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Grade 9 mathematics teachers' strategies to address mathematical proficiency in their teaching of linear equations

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Abstract

Developing mathematical proficiency is essential for learners to effectively engage with and apply mathematical concepts. This study investigates how mathematics teachers support learners in achieving proficiency in solving linear equations, drawing on the mathematics teaching and learning framework, which promotes a learner-centered and interactive approach to mathematics instruction. The study explores how teachers draw upon the five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition to enhance learner understanding of algebraic equations. A qualitative research approach with a case study design was employed to explore teachers' instructional practices. Data were collected from two grade 9 mathematics teachers in the Gauteng North District, South Africa. Classroom observations were conducted through video recordings of lessons, followed by in-depth interviews to gain further insights into teachers' strategies and pedagogical choices. A deductive analysis approach was used, with the five strands of mathematical proficiency serving as predefined themes for data interpretation. The findings reveal that while teachers focus extensively on developing learners' PU, there is limited emphasis on the other strands of proficiency. As a result, learners may struggle to develop a deeper CU, strategic problem-solving skills, and AR abilities, which are crucial for mathematical competence. These findings suggest that a more balanced instructional approach is necessary to foster all aspects of mathematical proficiency. It recommends that teachers design and implement instructional activities that integrate all five strands of proficiency to ensure a comprehensive understanding of mathematical concepts. Additionally, professional development programs should be tailored to equip teachers with strategies that enhance learners' holistic mathematical development.

Keywords: mathematical proficiency, conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition

INTRODUCTION

Learners often struggle with algebra due to conceptual misunderstandings, weak foundational skills, and difficulties with abstract reasoning (Stemele & Jina Asvat, 2024). As a fundamental component of algebra, linear equations play a crucial role in upper secondary mathematics, serving as essential tools for representing and transforming abstract concepts. They also act as a gateway to advanced studies in fields such as engineering and architecture (Holmlund, 2024). The Department of Basic Education (DBE, 2012) recognizes the significance of linear equations in the global history of mathematics curricula. Given their foundational role, linear equations receive substantial emphasis in assessments, as they are integral to other key mathematical topics, including patterns, sequences and series, trigonometry, quadratic functions, and analytical geometry (DBE, 2012).

In South Africa, DBE (2018) mandates that teachers develop learners' mathematical proficiency by focusing on teaching mathematics for understanding, particularly when addressing topics such as linear equations. The concept of mathematical proficiency emphasizes the

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Contribution to the literature

- This article contributes to the literature on mathematics education by examining how grade 9 teachers in South Africa develop students' mathematical proficiency in solving linear equations.
- Unlike previous studies that primarily focus on procedural fluency (PU), this research uniquely employs five-strand framework—conceptual understanding (CU), PU, strategic competence (SC), adaptive reasoning (AR), and productive disposition (PD)—as a lens to analyze instructional strategies.
- Using a qualitative case study approach, the study provides in-depth insights from classroom observations and teacher interviews, revealing an overemphasis on PU at the expense of deeper conceptual engagement and reasoning. Its findings highlight the need for a more balanced instructional approach and recommend professional development initiatives to enhance teachers' ability to foster all aspects of mathematical proficiency.

Table 1. Content area achievement comparison between South African and international learners (% correct) (all TIMSS items) (Mosimege et al., 2017, p. 11)

Content area	South Africa (% correct)	International (% correct)
Number	21	44
Algebra	20	37
Geometry	19	37
Data & chance	26	47

need for a deep and lasting understanding of mathematical concepts. To support this objective, the DBE (2018) introduced the *Mathematics teaching and learning framework: Teaching mathematics for understanding*. This framework aligns with Kilpatrick et al.'s (2001) five strands of mathematical proficiency: CU, PU, SC, AR, and PD. However, while the DBE framework directly incorporates the first four strands, it integrates PD through learning-centered classroom practices. The development of this strand is seen as a gradual process that emerges as learners engage in these practices.

Despite the establishment of the DBE (2018) framework and its alignment with Kilpatrick et al.'s (2001) strands of mathematical proficiency, learner performance in mathematics remains alarmingly low, both nationally and internationally. Data from various assessments, including the Trends in International Mathematics and Science Study (TIMSS) 2019, indicate persistent challenges in mathematics achievement (Mullis et al., 2020). According to Mullis et al. (2020), South African learners underperformed compared to their international counterparts across all content domains, particularly in algebra. **Table 1** presents a comparative analysis of mathematics performance between South African and international learners based on TIMSS 2019 results.

South African learners' performance in algebra (20%) was significantly lower than the international average (37%), highlighting a major crisis in mathematics teaching and learning. Even internationally, algebra and geometry scores were below 50%, indicating that these areas present challenges for learners worldwide. Further

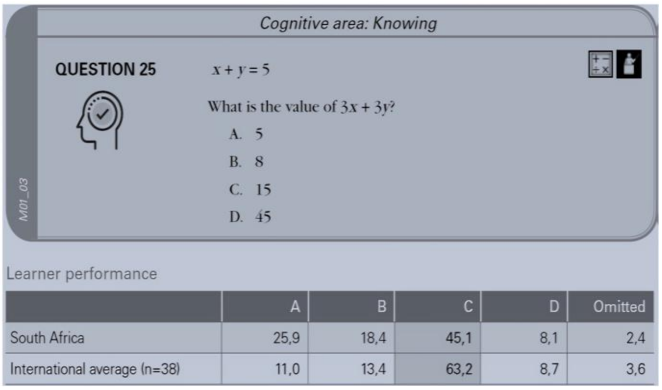

Figure 1. TIMSS item-by-item analysis of algebra performance (Mosimege et al., 2017, p. 39)

analysis of the TIMSS 2019 results (Mullis et al., 2020), as shown in **Figure 1**, illustrates the specific difficulties South African learners face in algebra.

On average, only 45% of South African grade 9 learners answered a specific algebra question correctly. Many learners struggled to recognize that the expression $3x + 3y$ is a multiple of $x + y$ and failed to balance the equation. Additionally, 26% of learners incorrectly selected option A, likely assuming that factoring out the common factor 3 from $3x + 3y$ would automatically lead to the equation $x + y = 5$. This pattern suggests a fundamental lack of CU, which may stem from an overemphasis on PU rather than deeper reasoning in mathematics instruction.

Given these persistent challenges, South African learners will likely continue to underperform in mathematics unless teachers strengthen learners' mathematical proficiency by prioritizing CU alongside PU. Despite the DBE framework's guidance, many teachers struggle to implement effective strategies for teaching linear equations conceptually. Nkundabakura et al. (2024) emphasize the need for continuous professional development programs that empower teachers with meaningful strategies to enhance their instructional practices.

Against this backdrop, this study aims to explore the strategies mathematics teachers use to develop learners' mathematical proficiency—including PU, SC, CU, AR,

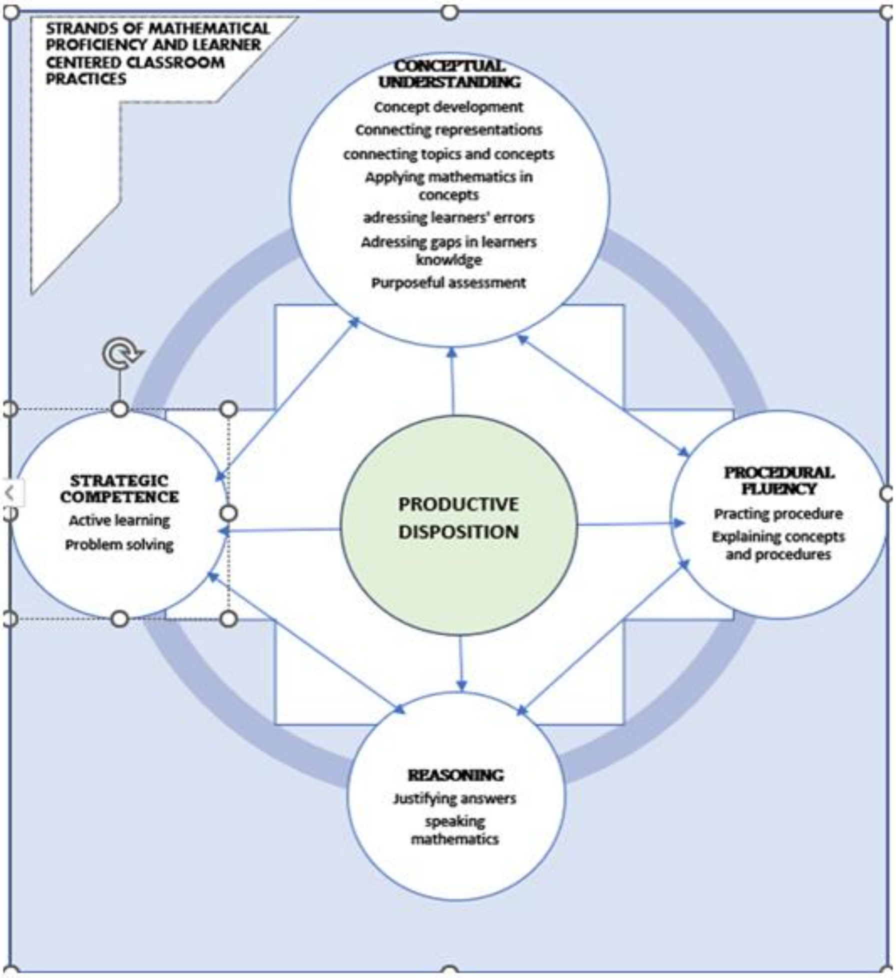

Figure 2. Conceptual framework on strands for mathematical proficiency (Adapted from Kilpatrick et al.'s, 2001 intertwined strands of proficiency)

and PD—when teaching linear equations. Specifically, the study is guided by the following research question: What strategies do mathematics teachers utilize to address mathematical proficiency—including PU, SC, CU, AR, and PD—when teaching linear equations?

This investigation seeks to contribute to the ongoing discourse on mathematics education by identifying instructional practices that can enhance learner outcomes and bridge the gap between policy suggestions and actual classroom implementation.

CONCEPTUAL FRAMEWORK

This study is grounded in Kilpatrick et al.'s (2001) intertwined proficiency model, which conceptualizes mathematical proficiency as comprising five interdependent and interwoven strands: CU, PU, SC, AR, and PD. Kilpatrick et al. (2001) describe these strands as "intertwined" because they are deeply connected, representing different but essential aspects of a unified mathematical competency. Achieving mathematical proficiency requires the simultaneous development of all five strands rather than focusing on just one or two. If any strand is neglected, the overall effectiveness and efficiency of mathematics teaching and learning are compromised.

Intertwined Strands of Mathematical Proficiency

Figure 2 illustrates how this study envisions the conceptual framework. Unlike a hierarchical model, where certain skills are prioritized over others, this framework maintains that all strands are equally important and function collaboratively. The arrows in the diagram indicate the reciprocal relationships among the strands, emphasizing their interconnected nature.

This framework guided the study's exploration of how teachers develop mathematical proficiency in learners when teaching linear equations. Although instructional practices can be categorized under different strands, none of them can be applied in isolation. Each strand contributes to a comprehensive understanding of mathematics, ensuring that learners develop the necessary skills, reasoning abilities, and positive attitudes to engage meaningfully with mathematical concepts. The strands are detailed as follows.

CU

CU refers to a learner's comprehension of mathematical concepts, operations, and relationships (Kilpatrick et al., 2001). It extends beyond memorization, allowing learners to grasp the underlying principles that connect different mathematical ideas (Cavanagh & Kiersch, 2023). In the context of linear equations, CU involves recognizing the relationships between symbols, expressions, and their representations. Learners with strong CU can connect linear equations to other mathematical concepts, such as sequences, patterns, and functions, and apply multiple representations when solving problems (Tashtoush et al., 2024).

Research indicates that a robust CU in algebra requires building on prior knowledge. For example, proficiency in solving linear equations depends on foundational skills such as finding multiples, factoring, and understanding the properties of numbers (Chen & Wu, 2026). İşözen and Kaçar (2026) further emphasize that conceptual knowledge develops when learners connect new mathematical ideas with pre-existing knowledge. Effective mathematics instruction should therefore integrate real-life applications and interdisciplinary connections, enabling learners to see the relevance of mathematical concepts in broader contexts (Amalia et al., 2024).

PU

PU is the ability to execute mathematical procedures flexibly, accurately, efficiently, and appropriately (Kilpatrick et al., 2001). It involves mastering algorithms, formulas, and symbolic manipulations while ensuring that procedural steps align with CU (İşözen & Kaçar, 2026). In the case of linear equations, PU means systematically applying algebraic operations to find solutions while understanding why these steps work.

PU is not independent of CU; rather, the two are intertwined. Learners must first grasp the underlying principles of linear equations before effectively applying procedures. Conversely, repeated practice with procedures helps reinforce CU, demonstrating the dynamic relationship between these two strands. Thus, effective instruction should balance conceptual and procedural learning to ensure learners develop both accuracy and understanding in mathematical problem-solving (İşözen & Kaçar, 2026).

SC

SC refers to the ability to formulate, represent, and solve mathematical problems using appropriate methods and strategies (Kilpatrick et al., 2001). It involves problem representation, selecting suitable problem-solving approaches, and applying strategies effectively (Copur-Gencturk & Doleck, 2021). Viro and Joutsenlahti (2020) asserts that SC is demonstrated when learners can adapt learned concepts to new contexts.

In teaching linear equations, SC entails representing word problems algebraically, choosing appropriate methods (e.g., substitution or elimination), and employing various strategies to solve problems. Additionally, SC is closely linked to mathematical communication, as learners must articulate their problem-solving processes clearly. Research by Ya-amphan et al. (2024) highlights that learner-centered classrooms encourage mathematical communication, fostering diverse problem-solving approaches. Consequently, instruction should incorporate tasks that require students to model problems, test different strategies, and justify their solutions, ensuring that SC is effectively developed.

AR

AR refers to the capacity for logical thought, reflection, explanation, and justification in mathematical problem-solving (Kilpatrick et al., 2001). It involves making sense of mathematical concepts, identifying patterns, and explaining reasoning processes. Learners with strong AR can analyze mathematical relationships, recognize connections between different problem-solving methods, and justify their answers using logical arguments.

In the context of linear equations, AR enables learners to reason through different solution strategies, explain why a particular method works, and critically evaluate their results. Encouraging learners to reflect on their problem-solving approaches strengthens their ability to generalize concepts and apply them in various mathematical contexts. For AR to be effectively developed, teachers must design tasks that prompt learners to justify their thinking, evaluate multiple approaches, and engage in meaningful mathematical discourse.

PD

PD is a habitual inclination to see mathematics as sensible, useful, and worthwhile, accompanied by a belief in one's ability to succeed in mathematics (Kilpatrick et al., 2001). Learners with a PD are more likely to persevere in problem-solving, remain engaged in learning, and develop confidence in their mathematical abilities.

Zambak and Tyminski (2023). suggests that fostering a PD involves helping learners connect mathematical concepts to real-life contexts, making mathematics meaningful and relevant. When learners recognize the applicability of linear equations in various domains, such as economics, physics, and engineering, they are more likely to develop a positive attitude toward the subject. Teachers play a critical role in nurturing this disposition by creating a supportive learning environment, encouraging curiosity, and promoting a growth mindset.

A PD also manifests in learners' willingness to engage with challenging problems and persist until they reach a solution. In the case of linear equations, this means developing self-confidence, resilience, and a willingness to experiment with different solution strategies. As learners gain a deeper understanding of mathematical concepts, their confidence grows, leading to increased engagement and motivation.

Kilpatrick et al.'s (2001) intertwined strands of mathematical proficiency provide a comprehensive framework for understanding how learners develop mathematical competence. The strands—CU, PU, SC, AR, and PD—are interdependent and must be cultivated simultaneously to achieve meaningful mathematics learning.

In the context of teaching linear equations, this study argues that teachers must implement instructional strategies that integrate all five strands rather than focusing predominantly on PU. While procedural skills are essential, they must be reinforced through CU, strategic problem-solving, logical reasoning, and the development of a PD toward mathematics. By aligning instructional practices with this framework, educators can enhance learners' mathematical proficiency, ultimately improving overall performance and engagement in mathematics.

The Notion of Mathematical Proficiency

Mathematical proficiency is a fundamental aspect of mathematics education, encompassing both logical reasoning and imaginative thinking to develop learners' mathematical knowledge and prepare them for future applications in mathematics and other disciplines. According to Kilpatrick et al. (2001), teachers play a critical role in fostering an environment that promotes mathematical proficiency, necessitating a deep and comprehensive understanding of mathematics. Effective mathematics instruction requires teachers to possess a high level of competence in designing and implementing strategies that facilitate learners' comprehension, competence, and appreciation of mathematical concepts.

Mathematical proficiency for teaching is dynamic and can be observed in the instructional decisions made by educators. This includes interpreting learners' understanding of mathematical concepts, employing diverse representations of mathematical ideas, assessing learners' comprehension levels, and applying various teaching methods and strategies (Kilpatrick et al., 2001). Mathematical proficiency consists of five interrelated strands that guide the teaching and learning process: CU, PU, SC, AR, and PD. These strands form the basis of this study's conceptual framework and are explored in detail in the next section.

The Concept of Linear Equations

Linear equations represent a foundational algebraic concept essential for higher mathematical learning. Tossavainen et al. (2011) define an equation as a mathematical expression in symbolic form that asserts the equality of two quantities. For example, in the equation $5x + 4 = 9$, the left-hand and right-hand sides are equal, and the variable's exponent is 1, making it a linear equation.

Before introducing linear equations, learners are typically exposed to arithmetic equations, such as $5 + 4 = 9$ (Asare, 2026). Extending this to an equation like "---- $+ 4 = 9$" introduces the concept of an unknown value, laying the groundwork for algebraic reasoning. Allsopp et al. (2016) identify three key components essential to understanding linear equations: number patterns, equality, and variables. A strong grasp of the equal sign and variable concepts is critical to mastering linear equations. Linear equations can be represented in multiple forms, including flow diagrams, algebraic formulas, word problems, and graphical interpretations. This study examines how learners are exposed to these representations and the teaching strategies used to strengthen their SC.

Strategies Used by Teachers to Teach Linear Equations

The effectiveness of teaching and learning approaches significantly influences students' learning outcomes in mathematics. Hidayat and Setyawan (2020) emphasize that teachers' choice of instructional strategies impacts learners' ability to grasp mathematical concepts. Effective teaching requires the adaptation of teaching styles to suit learners' needs. Van de Walle (2016) suggests several approaches for teaching mathematics, including drawing diagrams, acting out problems, and using models. These techniques help learners create visual representations, making abstract mathematical concepts more tangible and accessible. Mainali (2021) supports this view, arguing that hands-on learning enhances CU and engagement. Among the strategies used in teaching linear equations are pattern recognition and problem simplification:

- **Pattern recognition:** This approach focuses on identifying patterns in the steps and methods used to solve linear equations (Goldin, 2020). It enhances PU and SC by helping learners recognize consistent problem-solving methods. Teachers structure lessons to help students comprehend proper procedures, allowing them to develop their own techniques based on their CU.
- **Problem simplification:** This strategy involves breaking complex problems into manageable parts, facilitating problem-solving (Ishak et al., 2021). Simplification aids conceptual knowledge by ensuring that learners understand each step

Table 2. Biographical information of participants

Teacher's name	Qualifications	Experience	School's name
Teacher 1	B.Ed. degree in mathematics & life sciences	4 years	School 1
Teacher 2	B.Ed. honors in physical sciences	3 years	School 2

before progressing. It also strengthens PU by encouraging methodical problem-solving and accommodates learners with different cognitive abilities. By initially presenting problems that most learners can solve, teachers scaffold learning, fostering a PD toward mathematics.

Challenges in Teaching Linear Equations

Research on the teaching of linear equations is extensive globally (Kilpatrick, 2011; Poon & Leung, 2010). Studies highlight that learners frequently struggle with the mathematical curriculum, particularly when introduced to new algebraic concepts. However, there is limited research on mathematical proficiency in teaching linear equations in South Africa (Machaba, 2017; Mbonambi & Bensilal, 2014).

Mbonambi and Bensilal (2014) report that many grade 12 learners in South Africa lack essential mathematical skills for problem-solving, suggesting that competency-based instruction in algebra is often insufficient. This deficiency indicates that teachers may not emphasize mathematical proficiency when teaching algebraic concepts. Given this gap, the present study aims to investigate the strategies employed by grade 9 mathematics teachers to develop mathematical proficiency in learners when teaching linear equations in South Africa.

METHODOLOGY AND METHODS

Research Design

This study employed a qualitative research approach, which seeks to analyze human experiences from an insider's perspective, also known as an 'emic' perspective (Galperin et al., 2022). A case study design was selected to explore how mathematics teachers develop learners' mathematical proficiency in linear equations. This design enabled the researchers to collect rich qualitative data through observations and interviews, allowing for an in-depth understanding of classroom practices.

Due to the restrictions imposed by the COVID-19 pandemic, physical classroom observations were not feasible. Consequently, the study relied on video recordings provided by teachers to conduct remote classroom observations of grade 9 mathematics lessons. A transcript of each lesson was compiled, and an observation checklist was used to systematically organize and analyze the data. The checklist was informed by the conceptual framework of mathematical proficiency strands, ensuring a structured evaluation of teaching practices. Furthermore, the descriptors of mathematical proficiency strands assisted in interpreting the collected data.

Sampling and Participants

The study was conducted in two secondary schools located in the Gauteng North District of Gauteng Province, South Africa. This district comprises 21 schools. The selection of participants was done through purposive sampling, allowing the researcher to choose teachers familiar with the research phenomenon. Two grade 9 mathematics teachers were selected based on their qualifications, teaching experience, and the schools they represented. Video-recorded lessons were analyzed, followed by semi-structured interviews with the teachers to gain further insights into their instructional practices.

Biographical Information of Participants

Table 2 shows the biographical information of participants.

Teacher 1 and teacher 2 had different qualifications and levels of experience. Teacher 1 had a four-year undergraduate degree, whereas teacher 2 had pursued an additional honors degree specializing in physical science. This selection provided a comparative perspective on how qualifications and experience influenced teaching practices and the development of mathematical proficiency in learners.

School Context

School 1 is located in a semi-rural area and is classified as a quintile 2 school, meaning it is a no-fee-paying institution. In contrast, school 2 is situated in an urban area and falls under quintile 4, indicating that it is a well-resourced school where fees are payable. School 2 benefits from the mathematics, science, and technology grant, which provides advanced technological resources for both teachers and learners. The selection of these two schools allowed for an examination of how different teaching environments impact the development of mathematical proficiency.

Data Collection Methods

The study employed two primary data collection methods:

- **Classroom observations (video recordings):** Teachers recorded and submitted videos of their grade 9 mathematics lessons on linear equations. These recordings were transcribed, and an observation checklist—aligned with the mathematical proficiency strands—was used to analyze instructional strategies.

- **Semi-structured interviews:** After the video analysis, semi-structured interviews were conducted with the teachers to clarify observed teaching strategies and to understand their approaches to fostering mathematical proficiency.

Data Analysis

A deductive approach was employed to analyze the data. Kilpatrick et al.'s (2001) mathematical proficiency framework, which consists of five strands—CU, PU, SC, AR, and PD—was used as a pre-conceived thematic structure. The strands and their indicators guided the development of the observation checklist and the interpretation of the data. Researchers reviewed the video lessons, identifying key instructional elements that were further explored in the interviews.

Ethical Considerations

Ethical clearance for the study was granted by the University of South Africa's College of Education Ethics Review Committee (reference: 2021/06/09/36460125/39/AM). The research adhered to strict ethical protocols, including obtaining permission from the Gauteng Department of Education to conduct the study in government schools. Approval was also secured from school administrators to ensure alignment with institutional policies. Both grade 9 learners and teachers were fully informed about the study, with voluntary participation emphasized. To maintain confidentiality and anonymity, pseudonyms were used for schools and participants. Additionally, participants were assured that their involvement in the study would have no impact on their coursework, professional evaluations, or school assessments.

This research methodology outlines a robust qualitative framework for examining how mathematics teachers develop learners' mathematical proficiency in linear equations. The combination of video observations and semi-structured interviews ensured a comprehensive analysis of instructional strategies within different educational settings. By employing Kilpatrick et al.'s (2001) framework, the study provided a structured approach to evaluating teaching practices, ultimately contributing to the broader discourse on mathematical proficiency in secondary education.

CU

CU requires learners to grasp the meaning of mathematical concepts, not just how to manipulate numbers. Teacher 1 made some effort to define an algebraic equation but did not fully develop conceptual link.

> **Teacher 1:** Algebraic equations are statements that have two equal expressions. Which is true, right?
>
> **Class:** Yes.
>
> **Teacher 1:** If something is an equation, it needs an equal sign?
>
> **Class:** Yes.
>
> **Teacher 1:** On each side of an equal sign, there is an expression, we have this one, and we also have this one; they are expressions. This one has two terms; it is a binomial, and this one [is] a monomial.

Here, the teacher correctly defines an equation and briefly describes its components. However, the explanation is procedural and surface-level, as it focuses on the presence of an equal sign rather than its relational meaning (i.e., that both sides represent the same value). Learners are not asked to apply their understanding to different contexts, nor are they engaged in a discussion about why equations work the way they do.

A more conceptually rich approach could have included real-world examples or multiple representations (e.g., balance scales and function tables) to illustrate how equations maintain equality. Instead, the teacher's approach suggests that equations are merely symbols to manipulate rather than meaningful relationships between quantities.

PU

PU was the most developed strand in the lesson. The teacher guided learners step-by-step through solving an equation, reinforcing correct procedures and ensuring that they followed them consistently.

> **Teacher 1:** So, we have the first one here, the easiest of them all. I think everyone can do this one. It says, the example, solve for the unknown: $x + 1 = 3$, I think everyone can solve for x, right?
>
> **Class:** Yes.
>
> **Teacher 1:** Can I have someone to solve for x? This one, anyone can do it (learner writes on the board).
>
> **Teacher 1:** So, we have the unknown; what is the unknown? The variable there?
>
> **Class:** x.
>
> **Teacher 1:** We also have a constant on the same side of x, and we do not need that constant. How do we remove it?
>
> **Class:** By subtracting one.

> **Teacher 1:** Because we have a positive one, we are subtracting a one. On the other side, we have a three; we also subtract one. We have $x + 1 - 1$ gives us zero, $3 - 1$ gives us 2, $x = 2$. I think everyone can do this, right?
>
> **Class:** Yes.

Here, the teacher systematically guides learners through each step, reinforcing correct arithmetic and algebraic manipulation. Learners follow the same structure throughout the lesson, ensuring procedural accuracy. No discussion on why subtraction is the correct operation. The inverse operations were stated, not explored in depth. Only one method was demonstrated. Learners were not encouraged to think critically or consider alternative strategies. Lack of connection to real-world applications. While students learned "how" to solve the equation, they were not given an opportunity to see where equations are used in real life, limiting engagement.

SC

SC refers to the ability to select and apply various methods to solve a problem. However, teacher 1 demonstrated only a single strategy—the use of additive and multiplicative inverses—while explicitly stating that alternative methods were intentionally excluded. This limitation is evident in the interview excerpt below.

> **Researcher:** During all the lessons that you conducted, I noticed that you used only one strategy for solving linear equations. Were there no other methods that could have been used?
>
> **Teacher 1:** There are several methods that can be used, but I preferred the additive and multiplicative inverse because it is easy to understand. I did not want to confuse the learners with many strategies, and they end up not knowing which one to use. Again, the issue of time was a problem because to show them all the methods was going to need a lot of time.

Here, the teacher prioritizes efficiency over flexibility, assuming that exposing students to multiple methods would cause confusion. However, SC requires developing problem-solving flexibility, which helps learners choose the most appropriate method based on the context. The teacher acknowledges trial-and-error and transposing methods but dismisses them without offering learners a chance to explore their strengths and weaknesses. Learners did not engage in discussions comparing different methods, which could have enhanced their critical thinking skills.

AR

AR entails justifying solutions and explaining the logic behind mathematical procedures. However, in the lesson, neither the teacher nor the learners provided any justifications for their methods or solutions. This is evident in the interview transcript below.

> **Researcher:** When solving the linear equations problem, there was no indication of justifying the chosen strategy or the procedures followed. Was there a reason for doing so?
>
> **Teacher 1:** I was not aware that I must give a reason for every step that I was doing. I took for granted that learners memorize what I have said, and they will be able to apply it when I give them problems to solve. What was important for me was that learners could use the method that I gave them and get the correct answers.

This response suggests that the teacher equated correct answers with understanding, rather than emphasizing the reasoning process. Without explanation, learners may struggle to apply the concept in unfamiliar situations. The teacher could have asked, why does this method work? or Can you explain why we subtract one? Justifications and counterexamples could have been explored to build logical reasoning skills.

PD

PD involves fostering **confidence** and **interest** in mathematics. Teacher 1 attempted to **boost learners' confidence** by reassuring them that the problems were simple. This is evident in the interview transcript below

> **Teacher 1:** I think everyone can do this, right?
>
> **Class:** Yes.
>
> **Teacher 1:** Anyone who has a question about this? No one! Easy, right?
>
> **Class:** Yes.

While the teacher encouraged participation, these statements do not necessarily cultivate deep engagement or intrinsic motivation. Learners agreed passively rather than actively exploring mathematical ideas. Instead of telling learners a problem is "easy," the teacher could have asked, Can someone explain why this is simple? More open-ended questions could have encouraged curiosity and discussion.

Teacher 1's approach was highly procedural, emphasizing fluency in solving equations while neglecting reasoning, strategy, and conceptual depth. A more balanced approach—incorporating multiple strategies, justifications, and conceptual connections—

would better develop mathematical proficiency in learners.

- **Teacher 2:** Data presentation from the lesson observation and interviews.
- **Day 2:** Algebraic equations: Introduction to word problems 45 minutes.

On day 2, the teacher started the lesson by introducing the day's topic and reminding the learners about the concepts that were dealt with in the previous lesson.

CU

CU involves grasping mathematical concepts, operations, and relationships rather than just applying procedures. While the teacher provided clear explanations of key concepts (such as the difference between expressions and equations), the approach was largely teacher centered. This is evidenced in the extract below.

> **Teacher 2:** So, remember last time we did algebraic expressions, algebraic equations, and we looked at the difference between algebraic expressions and algebraic equations, where the expression is made up of variables, numbers and constant and coefficient. So, these are what make up algebraic expressions. Algebraic equations are made up of the same things, but on algebraic equations, there is an equal sign. That is the difference, right? What does the equal sign mean?
>
> **Class:** Both sides are equal.
>
> **Teacher 2:** It means that both sides are equal. That means the left-hand side is equal to the right-hand side. Instead of asking learners to recall and explain the previous lesson's content in their own words, the teacher spoon-fed them information. This limited their opportunity to actively construct their understanding of algebraic equations.

PU

PU is the ability to apply procedures efficiently, accurately, and flexibly. The excerpt below highlights how PU was the dominant focus of the lesson

> **Teacher 2:** Example: A number increased by six equals ten. So, a certain number that is unknown is increased by six, equals ten. So, we take a number as an x, we say let a number be x ... But since it is not indicated as an exact number, that number is increased by six. Which sign or operation can we use to indicate an increase?
>
> **Learner D:** Multiplication and addition.
>
> **Teacher 2:** We can use addition, right?
>
> **Class:** Yes.
>
> **Teacher 2:** Therefore, that means x is a number which is increased by six and equal; equal is a sign, right?
>
> **Class:** Yes.
>
> **Teacher 2:** Equals to ten.

This excerpt highlights how **PU** was the dominant focus of the lesson. The teacher guided learners step-by-step through setting up equations, ensuring they followed the correct **procedures**. The teacher also corrected learners immediately if they made a mistake. However, the lesson lacked emphasis on **alternative methods** or **deeper understanding**, as learners were only guided through one approach.

SC

SC is the ability to formulate, represent, and solve mathematical problems effectively. SC was present to some extent because learners were given **opportunities to convert word problems into equations** this is evidenced in extract below.

> **Teacher 2:** Example: Three times a certain number by five equals thirty. I want you to try this one on your own. Can you create an equation from that? (learner writing the answer on the board)
>
> **Teacher 2:** Let us do another one. We try to set up the equation. Example: A certain number increased by five equals thirty (learner writing the answer on the board, class clapping hands).

However, the **teacher did not encourage multiple strategies** or allow learners to explore different ways to represent the problems. They were expected to follow a single approach, meaning their **problem-formulation skills were not fully developed**.

AR

AR involves explaining and justifying mathematical reasoning and solutions. AR was not addressed, as there were no instances where the teacher or the learners justified the answers that they provided. For example,

> **Learner F:** Sir, it is not written like that but is $40x = 2000$.
>
> **Teacher 2:** It is $x \times 40 = 2000$. What is it that we are going to do to get the number of bouquets?
>
> **Learner F:** Divide by forty on both sides.

Although learners solved problems correctly, there were no moments where they were asked to justify their answers. When a student questioned the notation of the equation (*40x* vs. $x \times 40$), the teacher corrected them without discussion or reasoning. If AR were emphasized, the teacher could have asked learners to explain why both notations are mathematically equivalent.

PD

PD is the belief in one's ability to succeed in mathematics and view it as useful and worthwhile. There was an evidenced where learners cheered each other on by clapping their hands for the correct answer. (Learner writing on the board. There was an argument between learners on how a doubled number is written, whether it is 3^2 or 3×2.). Teacher 2 says *"$3^2 = 3 \times 3 = 9$ and $3 \times 2 = 6$. Therefore, the equation is $3 \times 2 + x = 50$."* There were some positive moments of engagement, such as learners clapping for their peers. However, students were not encouraged to explore their own thinking or come up with different solution methods. Instead, the teacher quickly corrected misconceptions without letting students fully debate their reasoning. This limited the development of their confidence and sense of ownership over the learning process.

DISCUSSION

The aim of this study was to explore grade 9 mathematics teachers' strategies to address mathematical proficiency in their teaching of linear equations. This discussion provides a comprehensive examination of the findings regarding the strategies teachers used to address CU, PU, SC, AR, and PD.

Addressing CU in Teaching Linear Equations

CU in this study was assessed through the teachers' use of prior knowledge, explanations of key concepts, and the ability to relate linear equations to real-life situations. The findings indicate that neither teacher acknowledged learners' prior knowledge, which limited opportunities for meaningful engagement. This contradicts research by Fandakova and Bunge (2016), which emphasizes the importance of activating prior knowledge in long-term memory to create meaningful connections with new concepts.

Although both teachers provided clear explanations of the terminologies used in linear equations, these explanations were teacher-centered, restricting learner participation. According to Walkington et al. (2019), allowing students to articulate their understanding fosters mathematical language acquisition. Furthermore, the equal sign was primarily treated as a separator rather than a symbol of equivalence, which is a fundamental misunderstanding that can hinder learners' ability to solve equations effectively (Kieran et al., 2016).

Additionally, the study found that the relationship of linear equations with other mathematical concepts was minimally addressed. Research suggests that fostering relational understanding (Adler et al., 2000) enhances knowledge transfer and meaningful engagement. These findings highlight the need for a more interactive approach where students actively engage in constructing meaning rather than passively receiving information.

Addressing PU in Teaching Linear Equations

PU was the strongest aspect of the teachers' instruction, as both ensured that learners followed correct procedures efficiently and accurately. However, the emphasis was predominantly on obtaining the correct answer rather than fostering deeper understanding. This aligns with concerns raised by Inayah et al. (2020), who argue that PU should be developed alongside conceptual comprehension to avoid rote learning.

Despite the strong emphasis on procedure, learners had limited opportunities to explore multiple solution strategies. Teacher-centered approaches dominated the lessons, minimizing student engagement and discouraging error analysis, which is critical for learning. Bay-Williams (2020) emphasizes the importance of linking PU with CU, suggesting that teachers should integrate varied strategies to promote flexible thinking.

Addressing SC in Teaching Linear Equations

SC involves the ability to solve problems using different strategies. The study found that teacher 1 relied solely on the additive and multiplicative inverse approach, while teacher 2 introduced an additional method—trial and error. However, in both cases, students were not encouraged to explore alternative solutions independently. Algani (2019) argues that teachers should incorporate innovative strategies to cater to diverse cognitive abilities and foster deeper problem-solving skills.

Teacher 1's view that exposing learners to multiple strategies would cause confusion is concerning, as research indicates that varied approaches enhance problem-solving flexibility (Sari et al., 2019). Similarly, Tachie (2019) highlights the importance of metacognitive abilities in mathematical problem-solving, underscoring the need for learners to engage with diverse strategies rather than adhering to a rigid, teacher-imposed method.

Addressing AR in Teaching Linear Equations

AR requires students to justify their mathematical reasoning. However, the study found that neither teacher emphasized justification, with explanations being largely omitted or superficial. Teacher 2 briefly mentioned the BODMAS rule, but there was no in-depth discussion on why particular steps were taken. This contradicts findings by Bieda et al. (2013), who stress that reasoning enhances comprehension and problem-solving skills.

Teachers' reluctance to probe students' reasoning stems from their belief that correct answers imply understanding. However, research suggests that encouraging justification helps learners internalize concepts rather than memorize procedures. The absence of justification means that learners may struggle to apply their knowledge in unfamiliar contexts, reinforcing the need for instructional practices that promote deeper reasoning.

Addressing PD in Teaching Linear Equations

PD refers to fostering learners' confidence and interest in mathematics. The study found that neither teacher encouraged the extension of knowledge from the known to the unknown, resulting in a passive learning experience. Research by Hasbi et al. (2019) emphasizes the importance of mathematical connections, suggesting that learners should build on prior knowledge to enhance understanding.

Additionally, learners were not given opportunities to develop their own problem-solving methods. Both teachers dominated the lessons, with minimal learner initiative encouraged. Kopel et al. (2021) argue that fostering creativity in problem-solving enhances learners' engagement and resilience. By not allowing students to explore their own approaches, teachers inadvertently limited their ability to develop independent mathematical thinking.

Teacher 1's belief that learners lack initiative and are reluctant to answer questions is problematic, as it reinforces passive learning. Research highlights the importance of creating an environment where students feel empowered to explore mathematical ideas independently (Awofala et al., 2020). Similarly, Kurniawati et al. (2022) advocate for teaching approaches that encourage collaboration, critical thinking, and creativity—essential skills for 21st century learning.

The findings suggest that while PU was effectively addressed, other strands of mathematical proficiency—CU, SC, AR, and PD—were insufficiently emphasized. A shift toward a more learner-centered approach is necessary to enhance engagement, reasoning, and problem-solving skills. Teachers should integrate multiple solution strategies, encourage justification, and foster a supportive environment where students take initiative in their learning. Future research should explore professional development initiatives that equip teachers with strategies to promote all dimensions of mathematical proficiency effectively.

CONCLUSION AND RECOMENDATIONS

The findings of this study highlight the predominant focus on PU in the teaching of linear equations among grade 9 mathematics teachers. While PU is essential for accuracy and efficiency in mathematical problem-solving, the study reveals that CU, SC, AR, and PD were insufficiently emphasized. Teachers primarily employed teacher-centered approaches, limiting learners' opportunities to explore alternative strategies, justify their reasoning, and develop independent problem-solving skills. Additionally, the failure to incorporate learners' prior knowledge and real-world applications restricted their ability to connect mathematical concepts meaningfully. The lack of emphasis on AR and justification of solutions further suggests that students may struggle to apply mathematical principles flexibly in unfamiliar contexts.

To enhance mathematical proficiency, it is recommended that teachers adopt a balanced instructional approach that integrates all five strands of mathematical proficiency. Teachers should actively engage learners in discussions, encourage multiple solution strategies, and provide opportunities for students to justify their reasoning. Professional development programs should be designed to equip teachers with effective pedagogical strategies that move beyond PU and foster deep CU and AR. Furthermore, integrating real-life applications of linear equations can improve learners' PD, fostering confidence and motivation in mathematics. By implementing these recommendations, educators can cultivate a more comprehensive and learner-centered approach to mathematics instruction, ultimately improving students' mathematical competence and engagement.

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