However, these studies also show a critical limitation. Specifically, few studies examined PSTs' backward conceptual connections of addition and subtraction to counting strategies and whole number composition and decomposition (Ma, 1999; Wu, 2011), and forward conceptual connections to the inverse relationship between addition and subtraction, associative and commutative properties of addition central to students' learning of multiplication, division, fractions, and algebra (Reigosa-Crespo et al., 2013; Wu, 2011). To address this limitation, the present study is explicitly aimed to examine PSTs' backward and forward conceptual connections of addition and subtraction concepts beyond base-10 place value and regrouping.

METHOD

Participants and Context

Participants of this study were 43 PSTs from different course sections of the only mathematics methods course required in an elementary teacher education program at a southwestern U.S. research university. As discussed later, we relied on video presentations to collect our data. Thus, participants were selected for the study based on whether their video presentations on whole number addition and subtraction were available and had good visual and audio quality. These participants were all females, with 81% being between the ages of 18 and 24 years, 74% were Caucasian, the rest were Hispanic or Latino, and a few were African American. This sample reflects the typical population of U.S. PSTs, primarily young and Caucasian females (AACTE, 2013; Kim & Corcoran, 2017; National Center for Education Statistics [NCES], 2012; Zumwalt & Craig, 2005).

The mathematics methods course from which participants were selected was designed using an approach, where each module was aligned with major mathematics topics covered by state-level elementary mathematics curriculum standards (Texas Education Agency, 2013). The course was similarly implemented across all course sections to develop PSTs' mathematics knowledge for teaching (Ball et al., 2008; Hill et al., 2008). A three-week module covering addition and subtraction with whole numbers was one of the five modules in the course. It provided several resources and activities to influence PSTs' horizon knowledge for teaching whole number addition and subtraction.

First, PSTs were required to familiarize themselves with the state-level EC-6 curriculum standards before class sessions (Texas Education Agency, 2013). This included the mathematics concepts related to whole numbers that students learn before and after addition and subtraction. Prior knowledge concepts may comprise base-10 place value, regrouping, composing and decomposing numbers, and forward, backward, and skip counting, usually taught to elementary students in grades K-2. The upcoming concepts that students will learn in later grades include the inverse relationship between addition and subtraction and the associative and commutative properties of addition, which students will learn in grades 3-6 (Texas Education Agency, 2013). During the mathematics methods class sessions, the instructors engaged PSTs in discussing the requirements for addition and subtraction instruction, identifying both the backward and forward conceptual connections. Unpacking standards helped PSTs understand the related backward and forward conceptual connections in the curriculum, which enhanced their understanding of the coherence of the curriculum (Cuoco & McCallum, 2018).

Second, they were also required to view 10 online videos, where teachers demonstrated how to teach whole number addition and subtraction involving various kinds of connections between addition and subtraction operation to other mathematics concepts before class sessions. During the class session, they identified the kinds of representations that teachers used in the videos for both the backward and forward conceptual connections of addition and subtraction to other mathematics concepts and operations. Then, they were asked to critique these approaches and justify their critiques in groups. The teachers demonstrating in the video provided modeling for PSTs, which helps them better understand how to make connections in their teaching (Blomberg et al., 2011; Kang & van Es, 2019).

Finally, they were also required to read the relevant chapters of the course textbook to understand different approaches and tools that can be used to teach addition and subtraction, such as using the make 10 approaches with ten frames, using the traditional algorithm for adding and subtracting single and multi-digit whole numbers, and composing and decomposing numbers with the support of number lines, base-10 blocks, and place value mats (Van de Walle et al., 2013). Then, in the class sessions, they were asked to discuss using these instructional approaches and tools to design and justify word problems involving whole number addition and subtraction with consideration of both the backward and forward conceptual connections of addition and subtraction to other mathematics concepts and procedures individually. They were then asked to share and justify their designs with the class and answer questions from their instructor and peers about their outcomes. Reading chapters of the textbook and discussions allowed PSTs to understand the strategies used in whole number addition and subtraction (Valtonen et al., 2021).

Data Sources and Analysis

Previous studies examining PSTs' horizon knowledge have mainly used structured survey or interview instruments (e.g., Glidden, 2008; Matthews & Ding, 2011; Tarim & Artut, 2013; Thanheiser, 2009; Thanheiser, 2012; Thanheiser & Melhuish 2019). Because the information obtained through surveys and interviews are often self-reported and thus relatively subjective, scholars have argued that these traditional approaches disconnect research findings from real-world teaching practices (Kersting et al., 2010). In this study, we relied on information manually coded using PSTs' video presentations.

Compared to information obtained through surveys and interviews, the information provided by video analysis tends to be more objective (Ruhleder & Jordan, 1997). As an outcome of their learning in the module, PSTs were required to record 6-12 minutes videos showing how they teach whole number addition and subtraction. In their presentations, each participant was required to design a single-step word problem involving multi-digit addition and subtraction that used two two-digit numbers or one two-digit number and one three-digit number to demonstrate how they could solve their word problem using at least two of the approaches that they learned in the module. Based on the curriculum, any problem could be theoretically connected to all the backward or forward concepts in **Figure 1**.

In the video demonstration, they were asked to align their instruction with the state-level curriculum standards for addition and subtraction and relevant mathematics concepts for a particular grade level (Texas Education Agency, 2013). Moreover, they were asked to make connections to other important mathematics concepts and operations in their demonstrations, although an exact number of connections was not required. In addition, they were required to identify relevant students' misunderstandings with addition and subtraction and how they would help students resolve their misunderstandings.

The analysis of video presentations in this study was conducted in several ways following the suggestions of video analysis (Stigler et al., 2000), which provides the opportunity for researchers to reexamine data repeatedly and, thus, develop more reliable evidence for supporting their claims (Bottorff, 1994; Grimshaw, 1982). First, a coding rubric (**Appendix A**) was developed to include each backward and forward connection related to whole number addition and subtraction. Based on the rubric, each video presentation was scripted and coded to identify the themes regarding teachers' horizon knowledge strand for teaching (Hill et al., 2008) in light of the backward and forward conceptual connections (Montes et al., 2013).

Second, the results from the above analysis were then coded to identify each kind of connection and inaccurate use of connections in the presentations as well as the examples representing these connections and inaccuracies following qualitative research methods (Merriam, 1998).

Next, we compared the difference between groups in terms of the number of participants who made connections to at least one of the concepts (listed in **Table 1**) and that of participants who did not, using Chi-square tests with sample weighting (Shavelson, 1996). As suggested by Fisher (1955), we used Fisher's exact statistics to compare the differences for cases, where numbers were below 10.

Finally, typical examples of these connections and inaccuracies were also carefully described and used to support the connections and inaccuracies in the video presentations using screen captures from the video presentations. To establish the reliability and validity of our analysis, a research team was formed, including two experienced researchers and several graduate assistants. They worked together to develop and discuss the coding process. In this study, reliability refers to the degree to which the coding procedure is consistent over time and

Table 1. PSTs' connections of whole number addition & subtraction to other mathematics concepts (n=43)

Types	Connections	Total: n (%[b])	TA: n (%[b])	TS: n (%[b])
Connections to concepts & operations learned				
Base-10 number system	Connection to place value	29 (67.4%)	22 (51.2%)	7 (16.3%)
	Errors observed[a]	7 (24.1%)	5 (17.2%)	2 (6.9%)
	Connection to regrouping	24 (55.8%)	17 (39.5%)	7 (16.3%)
	Errors observed	9 (37.5%)	7 (29.2%)	2 (8.3%)
	Connection to composing & decomposing to make 10	16 (37.2%)	10 (23.3%)	6 (14.0%)
	Errors observed	0 (0.0%)	0 (0.0%)	0 (0.0%)
	Connection to at least one of three ideas	37 (86.0%)	29 (67.4%)	8 (18.6%)
	Connection to one idea only	14 (32.6%)	13 (30.2%)	1 (2.3%)
	Connection to two ideas only	14 (32.6%)	12 (27.9%)	2 (4.7%)
	Connection to three ideas only	9 (20.9%)	4 (9.3%)	5 (11.6%)
Counting	Connection to forward counting	18 (41.9%)	17 (39.5%)	1 (2.3%)
	Errors observed	1 (5.6%)	1 (5.6%)	0 (0.0%)
	Connection to backward counting	6 (14.0%)	2 (4.7%)	4 (9.3%)
	Errors observed	0 (0.0%)	0 (0.0%)	0 (0.0%)
	Connection to skip counting	18 (41.9%)	12 (27.9%)	6 (14.0%)
	Errors observed	0 (0.0%)	0 (0.0%)	0 (0.0%)
	Connection to at least one of three counting strategies	25 (58.1%)	19 (44.2%)	6 (14.0%)
	Connection to one counting strategy only	8 (18.6%)	7 (16.3%)	1 (2.3%)
	Connection to two counting strategies only	17 (39.5%)	12 (27.9%)	5 (11.6%)
	Connection to three counting strategies only	0 (0.0%)	0 (0.0%)	0 (0.0%)
Connections to concepts & operations to be learned				
Inverse relationship	Connection to inverse relationship	4 (9.3%)	0 (0.0%)	4 (9.3%)
	Errors observed	0 (0.0%)	0 (0.0%)	0 (0.0%)
Properties of numbers & operations	Connection to associative property	6 (14.0%)	5 (11.6%)	1 (2.3%)
	Errors observed	0 (0.0%)	0 (0.0%)	0 (0.0%)
	Connection to commutative property	4 (9.3%)	4 (9.3%)	0 (0.0%)
	Errors observed	0 (0.0%)	0 (0.0%)	0 (0.0%)
	Connection to at least one property	7 (16.3%)	6 (14.0%)	1 (2.3%)
	Connection to one property only	4 (9.3%)	3 (7.0%)	1 (2.3%)
	Connection to two properties only	3 (7.0%)	3 (7.0%)	0 (0.0%)

Note. TA: The number of PSTs who teach addition; TS: The number of PSTs who teach subtraction; [a]Percentage for errors observed here & those in other places of the table are calculated on ratio of respective numbers of participants having errors with respective concept to total number of participants who presented that respective concept; & [b]Percentages for these numbers (all other percentages except percentage for errors) are calculated on ratio of respective numbers of participants who connect with certain concepts to total number of participants the accuracy of the measurement (Thorndike, 1997). To ensure reliability, each video was coded by two researchers independently. Then, we calculated initial intercoder reliability based on the percentage of agreement (Lombard et al., 2002; Neuendorf, 2017; Riffe et al., 2019). The initial agreement percentage was good (81.4%). The researchers then discussed those disagreements until all issues were solved.

Validity refers to the extent to which an instrument accurately measures the specific concept or content the researcher is attempting to assess (Thorndike, 1997). As discussed in the theoretical framework section, we derived the coding rubric from the curriculum standards (Texas Education Agency, 2013) and previous literature (Fuson, 2020; Sun et al., 2018; Thanheiser & Melhuish, 2019). The research team then collectively reviewed the rubric to ensure that the criteria used to identify each type of connection was consistent with the definitions provided by the curriculum standards and the literature. This process provides support for our theoretical validity.

RESULTS

Among the 43 PSTs, 35 (81.4%) of them taught addition, and only 8 (18.6%) taught subtraction. It's not surprising because adding on is a natural strategy compared to subtraction, and people practice adding quite often in different situations (Fuson, 1992). Therefore, it's natural for PSTs to choose teaching addition when they have a choice.

As shown in **Table 1**, most of them made backward conceptual connections, with 86% of the participants using at least one of the three base-10 number system concepts, and 58.1% of all participants made connections of addition and subtraction to at least one of the three counting strategies. Our results also revealed that only 9.3% of the participants made forward conceptual connections between addition and subtraction to the inverse relationship between addition and subtraction in their presentation (**Table 1**), and 16.3% of the participants using at least one of the two properties of numbers and operations.

Backward Conceptual Connections

Backward conceptual connections to base-10 number system

Our analysis revealed several results relating to participants' backward conceptual connections of whole number addition and subtraction to base-10 place value, regrouping, and composing and decomposing numbers to make 10. First, most participants connected whole number addition and subtraction to base-10 number system concepts. As shown in **Table 1**, 86% of the participants used at least one of the three base-10 number system concepts (i.e., place value, regrouping, and number composition or decomposition) as they presented their problem and its solutions with 32.6%, 32.6% and 20.9% of them connecting to one, two, or three of these concepts respectively. Results of a Chi-square test revealed a significant difference between the proportion of participants who connected to at least one of the three base-10 number system concepts and those who did not (Pearson $\chi^2[1, N=43]=22.35, p<0.01$).

Second, while most participants made connections of whole number addition and subtraction to base-10 place value and regrouping, relatively fewer participants (37.2%) made connections to number composition and decomposition. As shown in **Table 1**, 67.4% and 55.8% of the participants connected addition and subtraction to base-10 place value and regrouping respectively. A typical example of participants connecting addition and subtraction to base-10 place value and regrouping concepts was offered by participant A (**Figure 3**).

In demonstrating how to add 128+67, participant A explained that in 128, the 8 was in ones place, 2 was in the tens place, and 1 was in hundreds place. Similarly, in 67, the 7 was in ones place and 6 was in tens place. When adding 128 and 67, she stressed that one needed to combine 8 and 7 in the ones place of 128 and 67, then the 2 and 8 in the tens place of the two numbers so on. Since adding 8 and 7 in the ones place resulted in 15 ones, which is more than 10, thus, regrouping was necessary by decomposing 15 into 1 ten and 5 ones and adding 1 ten to the tens place while leaving 5 ones in the ones place. She then added 6 and 2 in the tens place initially and 1 in tens place from the above calculation together

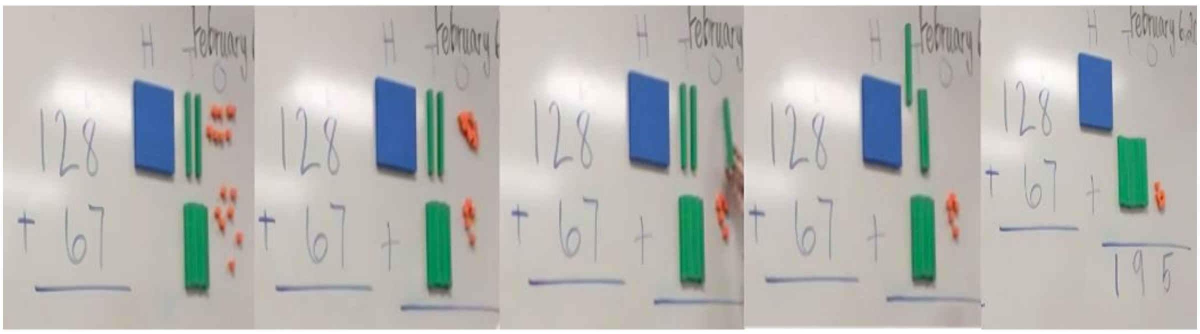

Figure 3. Backward conceptual connection to base-10 place value and regrouping (Source: Screenshots for participant A's presentation)

to make 9 in the tens place. Finally, she left to put 1 in 128 in the hundreds place to make 195 as the result.

Third, a number of participants showed inaccuracies when connecting addition and subtraction to place value and regrouping. Based on **Table 1**, about 24.1% and 37.5% of participants who made connections with addition and subtraction to place value and regrouping respectively, did so with inaccuracies. For example, participant B displayed a typical inaccuracy in connecting addition and subtraction to regroup when showing how to add 64 and 63. In demonstrating how she used regrouping to add 6 in the tens place of 64 and 6 in the tens place of 63, she explained the process of regrouping the result from 6+6 into 1 in hundreds place and 2 in tens place as "carry" and "trade" without a clear explanation of why 10 units in the ten's place became 1 unit in the hundred's place. Participant B stated "there's twelve ten-longs total in this column, since we did add up to ten, we want to trade ten of this ten-longs into a one hundred block." Additionally, participant C showed an inaccuracy in connecting addition and subtraction to base-10 place value when she added 24 and 112. She explained that one would "add 2 and 4 to get 6", "1 and 2 to get 3", and "add 1 and 0 to get 1" without a clear distinction in the place values when adding tens and ones together in each case.

Results of a Chi-square test revealed no significant difference between the proportions of participants who connected addition and subtraction to regrouping accurately (72.7%) and those who displayed inaccuracies (27.3%) when making this connection (Pearson $\chi^2[1, N=24]=1.50, p=.22$). The same test indicated a significant difference between participants who connected addition and subtraction to place value accurately and who displayed inaccuracies in making such a connection (Pearson $\chi^2[1, N=29]=7.76, p<0.01$).

Backward conceptual connections to counting

Several results also emerged from our analysis of participants' backward conceptual connections of addition and subtraction to various counting strategies. First, as shown in **Table 1**, more than half (58.1%) of the participants connected addition and subtraction to at least one of the three counting strategies, with 18.6% connecting to only one counting strategy, 39.5% making connections to two of the three counting strategies, and 0% connecting to all three counting strategies. A Chi-square test indicated no significant difference between the proportions of participants who connected addition and subtraction to at least one counting strategy and those who were unable to connect to any counting strategy (Pearson $\chi^2[1, N=43]=1.14, p=.29$).

Second, more participants connected addition and subtraction to forward counting and skip counting strategies than those who connected backward counting. **Table 1** shows that 41.9% and 41.9% of the participants connected addition and subtraction to forward counting and skip counting operations, respectively. For example, in showing how to add 14 and 12 together, participant D started at 14 on a number line and forward counted 12 positions to get to 26. Participant E connected addition and subtraction to skip counting in her presentation in which, she demonstrated how to find the result for 39 +32 by starting at 39 on the number line and then skip counting by ten three times, and then forward counting by one twice to get 71.

In comparison, only six (14.0%) participants made connections between addition and subtraction to backward counting. An example of connecting whole number subtraction to backward counting is, in showing how to calculate 106-15, participant F started at 106 on the number line, and backward counted by 5 three times to get 91.

Third, very few inaccuracies were identified when participants connected addition and subtraction to the three counting strategies. As shown in **Table 1**, no participants displayed any inaccuracies in connecting to all three counting strategies except for participant G, who, after skip counting by two six times starting from 122, failed to forward count by one when explaining how to solve the problem, 122+13.

Forward Conceptual Connections

Forward conceptual connection to the inverse relationship between addition and subtraction

Our analysis revealed that only a few participants made forward conceptual connections between addition and subtraction to the inverse relationship between addition and subtraction in their presentation; only 4 (9.3%) of the participants did so, and none of these presentations displayed any inaccuracy based on **Table 1**. Participant H demonstrated a typical example of how to make this connection by showing how to get the result for 200-72 (**Figure 4**).

In this case, she first labeled 72 and 200 on a number line. Starting at 72, she jumped eight positions to 80, then two 10 units to get 100, and finally, jumped 100 units to 200. Afterward, she added the sizes of her jumps,

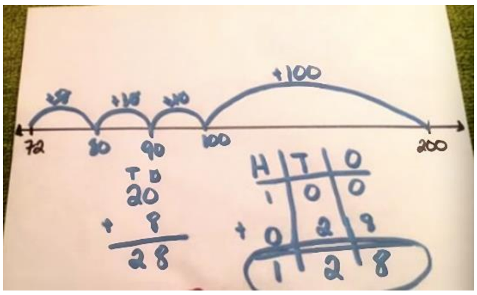

Figure 4. Forward conceptual connection to inverse relationship between addition & subtraction (Source: Screenshot for participant H's presentation)

8+10+10+100, to obtain the difference between 200 and 72.

Forward conceptual connection to properties of numbers and operations

Our analysis further indicated that only a few participants made forward conceptual connections between addition and subtraction and the associative and commutative properties of addition. **Table 1** showed that only 7 (16.3%) of the participants connected addition and subtraction to at least one of the two properties, with no inaccuracy observed in their presentations. A Chi-square test indicated a significant difference between the proportions of participants who connected addition and subtraction to at least one property, the associative or commutative property, and those who did not (Pearson $\chi^2[1, N=43]=19.56$, $p<0.01$). A common example of how participants made connections to the associative and commutative properties of addition came from participant I's case, as she demonstrated how to find the solution to 102+79 using the associative property of addition (**Figure 5**).

She first decomposed 102 into 100 and 2, and 79 into 71 and 8. Following the commutative property of addition, she added 8 and 2 together to make 10. Then, she decomposed 71 into 70 and 1, and added 70 to 10 to make 80. Finally, using the associative property of addition, she added 100 and 80 together and then added 1 to the result of 100+80 to make 181.

DISCUSSION AND CONCLUSION

Backward Conceptual Connections

The findings suggest that PSTs popularly used backward conceptual connections between addition and subtraction to base-10, regrouping, and number composition and deposition concepts in their presentations. However, most limited their connections to base-10 place value and regrouping concepts, paying less attention to making connections to whole number composition and decomposition. As shown in the results section, more participants made connections of addition and subtraction to base-10 place value and regrouping concepts than those connected to number composition and decomposition. One explanation for this finding is that most PSTs view base-10 place value and regrouping concepts as important foundations for solving single-step addition and subtraction problems at lower grade levels (Canobi, 2004, 2009). As a result, they are likely to connect whole number addition and subtraction problems to base-10 place value and regrouping concepts.

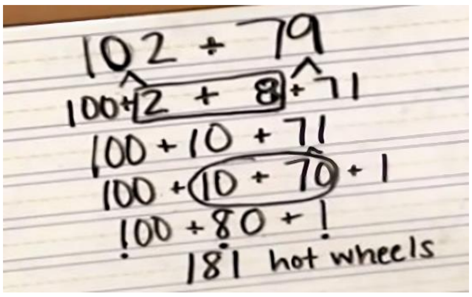

Figure 5. Forward conceptual connection to associative property of addition (Source: Screenshot for participant I's presentation)

Second, the study shows that although efforts to connect addition and subtraction to base-10 and regrouping concepts in their presentations were popular, many participants displayed inaccuracies when doing so. As shown in the results section, more than 25% of the participants displayed inaccuracies when connecting to base-10 place value and regrouping. This finding highlights a challenge for PSTs when helping their students develop a flexible understanding of the backward conceptual connections of addition and subtraction to the mathematics concepts beyond base-10 place value and regrouping as policymakers expected (National Mathematics Advisory Panel, 2008). As summarized in **Appendix B**, some inaccuracies were due to lacking a deep understanding of place value and regrouping. For example, when adding 64 and 63, one procedurally added "3 and 4", then "6 and 6 in the tens place resulting in 1 in the hundreds place and 2 in the tens place". Although the answer of 112 was correct, missing explaining why 10 units in the tens place became 1 unit in the hundreds place may confuse students why they needed to regroup. As a result, students may focus more on memorizing the procedure than understanding the logic. Another type of inaccuracy was potentially caused by the misuse of academic language. While students learn and will continue to learn the concepts in several school years, inaccurate use of language may lead to confusion. For example, some teachers may use "borrow" in their classes while others may use "regrouping." This inconsistency in the academic language will result in confusion among students. This finding offers important insights for teacher training programs to proactively adapt their mathematic method courses to incorporate the need to address inaccuracies observed in the present study. In addition, the present study also calls for future research to examine why PSTs develop a predominant focus on connections of addition and subtraction to base-10 place value and regrouping concepts. With more detailed knowledge about this problem, preserve teachers can better adjust their learning strategies to address the inaccuracies when making connections.

Third, this study demonstrates that participants paid much less attention to their backward conceptual connections of addition and subtraction to backward counting than forward and skip counting operations. As the results indicated, among the participants who made connections to at least one of the three counting strategies, more paid attention to forward and skip counting (41.9% and 41.9%, respectively) than to backward counting (14.0%). One possible explanation is, for most people, counting on/up is a more natural strategy than counting down (Fuson, 1992). As a result, PSTs may use fewer backward counting connections than forward counting connections.

This finding contributes to the much-needed understanding of PSTs' competence in connecting addition and subtraction to various counting strategies absent from the existing relevant literature (Thanheiser et al., 2013). The lacking connection between addition and subtraction and backward counting poses a challenge for PSTs when helping students learn addition and subtraction effectively and flexibly. Students need to effectively use backward counting along with forward and skip counting to understand the relationship between addition and subtraction (Wu, 2011), improving their addition and subtraction calculations effectively (Geary, Brown, & Samaranayake, 1991; Geary, Liu, et al., 1999). To provide more insights, this study calls for future research to explore:

(1) whether and to what extent PSTs' limited attention to connecting addition and subtraction to backward counting identified in this study exists among PSTs across different programs and

(2) how such limited attention affects their addition and subtraction teaching practices.

Forward Conceptual Connections

Overall, the results revealed that participants in this study made few forward conceptual connections of addition and subtraction to the inverse relationship between addition and subtraction, and the associative and commutative properties underlying addition. As the results showed, less than 10% and 15% of the participants demonstrated forward conceptual connections of addition and subtraction to the inverse addition and subtraction, and to associative and commutative properties of addition, respectively. One potential explanation for this finding is that although PSTs have developed content knowledge relating to both current topics and future topics in mathematics, their knowledge is fragmented instead of coherent (Selter et al., 2012; Sun et al., 2019). Therefore, they primarily focused on teaching current topics as assigned and connected the current teaching with the concepts they had recently taught. Their non-coherent knowledge limited their thinking about how the current teaching relates to student future learning. As a result, they paid relatively limited attention to future topics, resulting in fewer forward conceptual connections.

This finding provides preliminary insights into how PSTs understand and use connections of addition and subtraction to the basic principles and properties underlying addition and subtraction (Thanheiser et al., 2013). This finding uncovers a challenge for teacher education programs. Unlike envisioned by the reformed curriculum and policy initiatives, PSTs lack sufficient attention to utilizing teaching skills that will help students prepare for mathematics learning later in their schooling (Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000). Such a challenge has important practical implications because students' deep understanding and flexible use of inverse addition and subtraction and associative and commutative properties is central to preparing them for effective learning of multi-step whole number and fraction operations and algebra (Ma, 1999; Wu, 2011). Therefore, future studies can benefit from verifying whether and to what extent PSTs' insufficient attention to forward conceptual connections of addition and subtraction prevails among PSTs across different programs. Furthermore, it is crucial to investigate the underlying mechanisms that lead to their insufficient attention to such connections and how lacking these connections will affect their teaching practices.

In sum, the findings suggest that PSTs are more likely to make backward conceptual connections than forward conceptual connections. While mathematics learning is longitudinal, students' learning is always built on previous knowledge and lays the foundations for future learning (Selter et al., 2012). Therefore, the present study highlights the need to closely monitor PSTs' development of horizon knowledge, especially those related to forward conceptual connections. When they observe that knowledge is needed to enhance either backward or forward connections, PSTs should strive to acquire such knowledge and, more importantly, enhance the effective utilization of this knowledge. For teacher training programs, they need to proactively design and dynamically adjust their training programs to serve the needs of preserve teachers to help them not only develop sufficient horizon knowledge for future teaching but also improve the effectiveness of using this knowledge in teaching practices.

Limitations

This study relied on PSTs' video presentations to assess PSTs' horizon knowledge for teaching addition and subtraction. Although those video presentations tend to reflect the horizon knowledge they may demonstrate in their teaching practice better than paper and pencil assessment (Clement, 2000; Stigler et al., 2000), they may not be used exclusively to represent their actual addition and subtraction teaching practices. Thus, the findings from this study may not be directly generalized to actual teaching practices on addition and subtraction with whole numbers. Second, because only the final video presentation in the module was used to identify the participants' horizon knowledge for teaching addition and subtraction, options were limited to examine if other factors other than the readings and activities included in the module had contributed to the horizon knowledge. For example, PSTs may engage in self-learning activities outside of the classroom, which could potentially impact their horizon knowledge about making connections in teaching. Finally, because we did not have the opportunity to interact with the participants, we have limited information about why they chose to perform the activities (e.g., making certain connections) in their presentation. Future research can provide more insights by interviewing the participants to learn more details about their intentions and, more importantly, their knowledge of making those connections.

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Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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APPENDIX A

Table A1. Rubric for coding backward & forward conceptual connections to whole number addition & subtraction

	Backward conceptual connection						Forward conceptual connections		
	Base-10			Counting			Inverse relationship between addition & subtraction	Properties of numbers and operations	
	Place value	Regrouping	Composing/ decomposing to make 10	Forward counting	Backward counting	Skip counting		Associative property	Commutative property
Participant 1									
Participant 2									
Participant 3									
…									

Note. Coding: 1-Participant accurately connects whole number addition & subtraction to this concept; 1e-Participant connects whole number addition & subtraction to this concept, however with some inaccuracy; & 0-Participant does not connect whole number addition & subtraction to this concept

APPENDIX B

Table B1. Typical PSTs' inaccuracies when connecting whole number addition & subtraction to base-10 concepts

	Inaccuracy type	Example
Base-10 place value	Not naming digits of a number by correct place value (i.e., ones, tens, & hundreds) when performing operations	When explaining adding 24 & 112, one said: "add 2 & 4 to get 6", "1 & 2 to get 3", & "add 1 & 0 to get 1" without a clear distinction in place values when adding tens & ones together in each case
	Incorrect or non-mathematical explanations of concepts or procedures	In demonstrating how to add 64 & 63, one used regrouping to add 6 in tens place of 64 & 6 in tens place of 63, she explained process of regrouping result from 6+6 into 1 in hundreds place & 2 in tens place without a clear explanation of why 10 units in the ten's place became 1 unit in hundred's place
	Inaccurate language	Use "cube" instead of "unit" & use "stick" instead of "long/rod"
Regrouping	Inaccurate language	Use "carry", "trade", "borrow", "transform", "change", & "exchange" instead of "regrouping"