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Flexibly Applying the Distributive Law – Students' Individual Ways of Perceiving the Distributive Property

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Flexibility in transforming algebraic expressions is recognized as fundamental for a rich procedural knowledge. Here, flexibility in-depth is proposed as the ability to apply one strategy to a wide range of unfamiliar expressions. In this study, design experiments with four groups of students were conducted to support students' flexibility in-depth in regard to the application of the distributive law with the help of worked examples. The data from these experiments was analyzed qualitatively with the method of content analysis. Students' flexibility in-depth translates into their abilities to reconstruct the distributive property within an expression via its perceived relevant structural features. The students' use individual cornerstones for this reconstruction, for example worked examples or certain focal points in an expression like the multiplication sign. Transforming expressions and especially applying formulas to algebraic expressions is thus an interpretative and reconstructive process.

Keywords: algebra, procedural knowledge, distributive property, flexibility, algebraic symbols

INTRODUCTION

Procedural knowledge is needed for conceptual understanding, as conceptual and procedural knowledge go hand in hand and build upon each other (Kieran 2013; Rittle-Johnson & Schneider, 2014). Students need procedural knowledge for solving non-routine problems, as it allows students to understand quantities and their relations (Kilpatrick, 2001). From an epistemic perspective, procedural knowledge - understood as fluency in transforming algebraic expressions - is necessary to understand the objects that are represented by algebraic expressions (Lagrange, 2003). In recent years, procedural knowledge has been recognized as being inextricably related to conceptual knowledge. From this perspective, procedural knowledge has been advocated to be understood not only as students' knowledge of

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strategies and algorithms, but also as an understanding about the steps in a strategy which they can apply flexibly and „with critical judgement“ (Star, 2007, p. 133; c.f.
Baroody, Feil, & Johnson, 2007a).

In the research discourse about procedural knowledge and its conceptual connections, there has been an ongoing discussion on how students transform unfamiliar expressions: It has been suggested that students need a "structure sense" to do this (Hoch & Dreyfus, 2005; Rüede, 2012), an idea this paper follows. In the mentioned studies, structure sense is conceptualized in a very general way with no focus on specific strategies like the distributive law. This generality leaves room for the question on how students choose the most appropriate strategy from a set of strategies for a given expression (Rittle-Johnson, Star, & Durkin, 2009; Star & Rittle-Johnson, 2008). However, while these studies conceptualize strategy use in its width there is little research on the strategy use in-depth – that is: how do students use one strategy for different expressions and how do they transfer this strategy to unfamiliar expressions?

Here, I present results from a design research study, which promoted the flexible application of the distributive law in-depth, that is, the transfer of the distributive law to unfamiliar expressions. The results of the study suggest that "flexibility-in-depth" is connected to the ways in which students perceive properties in an expression, where students, who sustainably transfer the distributive law can perceive properties in a qualitatively different way.

State of the literature

- Students should be able to use different strategies flexibly and adaptively, so that a chosen strategy suits the problem.
- A student can apply a strategy flexibly when he, according to Rittle-Johnson & Star (2008, 566), has multiple strategies available and is able to select the most appropriate or efficient strategy.
- Transferring one strategy to an expression is an interpretative process, and thus another facet of flexibility that has not been explored yet. It is an open question how this interpretative process influences the application of a strategy.

Contribution of this paper to the literature

- As part of flexibility, students should be able to choose the most appropriate strategy. This paper argues that this process involves flexibility-in-depth, that is, the ability to transfer one strategy to a wider range of expressions.
- This paper employs a mixed-methods approach to explore different ways how students transfer the distributive law to different algebraic expressions.
- Students flexibility-in-depth in transferring the distributive law translates into their abilities to reconstruct the distributive property within an expression via its perceived relevant structural features.

THEORETICAL CONSIDERATIONS

Flexibility in strategy use in-width and in-depth

Kilpatrick understands flexibility as adjusting or creating „a method [...] to fit the requirements of a novel situation“ (Kilpatrick, 2001, p. 127). In order to do that, students have to take into account the subject in question, the context and the task (Heinze, Star, & Verschaffel, 2009). Star further understands flexibility as a strategy choice when a student (a) has multiple strategies available and (b) uses these strategies with relative efficiency (Star & Rittle-Johnson, 2008, p. 566). Baroody extends this definition to also include creativity and „transfer-adopting“ a strategy so that it meets the demands of a (novel) task (Baroody, Feil, & Johnson, 2007, p. 120). These two definitions suggest that flexibility has two dimensions: Flexibility-in-width is concerned with students having multiple strategies available and applying them flexibly, while flexibility-in-depth is concerned with students being able to transfer and adopt each strategy to unfamiliar expressions, perhaps in a creative way.

Perceiving properties of an expression for flexibility-in-depth

Flexibility is connected to the perception of properties. In arithmetic, perceiving the properties of numbers is a necessary precondition for students to develop their calculation strategies for these numbers (Beishuizen, 2001; Torbeyns, Smedt, Ghesquière, & Verschaffel, 2009; Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). It allows the students to activate strategies that fit to each number. These strategies are also further developed in the situation, in reaction to the properties of the expression or number at hand (Proulx, 2013; Threlfall, 2002). For the case of the distributive law, in an expression *15ab-5b*, students might perceive that *b* occurs in each term, and could start factoring out. However, after this step, students might change their perception of the expression and could now see that *5* can be factored out as well - perhaps even in connection with the b from the outset.

In Algebra, perceiving the properties of an expression is partially guided by visual clues from surface features that suggest the use of a strategy (Kirshner, 1989). But in order to use a strategy sustainably, students need to perceive structural properties of an expression that are related to the use of this strategy, which is an interpretative process (Rüede, 2012). Hoch and Dreyfus (Hoch & Dreyfus, 2004) show that seeing a structure in equations, for example by perceiving brackets as structural elements, might help students to choose the appropriate transformation. The background for these interpretations can differ: Arcavi (2005) suggests that students need to be able to give meaning to an algebraic expression in order to chose sustainable strategies for transforming an expression. Kieran (Kieran, 2011) argues in regard to "Early Algebraization", that students need to be able „to see a certain form in algebraic expressions and equations" (p. 596) in order to apply their strategies according to the structural and relational properties of an expression. These studies, in a broad sense, are concerned with flexibility-in-width, as they address students' ways of coordinating multiple algebraic strategies.

Still, some issues of flexibility-in-depth might be derived from these studies as well. The transfer of the distributive law to an expression like *15ab-5b* is interpretative in regard to perceiving properties of the expression: It depends upon the students individual ways of perceiving the structural properties of the expression whether or not *5*, *5b* or *b* is factored out. Perceiving structural properties might be related, as Kieran puts it, to see the form of the distributive law in the expression, that is, seeing which elements in *15ab-5b* might constitute a distributive property. This issue is at the heart of algebraic symbolism, as flexibility-in-depth empowers students to look structurally on an expression not based on visual clues, but with the help of historically established forms like the distributive property as background.

Research questions

In this paper, I address the question of how students flexibly apply the distributive law in-depth. More specifically, assuming that transferring a strategy is an interpretative process, how do students perceive properties of an unfamiliar expression while transferring a strategy to this expression?

CONCEPTUAL FRAMEWORK OF THE STUDY

Strategy use as an interpretative activity

Transforming algebraic expressions is regarded as an activity in which students mediate between the cultural praxes of the mathematics classroom and their individual notions and interpretations of the symbols and expressions (Radford, 2006). In this study, this is implemented pragmatically: When working on an expression, students can relate to cultural praxes by means of interaction with the teacher and artifacts (e.g. the task formulation) and to their own interpretations of the given expressions and their properties. Both come to life in the students' activities of transforming an expression; thus, the students' activities of transforming are the unit of analysis for reconstructing how students perceive properties. This, then, gives insight into flexibility-in-depth.

Design experiments to support students' perception of properties

This study is design-research based. Its design experiments are aimed to support students' in perceiving properties in algebraic expressions. Supporting students in perceiving properties can be accomplished by guiding students' attention, for example by sensitizing students for those properties that remain and those that change when a strategy is used, and to the invariant properties of an expression during such a transformation (Mason, 2004; Watson & Mason, 2006). In order to perceive relevant properties, students need previous knowledge about the specific properties related to a strategy, as previous knowledge constrains the application of a strategy (Newton, Star & Lynch, 2010).

Perceiving properties of an algebraic expression by guiding students attention can be accomplished with two teaching strategies: First, comparing the use of strategies with the help of worked examples, that is, exemplary solutions of a task, that include accompanying prompts can help students to perceive properties, when they are presented in direct connection to a similar task on the same page (Star & Rittle-Johnson, 2009, p. 421). Additionally, „comparing contrasting solutions seemed to support gains in procedural knowledge because it facilitated students' exploration and use of alternative solution methods "(Rittle Johnson & Star, 2007, p. 572). In order for students to use worked examples effectively, they have to fit to the previous knowledge of the students (Guo & Pang, 2011). Second, Rojano, Filloy and Puig (2014) illustrate, for the case of the strategy of substitution, that making references to previous transformations can help students to form a more sustainable strategy, that is, it helps students to transfer a strategy.

Task elements for supporting comparisons with worked examples

In this study, these two teaching strategies are implemented with the help of specific tasks. First, students are supported in using previous transformations of algebraic expressions by giving worked-out examples (in the following referred to as worked examples). For that, in task 2 of the study, students applied the distributive law to various expressions. For each expression, the distributive law was given in its original form, as either $a(b+c)=ab+ac$ or $ab+ac=a(b+c)$. This original form was given together with each expression, assuming that this closeness enhances the awareness of the possibilities for references to the distributive property. In task 3, which was constructed to support flexibility-in-depth, the students were asked to connect each given expression to a previously transformed algebraic expression from task 2. Each expression was given on a small sheet of paper, which allowed the students to actually place the expression close to their chosen worked example. Most of the time, after the students found a comparable worked example, they used this worked example as a guide to identify the distributive property in the given expression. This way, the worked example fostered the transfer of the distributive law and supported flexibility-in-depth.

Second, it was assumed that by giving the students the possibility to choose worked examples that were selfmade, the students flexibility-in-depth is supported. By choosing the most fitting worked example in line with their own interpretations of the distributive law students might identify more connections and shared properties between the expression at hand and the distributive property, as the worked example can act as an intermediate transfer-step. Furthermore, as Rojano et al. (2014) suggested, this situation helps students to relate to their previous transfer in task 2, where students already had transferred the distributive law based on its given original form. Accordingly, the students might use what they have learned from their previous transfer of the distributive law and apply this to the current given expression.

CONTEXT UND PARTICIPANTS

Design research study

This paper presents results of a design research study (Cobb, Confrey, & Disessa, 2003; Van den Akker, Gravemeijer, McKenney, & Nieveen, 2006) that was conceived to support students in becoming flexible in applying the distributive law to unfamiliar algebraic expressions in-depth. The study is based on a 90min. teaching intervention that consists of three tasks (as shown above). This teaching intervention is a product of two previous design circles. Here, results from the third task presented above are discussed, as in this task students are confronted with unfamiliar expressions without external support mechanisms.

The expressions given to the students in task 3 are presented in Table 2. These expressions are unfamiliar in the sense that the distributive property can be identified in more than one way. This way, flexibility-in-depth is needed, as the students had to first identify the distributive property in the expressions. For example, in *15ab-5b*, different terms can be factored out and accordingly the distributive property can be identified in more than one way.

Participants

Eight 8th graders participated in the study in four groups. The students were familiar with the distributive law, as they were introduced to it in the classroom around 2-4 weeks earlier. With the help of a pretest that consisted of transforming algebraic expressions and of giving a non-symbolic representation (verbal/geometric) for algebraic expressions, those students with a solid knowledge of the syntax (brackets, operation signs, rules like „-·-=+"etc., relational understanding of equations) and semantics (variable meanings) of algebra in familiar expressions were chosen. Based on the test items, the students did not yet show flexibility-in-depth or routine in applying the distributive law, that is, they showed problems in applying the distributive law in expressions with multiple ways to identify the distributive property.

Data collection and interview settings

The students were interviewed by specially trained interviewers. The training involved raising awareness for issues of flexibility-in-depth and the application of the distributive law as well as becoming aware of the central supporting mechanisms of the teaching intervention. Each interview/teaching intervention was videotaped and later transcribed. Each interview/teaching intervention was structured by the teaching material. The interviewers were encouraged to probe the students thinking and to not immediately intervene in unsustainable trans-formations.

Method of data analysis: Content Analysis

The data was qualitatively analyzed with the method of content analysis with both descriptive and inductive codes. By also employing quantitative approaches to the co-occurrence of codes, hypotheses about the data were generated (Mayring, 2015). The co-occurrence of codes led to the identification of relevant passages in the material, which were then analyzed in depth (comp. Figure 1, left side, for an overview of the steps of the analysis). The deductive codes were obtained based on the above presented theoretical considerations (Table 1).

Table 1. Codes and their descriptions (first step of analysis); one data segment can be coded with multiple codes

Code (in italics: not included in later steps of the analysis)	Code description / expected examples
Student perceives similar features / number of similar features	A student perceives features in a given expression, that in his view share similar properties. This may be connected to the perception of how often a similar property occurs. Expected examples: “there are three *c*’s.”; “In each term there is a *c*.”
Student perceives generality	A student perceives features that in his view suggest a general property. Expected example: “It doesn’t matter how the factor looks like”
Student perceives position or order of features	A student perceives features that are connected to the position or order of terms in the expression. Expected example: “the variable is in front of the brackets.”
Student perceives brackets	A student perceives brackets. Expected example: “there are brackets in the expression.”
Student perceives calculation sign	A student perceives a division, a multiplication, an additions etc. “there is a minus sign, so you have to ...”
Student perceives numbers and number expressions	A student perceives numbers or number signs. Expected Example: “there is a number in the term”.
Student perceives variables and variable expressions	A student perceives variables or variable expressions. Expected Example: “there is a variable in the term”.
Transformation mathematically sustainable	Transformation is correct; correct transformation was intended but was prevented by distractions (e.g. from interviewer or other student).
Transformation mathematically not sustainable	Transformation is based on correct rules, but are not applied correctly; correct transformation was intended, but is hindered by erroneous application of more basic transformations; transformation is based on unsustainable/incorrect transformation rules.
Student transforms according to interactions with interviewer	The interviewer influences the transformation of expressions. This way, the transfer of the distributive law is based on interactions with the interviewer.
Student performs rule-based transformations	A student transforms an expression by explicitly citing a rule (e.g. *4ab-4ab=0*; *a(b+c)=ab+ac*; “-·+=-“).
Student plans his transformation	A student first perceives features and then hypothesizes, which rules/ transformations might be applicable based on these features.
Student corrects his transformation	After an (unsustainable) transformation was made, the student corrects his transformation.

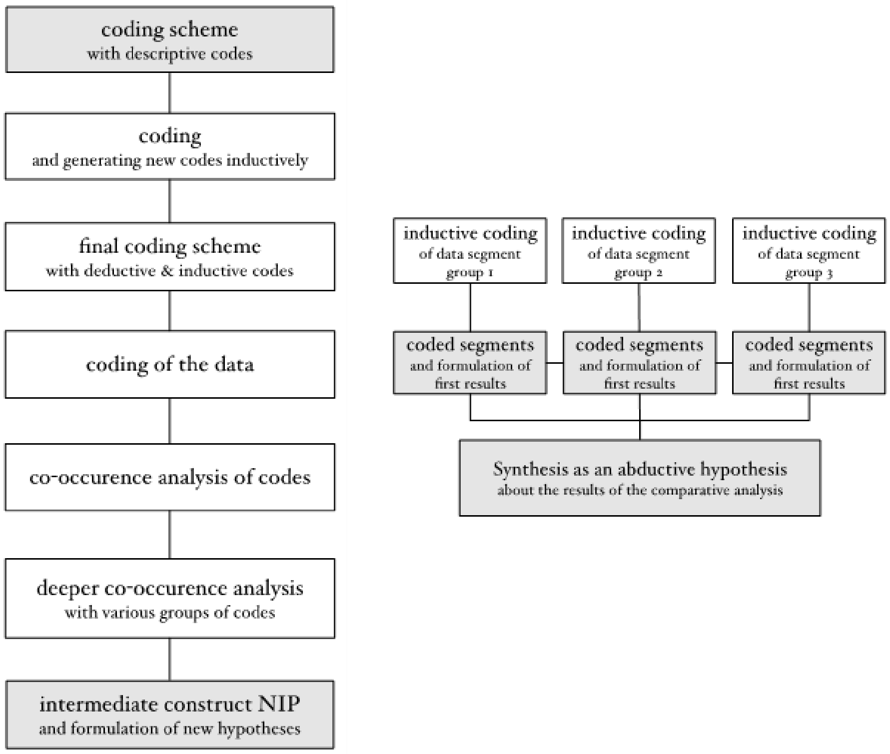

Figure 1. Steps of the analysis
Note: left; deductive analysis, right; comparative analysis. Grey marks those parts of the analysis that are documented here.

The unit of analysis was data segments in which students interpreted or transformed an expression or negotiated transformations in reference to their activities of transforming an expression. Each code is attributed to data segments interpretatively. When a data segment did not fit into the coding scheme, new codes were generated inductively for these segments. Inductive codes and descriptions were then added to the coding scheme.

The coded data segments were analyzed in regard to the co-occurrence of multiple codes. For example, all segments where retrieved where both the codes "perceiving variables" and „mathematically sustainable transformation“ occurred. It was assumed that this would result in groups of data segments, where certain connections and patterns between perceiving certain properties and (un-)sustainably transferring the distributive law could be uncovered. However, a comparative analysis of these groups of data segments led to no visible connections or patterns.

Accordingly, a deeper co-occurrence analysis was conducted that consisted of combining certain codes into new codes. These new codes were then employed in the co-occurrence analysis. Different combinations of codes were tested in an explorative way in order to uncover patterns in the data. This analysis suggested that the *number of identifiable properties* (NIP) that are identified by students relates to the sustainability of the transfer of the distributive law. The *NIP* is counted as the number of those properties, in which a given expression is different from the original form of the distributive law *ab+ac=a(b+c)* or *a(b+c)=ab+ac*, or, in other words, the number of properties which add complexity to uncovering the distributive property. As table 2 illustrates, this number ranges from 2-5. While the NIP, at first sight, might suggest

Table 2. Expressions in task 3 and those properties that add complexity to uncovering the distributive property

Expression in task 3 in the order as given to the students	Number of identifiable properties (NIP). (in italics: expressions where the distributive property can be identified in multiple ways)
xs-xt	• „-“ minus sign • differing variables, not chosen according to the relatively usual pattern *(xy+xz)*
(b+c)(-2)	• right-distributive instead of left-distributive (Position) • number instead of variable as factor • negative number as factor, thus in brackets
xy+xz+wx-vx	• more terms • differing variables / *x* in each term • minus sign
15ab-5b	• numbers and variables (both can be factors) • minus sign • number of variables in the terms differ
2x(4z+2vz-z)	• more terms • same variable in every expression in the brackets • factor composed of number and variable • numbers and variables in brackets • minus sign
4ab-4a(b+4ab)	• additional, structurally similar term 4ab that is not immediately related to distributive structure • negative factor *-4a,* • factor composition of number and variable • terms in brackets have different number of variables, numbers • same variables in all terms
15(ab-5b)	• factor only consists of a number • terms in brackets consist of variables and numbers • minus sign
a/2(x+2z)	• fraction as factor • divisor „2“ • term in brackets consist of numbers and variables

that a sustainable transfer of the distributive law is only dependent upon the number of properties that students perceive in a mechanical transformation process, the following analysis of transcript passages shows that it is a very rough measurement of how complex it is for students to interpretatively uncover the distributive property in an expression. Accordingly, it is here used as an intermediary construct for generating hypotheses.

Data was coded with „high perceived NIP"if students identify n (for expressions with NIP=2) to n-1 (NIP higher 2) properties of an expression. Data, in which fewer properties are identified, is coded with „low perceived NIP". The intermediate construct NIP suggests a connection between the sustainability of a transformation and the NIP that students perceive (table 3).

If students perceive a high NIP, this co-occurs with a sustainable application of the distributive law in all instances. If students perceive a low NIP, this co-occurs with both sustainable and unsustainable applications of the distributive law. If students perceive a high number of properties in an expression, they will very likely, according to the data, transfer the distributive law in a way that results in a sustainable application of the distributive law to an unfamiliar expression, at least for the first 4-5 expressions in task 3 (see table 4, left side). However, if students perceive only a low number of properties, the sustainability of the application of the distributive law seems to depend on other factors than the number of perceived properties of an expression. The following hypotheses, obtained by means of abductive reasoning, may „account for the phenomenon in question [in this case, the connection between the sustainability of a transformation and the NIP that students perceive as shown in Tables 3 and 4, A.M.]" (Clement, 2000, p. 554).

Hypothesis 1: Routine-building

Over the course of transforming the expressions, students develop a routine that helps them to only attend to the most relevant properties of an expression; this is consistent with table 4, where in the latter expressions, students are able to transform the expressions sustainably without the need to perceive a high number of NIP.

Hypothesis 2: Complexity of expressions

The expressions become gradually more complex. The last expressions are qualitatively more complex than the first expressions and can be solved only by more advanced students. In table 4, the more advanced students are those, that sustainably transfer the distributive law without the need to attend to all NIP.

Table 3. Sustainability of transformation in relation to NIP

Codes	**high perceived NIP**	**low perceived NIP**
Lacking Sustainability of transformation	0	7
Sustainability of transformation	14	12

Table 4. Connections between sustainability of transformation and NIP

	High perceived NIP	**Low perceived NIP**
xs-xt	4 (sustainable: 4)	0 (sustainable: 0)
(b+c)(-2)	3 (sustainable: 3)	1 (sustainable: 1)
xy+xz+wx-vx	3 (sustainable: 3)	1 (sustainable: 1)
15ab-5b	2 (sustainable: 2)	2 (sustainable: 2)
2x(4z+2vz-z)	1 (sustainable: 1)	3 (sustainable: 2)
4ab-4a(b+4ab)	1 (sustainable: 1)	3 (sustainable: 1)
15(ab-5b)	0 (sustainable: 0)	4 (sustainable: 3)
a/2(x+2z)	0 (sustainable: 0)	4 (sustainable: 2)

Hypothesis 3: Ways of perceiving features

The expressions become more complex in the sense that it is harder to identify the distributive property in these expressions; then, only those students who can attend to relevant (not all!) properties that are connected with this form can transform these expressions sustainably. In this case, the numbers in table 4 are an expression of the different ways in which students identify what is relevant in an expression.

Hypotheses 2 and 3 connect well to the fact that in the expressions four to seven (table 2) there are multiple ways to identify the distributive property, which adds complexity to applying the distributive law.

RESULTS

In the following, the common characteristics of each data segment group, as constituted by the fields in table 3, are illustrated with prototypical examples from the data segment groups, which incorporate most of the characteristics of its group. All episodes show students working on the expression *4ab-4a(b+4ab)*.

Characteristics of data segment group 1: Students apply the distributive law sustainably and notice a high number of properties

Students who transfer the distributive law mathematically sustainable while also perceiving a high number of identifiable properties in an expression are able to uncover the distributive property based on various, not necessarily structurally relevant properties. In the following example, Dennis and Can try to transform the expression *4ab-4a(b+4ab)*:

> 145 Dennis: Ahm [5 sec.], fits to A [*x(y+z)*] and B [*2a(b+c)*] I think, yes. It fits to both. Yes. Only that there comes also a minus with it. I would first calculate these [points at *4ab(b+4ab)*, speaks to Can]. The first result would be here *4ab*, this would be it, that would result in *4ab* as whole result. This would fit to both, then. Or what do you think? [3 sec.] It fits to both.
>
> [...]
>
> 149 I.: Can you say again why?
>
> 150 Dennis: Yes. Now it does not make a difference if there is a number in front of the number, I mean the number which is in front of the brackets. One just calculates *4ab* times *ab* and *4a* times *4ab*. One could just calculate this and the result here is 4ab [points at the second half of the expression *4ab-4a(b+4ab)*]. And the 4ab minus *4ab* results in *0*. *4* times *4* is *16*. *16ab* is the whole result. As I would calculate it. And here it is the same way if you calculate it. Only that there are numbers, and here only variable expressions. So this is then the same.

In this episode, Dennis identifies several features of the expression, namely the minus sign, the brackets, the composition of terms and especially the composition of the factor for the distributive property (turn 150, "the number which is in front of the brackets"). Furthermore, he uses worked examples to figure out how to transform the expression with the distributive law (turn 145, "I would first calculate these"). For that, he refers to two examples that are very close to the original form of the distributive law, namely *x(y+z)* and *2a(b+c)*. Here, Dennis has transferred the distributive law in a correct way (he corrects his error in later turns).

Dennis' flexibility in-depth is informed by worked examples that help him to first focus on the expression *4ab(b+4ab)* (turn 143). This way, he notices that the expression *-4a(b+4ab)* is structurally similar to both *x(y+z)* and *2a(b+c)*, except for the minus sign. In order to connect the worked examples and the expression, Dennis first abstracts from the composition of the factor of the distributive property (turn

150, „it does not make a difference if there is a number in front of the number, I mean the number which is in front of the brackets"). Thus, in this episode, the distributive law is not perceived as a ready-made object in the expression. Instead, the students transfer the distributive law, first, by uncovering the distributive property in the expression, guided by worked examples, and second, by adapting the factor. In other words, flexibility-in-depth in this group means to bridge the divide between the worked examples and the distributive property in the expression, where this distributive property is "concealed" by other variables, numbers, and a more complex composition of the terms in general.

Characteristics of data segment group 2: Students apply the distributive law unsustainably, if they cannot uncover the distributive property

Students who apply the distributive law in an unsustainable way are able to perceive some relevant properties of an expression, but they do not connect these properties in a way that helps them to uncover the distributive property in the given expression. In the following example, Silanur and Merve try to transform the expression *4ab-4a(b+4ab)*:

> 208 Silanur: Everywhere is *a* and everywhere [2 sec] so here is an *a*, here is an *a* and here is an *a* [underlines all *a*'s in the expression]. Here is a *b*, here is a *b*, and here is a *b* [underlines all *b*'s in the expression]. And everywhere *4* [underlines all *4*'s in the expression]. I guess everywhere 3 times.
>
> [...]
>
> 211 Merve: Should we take always two? Because it fits in everywhere. Or should we leave out the *4* then?
>
> [...]
>
> 215 I.: Why would you take two then?
>
> 216 Merve: Because else you can't put the *4* in the front, but you don't need that anyways?
>
> [...]
>
> 219 I.: Why can't you put the *4* in the front?
>
> 220 Merve: Because the *4* is not in the b [points at the *b* in (*b+4ab)*], I mean *b* isn't in it.

In this episode, Silanur and Merve identify common features in the terms, namely the occurence of *a, b* and of the „*4*". However, as the dialogue shows, they cannot decide which transformation may result from these features. Different to Dennis and Can, they do not refer to a worked example. Instead, Merve seems to be guided by the assumption that she needs to find a variable or number that "fits in everywhere" (turn 211). Furthermore, she wants to factor out the expression instead of expanding it, perhaps based on her identification of common features in the terms. In this episode, Silanur and Merve have not transferred the distributive law in a mathematically sustainable way.

Merve's transfer of the distributive law is based on the idea of factoring out. In comparison to Dennis' transfer, Merve does not interpret or adapt the identified features in a way that would help her to uncover the distributive property in the given expression. Instead, she struggles with factoring out: *4* is not „in" the expression with *b*, and thus, you cannot use *4* as a factor for factoring out („put [...] to the front", turn 216).

In this episode, the students' flexibility in-depth is lacking. The students try to find a direct link between certain features in the expression and factoring out. Different from Dennis in the previous group, Merve and Silanur do not uncover features, but orientate themselves at surface features that they associate with the distributive property. Furthermore, focusing in this way on factoring out seems to prevent Merve and Silanur to attend to properties that are relevant for transferring the distributive law, in this case the brackets. Accordingly, Merve and Silanur seem to lack flexibility-in-depth.

Characteristics of data segment group 3: Students apply the distributive law sustainably, if they can see the distributive property

Students who transfer the distributive law in a sustainable way and perceive a low number of properties, tend to hierarchically perceive features of an expression. An example for this is Anna, who transforms the expression *4ab-4a(b+4ab)*:

> 263 Anna: Because that is *4ab* then, and now I leave out the imaginary brackets [around *-4a(b+4ab)]* again, *4a* then brackets *b* plus *4ab*. […] And then I would simply this over here [points at *4ab* in the first half of the expression] I would leave it out for the time being, and calculate this then, because this would be for me [2sec.] Minus *4ab* and then times plus , because of the minus I would make minus. This would be in the small [imaginary] brackets again minus *16a* squared *b* [3 sec.] Yes this is then, is then yes minus in the big ones and then I would calculate it the way that then it is *4ab-4ab-16a* squared *b*.

In this episode, Anna identifies some features of the given expression, namely the composition of the terms and the minus sign – which she handles by inserting imaginary brackets. Furthermore, the other student Sonja indicates previously that „there is a multiplication sign [in front of this bracket]“ (turn 248). Anna inserts imaginary brackets around the expression like this: “(*-4a(b+4ab*))”. In this episode, Anna and Sonja transfer the distributive law in a sustainable way.

Annas flexibility in-depth is characterized by perceiving features in their connection to the distributive property. The perception of the minus sign and the multiplication sign guides Anna and Sonja to uncover the distributive property – in comparison to Dennis, these students do not need a worked example for this. In comparison to Merve, there is a form of adaption, as Anna uses brackets in an informal, yet structural way, for connecting the perceived properties with the distributive property in the expression. Thus, in this episode, perceiving certain properties in the expressions helps Anna and Sonja to uncover the distributive property, which then allows them to perceive other properties of the expression in a hierarchical and focused way: On the one hand, Anna postpones transforming the first term *4ab* („I would leave it out for the time being“). On the other hand, other properties of the expression are not even in focus, e.g. the specific properties of the terms in the brackets: Anna operates on them without the need make their properties explicit.

In summary, this transfer of the distributive law is based on a hierarchical perception of features that revolves around the identification of the multiplication sign and a subsequent structuring (imaginary brackets). In other words, flexibility-in-depth here includes being able to identify a central property of the distributive property (the multiplication sign) and using this central property as focal point for further considerations.

SUMMARY AND DISCUSSION

Students’ flexibility in-depth, that is, their abilities to transfer the distributive law to unfamiliar expressions, translates into their abilities to reconstruct the distributive property within an expression via its perceived relevant structural features. Reconstructing refers to the fact that the students in this study, as illustrated above, do not perceive the distributive law as a ready-made object, but, in a interpretation-guided movement in the expression, uncover the distributive property and start rebuilding the distributive law from there – within the confinements and given features of the expression. Within refers to the fact that the distributive law is not imposed on an expression, but is build bottom-up from existing/ perceived properties.

Flexibility in-depth is influenced by different factors. In line with the initially formulated hypotheses, these are the way students perceive properties, the complexity of expressions and routine-building:

Ways of perceiving features

It does not necessarily matter if students can perceive more features, but it does matter how students perceive features. As Merves and Silanurs episode (data segment group 2) suggests, students are generally able to perceive most of the features of an expression, but cannot necessarily use them to reconstruct the distributive property.

Routine-building

In turn 150 (data segment group 1), Dennis comes to the understanding that the exact composition of the factor in the distributive law is not necessarily relevant. In a similar way, Anna does not explicitly perceive the composition of terms where it is not necessary. Thus, building a routine is, according to these instances, connected to generalizations of some properties of an expression.

Complexity of expressions

The expression *4ab-4a(b+4ab)* is more complex as it has two properties that, in comparison to the other expressions in this study, add a new layer of complexity to uncovering the distributive property, namely that both factoring out and expanding seem to be equally viable. In this study, this complexity triggers students' flexibility in-depth.

Students' flexibility in-depth builds upon the quality of how features are perceived and not upon how many features are perceived. Students who can generalize some properties of an expression can focus on relevant features and do not need to perceive every feature of an expression. This way, they can cope with more complex expressions. Individual cornerstones for the transfer of the distributive law are, in this study, worked examples or certain focal points (e.g. the multiplication sign).

This paper suggests a two-dimensional landscape. One dimension is flexibility-in-width – having different strategies available. The other here proposed dimension is flexibility-in-depth – transferring one strategy to different, unfamiliar expressions. The transfer of the distributive law builds on activities of uncovering the distributive property, which is, in turn, connected to perceiving properties in an expression. But perceiving properties is only a necessary prerequisite for students to interpretatively uncover the distributive law. Thus, flexibility in-depth builds on interpretative activities. But what are the means for interpreting the properties of an expression? Are they confined to the expression at hand and its properties, that is, are they procedural? Or are they related to other sources for interpretation, e.g. geometric figures of references to real-world problems, that is, do they belong to conceptual knowledge? These questions hint at issues of flexibility in-depth in other mathematical areas where formulas are used in conceptual ways. For example, applying the binomial formula is related to issues of transferring the formula to yet unfamiliar expressions, to flexibility in-depth. One might think of quadratic functions and searching for their zeros. In such instances, flexibility in-depth is a necessary prerequisite for working conceptually, for interpreting algebraic expressions.

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