



# How can Students Generalize the Chain Rule? The Roles of Abduction in Mathematical Modeling

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The purpose of this study is to design a modeling task to facilitate students' inquiries into the chain rule in calculus and to analyze the results after implementation of the task. In this study, we take a modeling approach to the teaching and learning of the chain rule by facilitating the generalization of students' models and modeling activities. We assumed abductive reasoning to be one of the key factors which can support the generalization of students' models and modeling activities. We believe that analogical reasoning and diagrammatic reasoning are key factors in fostering students' use of abduction. As a result, we determined that the students' models and modeling activities were generalized to the chain rule by their use of abductive reasoning, and the students found the chain rule to be a generalized rule for describing changes of various quantities.

*Keywords:* mathematical modeling, abduction, analogy, diagrammatic reasoning, chain rule

## INTRODUCTION

The chain rule is one of the key concepts in calculus (Cottrill, 1999; Stewart, 2012). Students' difficulties in learning the chain rule have been reported in other studies. Studies have shown that most students do not recognize that they are applying the chain rule (Clark et al., 1997). The formal proof of the chain rule seldom sheds light on the meaning of the chain rule since it approximates an algebraic trick (Cottrill, 1999). Nevertheless, scant attention has been given to find an alternative way of facilitating students' inquiries into the chain rule.

The chain rule was intuitively simple for Newton and Leibniz, who are the inventors of calculus (Rodríguez & Fernández, 2010). For them, the heart of the chain rule was that it explains the compound ratio among variables (Guicciardini, 2003). In this respect, one option for facilitating inquiry into the chain rule is to

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provide students with opportunities to organize relationships among various changes of quantities (e.g., Lutzer, 2003).

The opportunity to participate in modeling activities motivates students' inquiry into the concepts of rate of change (Årlebäck, Doerr, & O'Neil, 2013) and the derivative (Park, Park, Park, Cho & Lee, 2013). Studies have also noted that students' models and modeling activities can be generalized to conceptual systems that signify mathematical relationships or regularities (Lesh & Harel, 2003). Hence, we would argue that it is possible to foster students' explorations of the chain rule by having them model relationships of changes of quantities and facilitating the generalization of their modeling activities.

In this study, we aim to design a modeling task to foster the generalization of student models and modeling activities by facilitating abductive reasoning, after which we analyze the students' modeling activities to identify how the generalization of the models and the inquiries into the chain rule are progressing. To accomplish this aim we focus on abductive reasoning to support the generalization of modeling activities, reflecting the Peircean perspective, which views abductive reasoning as closely related to the generalization of mathematical activities (Otte, 2006).

## THEORETICAL BACKGROUNDS

### Students' difficulties in learning the chain rule

The chain rule has been considered to be a notorious concept in calculus, and researchers have pointed out students' difficulties in learning the chain rule (Clark et al., 1997; Gordon, 2005; Maharaj, 2013). For example, Gordon (2005) noted that the chain rule is difficult to explain and so most students do not really see where it comes from (p. 195). He also argued that the chain rule is difficult to represent in symbols and awkward to put it into words, so most students do not remember or correctly apply it. It has also been reported that most students were not aware that they were using the chain rule (Clark et al., 1997).

Studies have pointed that one of key reasons for difficulties in learning the chain rule is that the chain rule is connected to various mathematical concepts. The genetic decomposition of the chain rule conducted by Clark et al. (1997) supports this statement. This study shows that understanding the chain rule involves at least the conception of function, function composition and decomposition, and rules of differentiation (Clark et al., 1997). Maharaj (2013) also reported that detecting embedded functions inherent in a problem situation was a key step and a difficult phase in learning the chain rule.

Lutzer (2003) argued that an alternative way to facilitate students' learning of the chain rule and enrich the meaning of the chain rule is to show a "motion presentation," which presents how the chain rule explains relationships among changes of quantities

### State of the literature

- Empirical studies show students' difficulties in learning the chain rule. Thus many authors claim the necessity of finding an alternative way to facilitate students' inquiry into the chain rule.
- Although some authors argue that adopting real contexts is an alternative way to support learning the chain rule, there have been few empirical studies on introducing real situations in learning the chain rule.
- The authors claim model generalization as an important process that can be used to support students' inquiry using modeling, but little is known about factors that will help facilitate model generalization.

### Contribution of this paper to the literature

- The paper presents a teaching experiment on integrating a modeling approach to learning the chain rule.
- The practical outcome is the design of a modeling task aimed at facilitating students' inquiry into the chain rule that is ready for use in classrooms. Additionally, deep empirical insights into the learning pathways along the intertwined the use of abduction and model generalization are given.
- This study makes contributions to local instruction theories on learning the chain rule and empirically confirms that analogical reasoning and diagrammatic reasoning are key factors for supporting the construction of abduction.

in real-world problem situations. Although Lutzer argued that adopting a real-world situation may provide students with richer understanding of the chain rule, he only presented motion interpretations of the chain rule that can be shown to the students in lecture rather than finding a way to facilitate students' inquiries into the chain rule. Studies have also pointed out that learning the chain rule should be connected to real-world problem situations (cf. Uygur & Ozdas, 2010), but there is little discussion on specific ways to promote students' learning the chain rule in connection with real situations.

Given that, we aim to find a way adopt a modeling perspective to facilitate students' inquiries into the chain rule that begin with real-world problem situations. Lesh and Doerr (2003) argued that mathematical modeling activity can encourage students' explorations on meanings of mathematical concepts or procedures in connection with real-world problem situations. As we have noted, because students' difficulties in learning the chain rule are a hard problem to resolve, research needs to find an alternative way to facilitate students' inquiries into the chain rule. Since a key aspect of the chain rule is that it explains the compound ratio among variables (Guicciardini, 2003), it is easier to understand the chain rule in connection with real-world problem situations (Lutzer, 2003; Uygur & Ozdas, 2010). The chain rule is considered to be a particularly appropriate mathematical concept to explore using a modeling perspective since "calculus serves as a basis of modeling and problem solving in applications" (Tall, Smith, & Piez, 2008, p. 208). In the following sections, we analyze the literature to examine how mathematical modeling can be applied to facilitate students' inquiries into the chain rule.

### **Inquiry into the chain rule by modeling and model generalization**

Blum et al. (2002) noted that "the objects, data, relations and conditions involved in it [the real-world] are translated into mathematics, resulting in a mathematical model," and mathematical modeling is "the process leading from a problem situation to a mathematical model (p. 153)." Casti (1989) pointed out that considering mathematical modeling as only a process of formularizing relationships among entities in a real-world situation is interpreting a mathematical model in a narrow sense from the Newtonian perspective of modern science. Although mathematical modeling may involve a process of describing the real world by using an established theory or rule (Frejd & Bergsten, 2016), it can also be applied to generate a new theory (Clement, 2009). Until the early 1900s, studies have argued that models and modeling are no more than exemplifications of theories or esthetic things (e.g., Carnap, 1939). However, Bailer-Jones (1999) considered modeling as a departure of hypothesis formation through organization of the real world from a hypothetical-deductive perspective. In other words, mathematical models have more productive potential than original problem situations and are not just copies or reductions of originals (Bailer-Jones, 1999).

In this respect, Lesh and Sriraman (2005) also pointed out that mathematical modeling has the potential to support mathematical inquiry in the educational context by fostering students' conjectures on underlying mathematical regularity in real-world situations. Lesh and Doerr (2003) noted that students' models can be developed to become a conceptual system beyond algebraic formulation of relationships among entities in a problem situation. Development of models into conceptual systems means that functions of mathematical models transit from describing particular situations to theories which are able to explain various generalized situations. Also, students' modeling activities can be decontextualized, integrated, and generalized from the concrete and particular (Lesh & Doerr, 2003).

Empirical studies show that mathematical modeling can activate students' inquiries into rate of change, the derivative, and the concept of the function (Årlebäck,

Doerr, & O'Neil, 2013; Park et al., 2013; Doorman, Drijvers, Gravemeijer, Boon, & Reed, 2012). Lutzer (2003) argued that modeling real contexts with change of variables had the potentials to encourage students' learning of the chain rule. However, there are still many criticisms of mathematics teaching and learning based on real-world situations, and one of main criticisms is that "it is often impossible to proceed from experientially real situations to mathematics" (Gravemeijer & Terwel, 2000, p. 792). Hence, a key issue in using mathematical modeling to design the teaching and learning of the chain rule is finding a way to help students to generalize their modeling activities from being a description of particular situation to learning the chain rule. Nevertheless, little is known about factors or ways to generalize students' mathematical models and modeling activities.

Thus, we first need to find theoretical factors and provisional mechanisms of generalization of mathematical models that will accomplish the aim of this study and draw implications for further research on adapting real contexts to mathematics teaching and learning. To be more specific, we examine a Peircean perspective on generalization and design a mathematical modeling task which reflects his discussions. Peirce pointed out that generalization is a dynamic process based on "observation and hypothesizing on particular cases" and "verification and revision of established hypotheses" and is not a linear or gradual process (Otte, 2006). The key factor of this dynamic process of generalization is *abduction* (Otte, 2006). In the following section, we review a Peircean perspective on abduction and generalization to find a way to facilitate generalization of models and modeling activities.

### **Model generalization by abduction**

Abduction is the process of forming an explanatory hypothesis A on observed surprising result B (C.P. 5.171), and it takes the form "if A is true then B would be a matter of course" (Prawat, 1999). Generalization is closely related to abduction (Otte, 2006). The key parts of generalization from a Peircean perspective are: First, generalization is closely related to abduction. Second, it progresses dynamically comes and goes between the particular and the general. The hypothesis established by abduction is a general rule since it explains various observed results, so it is a type of reasoning which comes and goes between the particular and the general (Otte, 2006). According to Peirce, "[abduction] is the first step of scientific reasoning, as induction is the concluding step" (C.P. 7.218). That is, generalization and establishment of a hypothesis are not direct results of induction or an accumulation of observed results. Rather, generalization proceeds from hypothesizing a provisional general rule that explains observed particular results, then inductively and deductively verifying and revising the initial abduction and hypothesis (C.P. 5.171). In this process, abduction also progresses by a cyclic process which involves generation of a provisional hypothesis and its verification and revision (Peng & Reggia, 1990).

Eco (1983) identified three kinds of abduction: overcoded, undercoded, and creative. Overcoded abduction is an automatic formation of a hypothesis. For example, if someone knows what *man* is in English, when the utterance "man" is heard its meaning is then interpreted automatically (p. 206). In this case, overcoded abduction occurs. If there are equiprobable rules to explain observed results, then undercoded abduction occurs. If there is no rule to explain observed results, we have no choice but to create a new rule. In this case, creative abduction occurs. Hence, the major focus of this study is finding a way to facilitate students to produce a new rule or generality from mathematical modeling of rates of changes of variables by the use of creative abduction.

The use of creative abduction is closely related to the use of analogy rather than making something out of nothing (Prawat, 1999). In other words, new hypotheses or

mathematical rules are formulated by analogizing and revising existing rules which are used in other similar contexts (Prawat, 1999; Sfard, 2008). Thus, we need to support students to analogize rules of a similar context to an unfamiliar situation in order to make a hypothesis.

On the other hand, creative abduction is also related to the use of diagrams. A diagram is “a representamen which is predominantly an icon of relations (CP 4.418),” and it is possible to construct an abduction by construction of a diagram, experimentation on it, and observation on the results of the experiment (Hoffmann, 2005). An icon is a sign which represents its object by relying on likeness to that object, and this likeness is aided by conventional rules (Otte, 2006). For example, since an algebraic formula works by conventional rules and represents relationships among constituent elements, it is both an icon and a diagram (Otte, 2006; Presmeg, 2005). An experiment on a diagram is the transformation of representations based on conventional rules, so the results of an experiment are assumed to have some degree of rationality (Hoffmann, 2004). Thus, diagrammatic reasoning is at the heart of the construction of creative abduction since surprising results drawn from an experiment on a diagram imply the existence of provisional rules or an explanatory hypothesis (Hoffmann, 2004). Hence, one way to promote students’ construction of creative abduction is to support students to organize a problem situation with diagrams and experiment on their diagrams.

As we synthesized the literature, we found that abduction was closely related to generalization, so we assumed that facilitating the use of abduction may support generalization of mathematical models and modeling activities. This study takes a step forward from the lack of discussion on identification of factors to foster the ability to generalize from modeling; we would argue that abduction is a key factor in this. Based on these findings, our design will be guided by the following three interrelated aspects of the generalization by abduction: First, we need to design modeling activity for students that fosters students’ use of analogy. The use of analogy has the potential to lead students to use similar mathematical rules from other contexts and revise them to create new mathematical rules (Prawat, 1999; Sfard, 2008). It is necessary for students to use rules in an approximate way and gradually revise rules that they already know even though these rules are not exactly suited to a given problem situation. Given that, we take into account the role of analogy in encouraging students to construct abductions that support the establishment of hypotheses to explain the results they have observed in mathematical modeling.

Second, it is important to support students’ construction of a diagram, experimentation on it, and observation of the results of that experimentation in order to foster their construction of abduction. These three steps constitute *diagrammatic reasoning* (Hoffmann, 2004), and we take account of the role of diagrammatic reasoning in supporting students’ identification of hidden relations in order to form new explanatory hypotheses based on modeling. In other words, we argue that diagrammatic reasoning enables students to draw new ideas to construct abduction (Otte, 2006). We also assume that diagrammatic reasoning may foster students’ participation in mathematical inquiry in order to form a new hypothesis by supporting the necessity and existence of a new explanatory hypothesis on the observed results.

Third, it is necessary for students to carry out modeling by dynamically coming and going between a particular situation and its mathematical model. Abduction can be constructed by observing only one result, whereas induction is the process of verification of abduction (Pedemonte & Reid, 2011). In a similar manner, Watson and Mason (2005) pointed out that one representative case can be used as raw material for inductive reasoning. In this respect, mathematical modeling, which aims at generalization by abduction, also needs to be begun by building a mathematical model

from careful observation of a representative situation before handling many cases and followed by verification of this model by examining other cases.

## METHODS

### Participants

The participants in this study were 20 students in the 11th grade with slightly above average levels of mathematics achievement. They had already learned polynomial derivatives, and they had not yet learned the derivatives of irrational functions and the chain rule. This classroom was a relatively average classroom where mathematics learning and teaching followed the current national curriculum. The students participated in an experimental class with a designed modeling task. The names of students were coded S1-S20.

The instructor, Mrs. Lee, a mathematics teacher who is not one of the authors, has been teaching mathematics at a high school for eight years. She took part in this study because of her interest in the development of mathematical thinking using modeling. She shares the belief that students' mathematical thinking ability can develop in a rich socially interactive environment. Like the researchers, Mrs. Lee also shared the belief that students' mathematical inquiry can be meaningfully supported by orchestrating modeling in the classroom, so the task was conducted as follows: (a) The teacher maintained a balance between (minimal) teacher guidance and (maximal) student independence, and the teacher's interventions were mainly strategic interventions that give hints to students on a meta-level: "Imagine the situation," "What do you aim at," "How far have you got," "What is still missing," and so on, as Blum and Borromeo Ferri (2009) emphasized. (b) Students were asked to represent their solving processes in various ways in order to foster students' diagrammatic reasoning. (c) As Maaß (2006) pointed out, modeling in the classroom can be effectively orchestrated by dividing students into several small groups and supporting each individual's inquiries reflecting on the small group interaction. Given that, 20 participants worked together in five groups consisting of four students each with rich communication and also worked on their own individual worksheets. (d) The teacher did not provide a solution or give hints that might lead to one; her role was rather to observe the students' individual work, to support a rich exchange of ideas within the sociocultural communication process, and to encourage self-reflection (Hitt & González-Martín, 2015). (e) An edited video clip of the movie which involved a scene of the room shrinking was shown to the students to motivate them and familiarize them with the modeling context.

### The task

The aim of the modeling task was to find the instantaneous rate of change of the length of the edge of a room based on the Spanish movie *Fermat's Room*, a movie well known to Korean students. The modeling task consists of three subtasks, as described below.

There is a shrinking room in *Fermat's Room*. The initial area of the room is 100 square meters and the area decreases by 3 square meters per minute.

1. Formularize the given conditions and explain why your formulas work.
2. Find the instantaneous rate of change of the length of the edge of the room after 7 minutes as accurately as possible and explain why your solution is reasonable. Also, conjecture about general formula to find the instantaneous rate of change of the length of the edge of the room. Then, represent the instantaneous rate of change of the length of the edge with the formulas established in Subtask 1 in order to describe the given conditions.

3. Verify whether the general formula set up in Subtask 2 can be applied to finding the instantaneous rate of change of the length of the edge of the room after 11 minutes and 13 minutes. Justify your generalized formula.

In this modeling task, the students are asked to find the instantaneous rate of change of the length of the edge of a room from the rate of change of area of the square room with consideration of the relationship among changes in three variables: time ( $t$ ), area ( $S$ ), and length ( $a$ ). This modeling task is thus considered to be designed to lead students' inquiry into the chain rule.

In Subtask 1, students are asked to formularize the conditions of the task, which means that students should build models of the problem situation. The intention of Subtask 1 is to support students' inquiry in multiple ways. First, Subtask 1 is designed to have students perceive the task and problem situation. Blum and Borromeo Ferri (2009) have shown that understanding of given problem situation precedes other steps of modeling. Given that, we first asked students to formularize the given conditions in order to understand the task and problem situation. Second, Subtask 1 was designed to encourage students' analogical reasoning, which supports construction of abduction (Prawat, 1999). We assumed that mathematizing the conditions of the given problem situation can be a starting point in the effort to use and modify the mathematical rules already known by the students. In other words, Subtask 1 was designed to lead students to perceive not only the given task but also the mathematical rules that can be applied to factors of the given situation by asking them to build models of the conditions of the task. Thus we expected that the students would analogically use these rules to build a model of the rate of change of the length of the edge since the conditions of the task are closely related to it. Third, Subtask 1 was designed to foster students' diagrammatic reasoning. As we reviewed, algebraic formulation is also a diagram, so transformation of this diagram by its conventional rules may provide students with clues towards making a new hypothesis (Hoffmann, 2004). Given that, we considered that asking students to build an algebraic diagram may promote students' diagrammatic reasoning.

In Subtask 2, students were asked to find the instantaneous rate of change of the length of the edge of the room ( $\frac{da}{dt}$ ) after 7 minutes. First, Subtask 2 was designed to have students conjecture about a generalized model by modeling the situation at a certain time. That is, it was designed to lead students to perceive and apply patterns that emerge when they are building a particular model and describe relationships among the factors of the problem situation. Subtask 2 also asked students to associate the result of this subtask with formulas established in Subtask 1. In this process, we believed that Subtask 2 could support students' diagrammatic reasoning. That is, it would foster students' transformation or experimentation on the diagrams constructed in Subtask 1 by asking them to resolve Subtask 2 using the result of Subtask 1.

Subtask 3 asked students to find the instantaneous rate of change of the length of the edge of the room after 11 and 13 minutes, a task which was designed to support inductive verification and revision of the models built in Subtask 2. We also asked students to justify their generalized model to encourage their deductive verification.

We asked students to carry out modeling by dynamically coming and going between "construction of models of a particular situation," and "conjecturing and justifying the generalized mathematical model" in the overall task. Subtask 1 asked students to build and handle a generalized model and Subtask 2 asked students to construct a particular model and examine particular data to conjecture about a new generality. Subtask 3 then asked students to examine and revise the provisional generality via particular data and to verify their models deductively. As we already reviewed, from Peircean perspective generalization progresses dynamically coming

and going between the particular and the general, so we asked students to come and go between the particular and the general model in the overall task.

We also focused on encouraging students' diagrammatic reasoning and analogical reasoning in the overall task. We first asked students to build diagrams in Subtask 1. Subtask 2 then asked students to associate the result of Subtask 2 with the rates of changes among the variables established in Subtask 1 to foster students' experimentation or construction of abduction. We also asked students to consider the relationship between the conditions of the given situation and their model construction in order to foster students' analogical reasoning in applying their rules to a new context (Subtasks 1 and 2).

### **Data collection and analysis**

The data analyzed consisted of the students' written answers, video recordings, and lesson observations during a 3-hour mathematics lesson in the summer of 2015. The collected data was analyzed in chronological order and divided into distinct but related episodes, as Cobb and Whitenack (1996) have suggested. Palha, Dekker, Gravemeijer and van Hout-Wolters (2013) suggested analyzing the results of a teaching experiment in three steps: The first step is a global analysis, which consists of selecting representative episodes in the whole data set. In the second step, a deeper analysis of each fragment is conducted. This step involves the identification of the processes and characteristics of students' inquiry progression in each episode in terms of the aims of the research. In the third step, systematic analysis of the whole data and verification of researchers' conjectures via the whole data set is performed. In this step, grounded theory techniques involving open and axial coding are employed to analyze the collected data (cf. Strauss & Corbin, 1998).

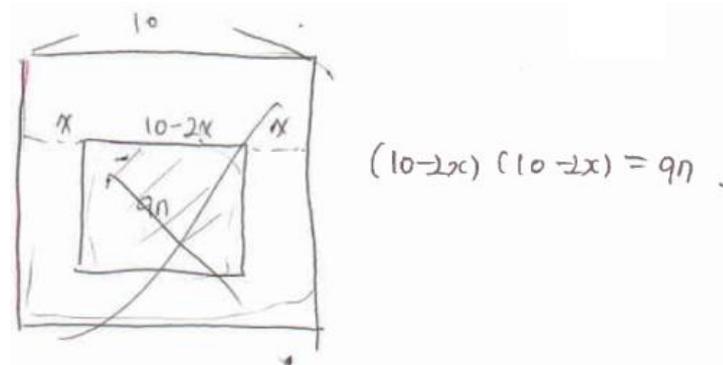
The aim of this study was to find a way to facilitate students' inquiries into the chain rule by fostering generalization of modeling activities and to investigate the roles of analogy, diagrammatic reasoning, and abduction by analyzing the implementation of the designed modeling task. We modified a data analysis technique from prior research in order to fit our research aim as follows: (a) We analyzed models built by students and how these models were revised during class with the aim of investigating students' inquiry into the chain rule by supporting the generalization of models. When resolving each subtask, students built different models, and the characteristics of the constructed models in each subtask were contrasted. Given that, we categorized three episodes which correspond to students' activities on each subtask from the whole data set. (b) We conducted a deeper analysis of each episode and identified students' use of analogy, diagrammatic reasoning, and abduction. We then investigated the relationship among students' reasoning, model construction and revision, and inquiries into the chain rule. To be more specific, we focused on the following: (i) whether analogical reasoning or diagrammatic reasoning facilitated the construction of abduction, (ii) whether the use of abduction supported generalization of mathematical models, and (iii) whether students' inquiries into the chain rule occurred by mathematical modeling. By doing so, we categorized the ways of students' model generalization and inquiry into the chain rule. (c) We conducted systematic analysis of the whole data and verified our conjectures on students' modeling processes in order to examine whether our analysis results could provide a coherent explanation of the whole data. If our hypothesis did not fit to the whole data, we returned to the previous step to analyze again.

## **RESULTS**

In this study, we observed how model generalization occurs and modeling supports students' inquiries into the chain rule. This chapter is organized into three parts: (a) We addressed students' modeling activities in Subtask 1. In this first episode, we especially focused on students' model construction of the instantaneous rate of change of the length of the edge of the room using an undercoded abduction that uses using analogy. (b) We identified how students generalized their modeling activities of the instantaneous rate of change of the length of the edge of the room in Subtask 2. In this episode, we addressed the students' two ways of model generalization and model construction. In the first type of modeling, we identified how the real context is related to the students' model generalization and their use of creative abduction. In the other type of modeling activities, we focused on students' model generalization of a particular model and their difficulties in generalization. (c) We identified how students generalized their models and modeling activities to produce the chain rule in Subtask 3. In this episode, we mainly focused on the students' construction of creative abduction based on their diagrammatic reasoning and their justification of the models.

### Episode 1: Model construction by undercoded abductions using analogy

As the students read the modeling task, their small group discussion first focused on how the room is shrinking. Based on viewing the video clip, the students assumed that the shrinking room remains square. Based on this assumption, students attempted to simplify the given problem situation using a visual representation. The students initially focused on the relationship between the length of the edge and the area of the square room. Figure 1 is S15's figural representation of the problem situation.



**Figure 1.** S15's visual representation of the problem situation

The relationship between the area ( $S$ ) and the length of the edge of the room ( $a$ ) is  $S = a^2$ , and a decrease in area meant a decrease in the length of the edge. Given that, the students considered the rate of change of the area ( $\frac{dS}{dt} = -3$ ) to be a key factor in finding the instantaneous rate of change of the length of the edge. The students then formularized the rate of change of the area of the square with respect to time and attempted to build models of the rate of change of the length of the edge. Their use of analogy was identified in their model construction. The students' modeling is categorized into four types, and the students' ways of using analogy were also identified as shown in Table 1.

With the exception of the second group of students, every student built at least two models of the given problem situation. The students mainly constructed models of the area with respect to time and the length of edge with respect to time (Type 1, 2, 3),

**Table 1.** Students' models and analogies in Subtask 1

Type	Model	Group	Analogy
Type 1	$S(t) = 100 - 3t$ $\rightarrow a(t) = \sqrt{100 - 3t}$	1, 3, 4, 5	-
Type 2	$S(t) = 100 - 3t$ $\rightarrow a(t) = 10 - \sqrt{3}t$ $\rightarrow a'(t) = \frac{1}{2}(100 - 3t)^{-\frac{1}{2}}$	1, 5	Analogize differential method of polynomial to rational functions
Type 3	$\frac{dS}{dt} = -3$ & $S = a^2$ $\rightarrow \frac{da}{dt} = -\sqrt{3}$ & $a(t) = 10 - \sqrt{3}t$	1, 2, 3, 4, 5	Analogize relationship between area and length of edge to relationship between $\frac{dS}{dt}$ and $\frac{da}{dt}$
Type 4	$S = a^2$ $\rightarrow S' = 2a$	1,4	-

and some students built models of the area with respect to the length of edge (Type 4). Although every student knew the relationship between the area and the length of edge, a relatively small number of students modeled this relationship.

The students who built the Type 1 model first formularized the area with respect to time from the given conditions of the task and then built a Type 1 model with consideration of the quadratic relation between  $S$  and  $a$ . The students could not directly differentiate the Type 1 model since they had not yet learned the differentiation rule for rational functions. Given that, some students differentiated this function by using the analogy from the differential rule of polynomials to differentiate rational functions. To apply the differential rule of polynomials, students first transformed  $\sqrt{100 - 3t}$  to  $(100 - 3t)^{\frac{1}{2}}$ , and then applied the differential rule of polynomials to obtain a Type 2 model. The students who built a Type 3 model assumed that the quadratic relation between  $S$  and  $a$  may hold between  $\frac{dS}{dt}$  and  $\frac{da}{dt}$ . Given that, these students first conjectured that  $\frac{dS}{dt}$  was  $\sqrt{3}$ ,  $-\sqrt{3}$ , and even  $\sqrt{-3}$ , and they considered  $\frac{da}{dt}$  to be  $-\sqrt{3}$  since the length of edge is decreasing. The students who built a Type 4 model conjectured that  $S' = 2a$  may hold based on  $S = a^2$ .

Since these students' modeling activities were based on the use of analogy, which is a plausible reasoning, their degree of certitude was not high. Hence, they tried to verify their models. Every student who built a Type 1 model also constructed a Type 3 model. Since they could not differentiate a Type 1 model, they tried to directly find  $\frac{da}{dt}$  from  $\frac{dS}{dt}$ . Thus they also constructed the length model of  $100 - \sqrt{3}t$ . However, this model did not accord with the Type 1 model, so they verified these models by finding the length of the edge at a particular time.

S5: By the way, I think I found it?  $10 - \sqrt{3}t$ ? But it does not make sense if I substitute.

S8: It doesn't make sense?

S5: If I substitute 1?  $10 - \sqrt{3}t$ , but if I square it, the square root sign still survive. However, since the initial area is 100 and is decreasing by 3 per minute, 97 should be the result [when squaring the length of the edge], but the square root still survives and it does not make sense.

The students' conjecture of  $\frac{da}{dt} = \sqrt{\frac{dS}{dt}}$  was rejected by a similar inductive verification by S5. Given that, the students who built a Type 1 model attempted to find  $a'(t)$  in an alternative way, and the students in Groups 1 and 5 conjectured that the Type 2 model was correct. Although  $a(t)$  was relatively easy to verify,  $a'(t)$  was difficult for students to verify.

After enough small group inquiries, the teacher orchestrated a whole-class discussion to share each group's models and modeling processes. The students who

did not reject the Type 3 model (Group 2) also identified that this model was not appropriate. The teacher postponed her judgment on the students' modeling processes, and students' inquiries on Subtask 1 were finished, although the Type 2 model was still vague for students.

Three key issues emerged from this section. First, model construction was closely related to the use of analogy. The students first formularized conditionals of the given situation and then tried to build models of the rate of change of the edge using these formulas. The students used analogy to apply the differentiation rules of polynomials or relationships between the area and the length of edge to model the given situation. These students' model construction can be interpreted as the use of undercoded abduction, since they selected existing rules to explain observed results. The students' use of analogy played a key role in constructing an abduction to form a hypothesis on the given situation. The students' used analogy to roughly broaden the usage of mathematical rules that they already knew to new context. Second, the students verified their models inductively. The students could not deductively verify their models and abductions. Given that, students examined particular cases to verify and reject their models. However, they could not provide reasonable arguments for rejecting models and could not verify the Type 2 model. Third, students conducted multiple modeling. They modeled the given situation in different ways to verify their use of abduction. They then found inconsistencies between two models (S5). As Cottrill (1999) pointed out, deductive proof of the chain rule is not easy work for students, so they seemed to be trying to verify their models by creating multiple models in order to compare two different abductions.

**Episode 2: Model revision and generalization by creative abductions in a real context**

Subtask 2 asked students to find the instantaneous rate of change of the length of the edge of the room after 7 minutes, so students were reminded of the definition of a derivative. The students found the instantaneous rate of change of the length of the edge in two ways as shown in Table 2.

The students who utilized the definition of a differential coefficient calculated the instantaneous rate of change of the length of the edge as shown in Figure 2.

The other students of Type 2 observed patterns of average rate of change to approximate the limit value of the formula in Figure 2, as shown in Figure 3. In Figure

**Table 2.** Ways of modeling in Subtask 2

Type	Modeling	Group
Type 1	Utilize definition of differential coefficient	2, 3, 4
Type 2	Observe patterns of average rate of change	1, 5

Handwritten student work showing the derivation of a derivative using the definition of a limit. The work includes the general formula for a derivative, a specific example with  $f(x) = \sqrt{4x+3}$ , and the final result  $\frac{3}{2\sqrt{14}}$ .

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$$\lim_{h \rightarrow 0} \frac{f(7+h) - f(7)}{h}$$

$$\lim_{h \rightarrow 0} \frac{\sqrt{4(7+h)+3} - \sqrt{4(7)+3}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{4h+31} - \sqrt{31}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{4h+31} - \sqrt{31}}{h(\sqrt{4h+31} + \sqrt{31})} = \lim_{h \rightarrow 0} \frac{-3}{\sqrt{4h+31} + \sqrt{31}}$$

$$= \frac{3}{2\sqrt{14}}$$

**Figure 2.** Worksheet of S11

3, two students rationalized the numerator of the fraction. S1 responded to the

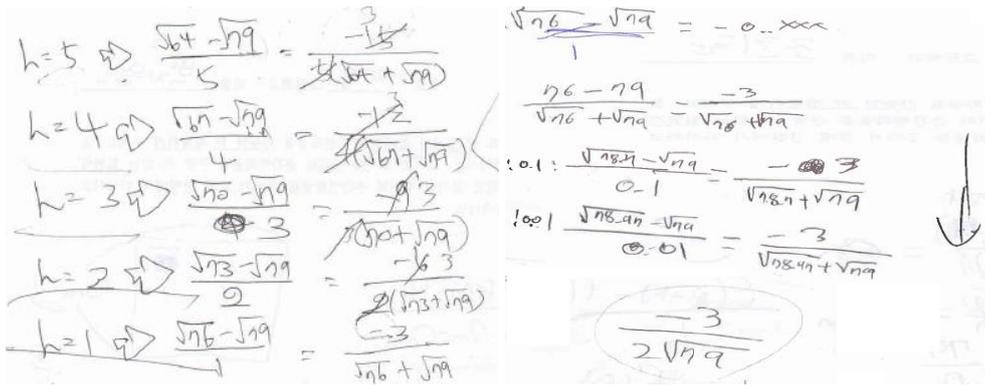


Figure 3. Worksheets of S3 (left) and S1 (right)

teacher’s question about his rationalization: “I wanted to rationalize assuming denominator was 1.” In the top right of Figure 3, the segment between  $\sqrt{76} - \sqrt{79}$  and 1 is colored in blue. This is the starting point for rationalizing the numerator of the fraction since its form is completely different from the others. Given that, S1 tried to adjust the forms of the formulas similarly, so he perceived  $\sqrt{76} - \sqrt{79}$  as  $\frac{\sqrt{76} - \sqrt{79}}{1}$ , and rationalized the numerator. As a result, students observed repetitive patterns of -3 in numerators as well as  $\sqrt{79}$  in denominators.

S3: You found it when h is 5?

S1: Yes.

S3: Tell me.

S1: The square root of 64 minus the square root of 79 over 5.

S3: The square root of 79?

S1: The square root of 79 is the same for all.

S3: Next, when is h 4?

S1: That is, increase 3 here [pointing at  $\sqrt{64}$ ] per minute.

S3: To the former number [in the numerator]?

S1: Increase 3 to the former, and the denominator keeps decreasing.

S3: Increase 3 per minute?

S1: 67, keep increasing.

S4: 70

S1: 7.1, minus...

S3: Hey, when h is 5, what is ... numerator? -1?

S1: What?

S3: Numerator

S1: Numerator? That ...

S3: Minus 1?

S1: What?

S3: If we rationalize ...

S1: I don't know. I didn't rationalize.

S3: I will give it a try.

S3: Minus 15, hmm. Ah, that is minus 3. 12 ... minus 3.

S1: Oops! Minus 3 continues!

S3: Uh? I got goose bumps!

S1: Uh? It is not when h is 0.1. When h is 0.1, -3. When h is 0.1, no, multiply by 10, then -3!

These students in Group 1 (S1-S4) observed a pattern of the average rate of change of the length of the edge and conjectured that the denominator will converge to  $\sqrt{79} +$

$\sqrt{79}$  and  $a'(7)$  is  $\frac{-3}{2\sqrt{79}}$ . In particular they found that the repeating number  $\sqrt{79}$  is  $a(7)$  and  $-3$  is  $\frac{dS}{dt}$ . From this inquiry, they conjectured  $a'(t)$  is as shown in Figure 4.

In Figure 4, S3's model of  $a'(t)$  was built by abduction of the results of her modeling of  $a'(7)$ , and this consisted of two abductions. Her first abduction was identified in hypothesizing that  $a'(7)$  was  $\frac{-3}{2\sqrt{79}}$  based on observation of patterns in Figure 3. She attempted to explain the observed pattern using  $\frac{-3}{2\sqrt{79}}$ . The second abduction was hypothesizing that  $a'(t)$  is  $\frac{\text{change of area}}{2 \times \text{length of the edge}}$  in Figure 4. The repeating value  $\sqrt{79}$  was the length of the edge after 7 minutes and was identified when modeling  $a'(7)$ . They also observed that values inside each square root sign were the areas at each time, so they saw that the value of the numerator was the area difference between each time they rationalized the formula. Hence, S1 found out that  $-3$  was always derived by dividing this numerator by the difference in time. Based on this process, students conjectured that  $a'(t)$  was as shown in Figure 4, and this can be interpreted as an example of creative abduction.

The student's construction of a creative abduction was closely related to the real context in the given task. It is important to note that S3 considered the numerator of  $a'(7)$  to be  $\frac{dS}{dt}$  and the denominator of  $a'(7)$  to be 2 times of length of edge. Since values of  $-3$  and  $\sqrt{79}$  had contextualized meanings and her observed pattern held for every time with the same context, S3 was able to easily generalize  $a'(7)$  to  $a'(t)$ . It is interesting that students who observed patterns of average rate of change (Type 2) produced more a generalized formula than students who used a generalized algebraic solution (Type 1).

Figure 5 represents the modeling process of S7 from Group 2. Since Subtask 2

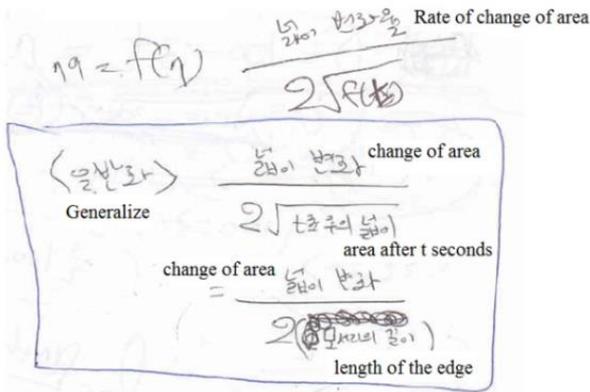


Figure 4. Worksheet of S3

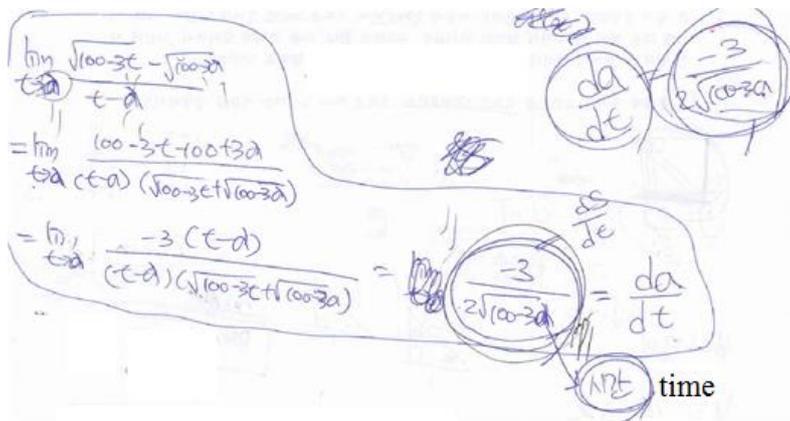


Figure 5. Worksheet of S7

asked students to represent  $a'(7)$  with the formulas established in Subtask 1, she represented  $-3$  as  $\frac{ds}{dt}$ . However, values involved in the formula in Figure 5 had relatively little potential for further inquiries in comparison with values in the formula in Figure 3. S7 implemented an algorithm to find the differential coefficient without considering the problem context, so she did not know why  $\frac{da}{dt}$  was related to  $\frac{ds}{dt}$ . After enough small group inquiries, the teacher orchestrated a whole-class discussion in order to share each group's models and modeling processes. There were two issues in whole-class discussion. First, students examined the model built in Subtask 1, which was not verified when resolving Subtask 1:  $a'(t) = \frac{1}{2}(100 - 3t)^{-\frac{1}{2}}$ . This model was rejected since it is not consistent with models built when resolving Subtask 2. The second issue was the students' attempts to explain the relationship between  $\frac{ds}{dt}$  and  $\frac{da}{dt}$ . They tried to find a reason why these two values were related in their models. Subtask 2 asked students to represent their models of  $a'(7)$  with formulas established in Subtask 1. Given that, the students focused on making a connection between  $\frac{da}{dt} = \frac{1}{2a} \times \frac{ds}{dt}$  and  $S = a^2$  as well as  $\frac{ds}{dt} = -3$ , which were from models that had still survived. However, they did not clarify the reason why the formula  $\frac{da}{dt} = \frac{1}{2a} \times \frac{ds}{dt}$  works and what it represented. The students' exploration of this issue will be addressed in the next section.

There are two key issues that emerge from this section. First, there was close relationship between "construction of an abduction" and "observations of the

modeling process and contextual interpretation of patterns that emerged in the modeling process.” This relationship is determined by students’ construction of an abduction of  $a'(t)$  based on the students’ observation and contextual interpretation of the pattern that emerged during the modeling of  $a'(7)$ . This can be interpreted as a creative abduction since students conjectured a generalized rule based on the observed result of creating a particular model. This creative abduction also resulted in the building of a generalized model.

Second, generalization of a model was relatively more difficult for students than generalization of the modeling process. As we have seen in Figure 5, interpreting the meanings of values involved in a model derived by an algorithm was not easy for students, so these students’ generalizations fell short of those of students who had conducted the other type of modeling. Although these students tried to use abduction (Figure 5), the explanatory potential was not so high.

Subtask 2 asked students to represent the instantaneous rate of change of the length of the edge with the formulas established in Subtask 1, and the students already knew several differential rules which could be applied to the formulas established in Subtask 1. Hence, students tried to directly derive their provisional solutions of  $a'(t)$  or  $a'(7)$  in Subtask 2 from the formulas of Subtask 1, but they could not derive their solutions of Subtask 2 by transforming formulas from Subtask 1 and applying their differential rules. The students were confused and this ambiguous situation motivated students’ further inquiries, which are addressed in the next section.

### **Episode 3: Model generalization and justification by creative abductions with diagrammatic reasoning**

Since Subtask 3 asked students to find the instantaneous rate of change of the length of the edge after 11 and 13 minutes, students first verified the conjectures they had formed in Subtask 2 inductively. Although the students conjectured that  $\frac{da}{dt} = \frac{1}{2a} \times \frac{dS}{dt}$ , they were still unsure of their conjecture since their hypothesis was derived by abduction, which is plausible. Given that, students tried to derive this model from  $S = a^2$  or  $\frac{dS}{dt} = -3$  with differential rules. The students of Type 1 who found  $a'(t)$  using an algebraic algorithm in Subtask 2 especially doubted both of the conjectures of students of Type 2 and interpreted -3 as  $\frac{dS}{dt}$ . Given that, they tried to differentiate the formula  $S = a^2$  to find a reason for interpreting -3 as  $\frac{dS}{dt}$ .

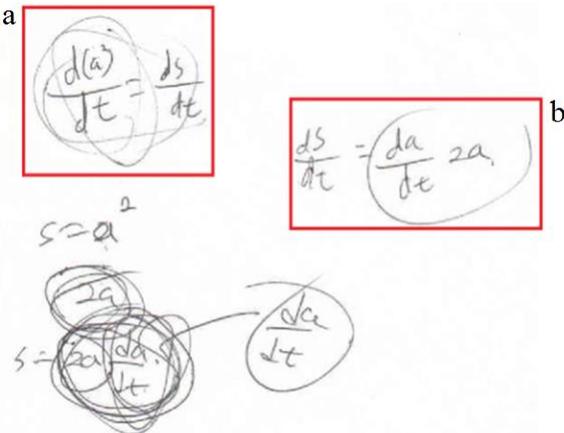


Figure 6. Worksheet of S11

In the left square of Figure 6, S11 was very careful to differentiate  $S = a^2$  with respect to  $t$ . In the right rectangle of Figure 6, he also represented a transformed version of conjecture formed by students in Subtask 2, whereas original one was in the form of  $\frac{da}{dt} = \frac{1}{2a} \times \frac{dS}{dt}$ . That is, he experimented on this algebraic diagram  $\frac{da}{dt} = \frac{1}{2a} \times \frac{dS}{dt}$  to obtain the diagram in the left square a, but the right one produced b. Students in other small groups also made similar attempts.

After enough small-group interaction, the teacher orchestrated a whole-group discussion to clarify the problematic part of their inquiries. S12 clarified his issue as following:

S12: Given result of differentiation on S it must be this. [He wrote the formula  $\frac{dS}{dt} = \frac{da}{dt} \times 2a$  on the blackboard and pointed.]. But if we use our brains, the differentiation of S supposed to be  $2a$ , isn't it? I think we need to think about why  $\frac{da}{dt}$  came out.

Interestingly, the students first reexamined their modeling processes in the subtasks to determine whether  $\frac{dS}{dt} = \frac{da}{dt} \times 2a$  is an appropriate model for the given situation. After several reexaminations of the modeling process, some students began to reinterpret the meaning of  $\frac{dS}{dt} = S' = 2a$  in S12's argumentation. S16, a member of Group 4 pointed out a problem in this argument.

S16: If we differentiate S with respect to a, we represent it  $\frac{dS}{da}$ , and this is  $2a$ . Thus, we substitute this into here  $[\frac{dS}{dt} = \frac{da}{dt} \times 2a]$ ,  $\frac{dS}{dt}$  is  $\frac{da}{dt}$  times  $\frac{dS}{da}$ . [He wrote this on the blackboard.] I think this formula can be canceled, but other teachers told us not to do that.

In our school mathematics, we do not adopt an infinitesimal perspective on calculus, so we deal with  $\frac{da}{dt}$  as a single symbol. Hence, S16 hesitated to interpret the meaning of his formula  $\frac{dS}{dt} = \frac{da}{dt} \times \frac{dS}{da}$ . After listening to his claim, S5 tried to add a supplement to his approach with drawing as shown in Figure 7.

S5: Then we can say  $\frac{dS}{dt}$  is  $\frac{dS}{da}$  times  $\frac{da}{dt}$ . Here, we need to know that the length of the edge changes as the value of t changes. Doesn't it? Also, if the length of the edge increases, the values of area change. Hence, the area changes with respect to time; this [she drew curved arrows under letters t, a, and s] is what happens. The steps are . . . After this [pointing to t] changes, then this [pointing to a] should be changed, so I wrote this [circling  $\frac{da}{dt}$ ] to represent the change of the length of the edge after the

changing value of  $t$ . Changing area after the change of the length of the edge is represented by this [circling  $\frac{dS}{dt}$ ]. The solution is derived like this.

Although S5's justification is informal, it is a valid justification since the chain rule  $\frac{d}{dt}(g(f(t))) = g'(f(t)) \times f'(t)$  is derived by finding a limit value of  $\frac{g(f(t+h))-g(f(t))}{f(t+h)-f(t)} \times \frac{f(t+h)-f(t)}{h}$  in school mathematics. Since this justification was informal and contextual, she examined her hypothesis by exemplifying on the simple differentiation of polynomials in two ways: applying her conjecture (the chain rule) and application of a differentiation rule of polynomials after expansion as shown in Figure 8.

There are two issues that emerge from this section. First, inconsistency between the result of diagrammatic reasoning and modeling triggered students' use of abduction. Deriving  $s' = 2a$  from  $S = a^2$  can be interpreted as an application of conventional rules of differentiation to an algebraic diagram. Thus, this attempt can be seen as the students' attempts to transform or experiment on diagrams. As

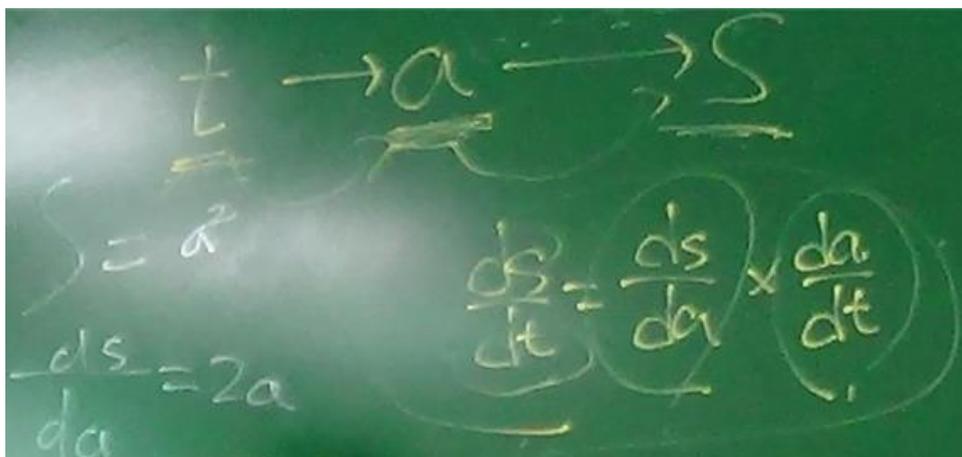


Figure 7. Figural representation of S5 (Only yellow letters are written by S5.)

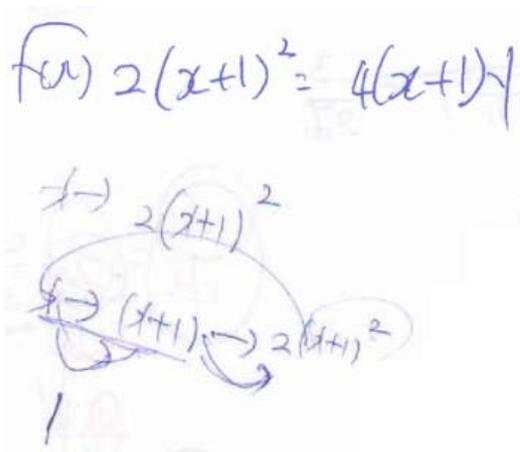


Figure 8. Worksheet of S5

Hoffmann (2004) pointed out, experiments on diagrams provided students with strong assurance since they are based on conventional rules (S12). Inconsistency between the results of experimentation on a diagram and students' models of the rate of change of length of edge raise an issue to students, as S12 addressed. The students first had confidence in the results of diagrammatic reasoning and reexamined their modeling process. However, since there was no problem in the modeling process they reinterpreted their experiments based on an algebraic diagram. They then determined that  $S' = 2a$  is partially valid via conventional rules of differentiation, but

their interpretation was not appropriate (S16). Given that, S16 and S5 used a creative abduction to formulate a comprehensive explanation, i.e., the chain rule, based on the results of diagrammatic reasoning and their mathematical modeling.

Second, the motion context which is implied in Leibnizian notation supported students' interpretation and justification of the chain rule. Although Leibnizian notations are considered to be conventional symbols (Presmeg, 2005), they still involve contextual and relational aspects (Cottrill, 1999). In this respect, Leibnizian notation functioned as an icon and a diagram for students. Based on the ratio relation that immediately emerged from Leibnizian notation, the students hypothetically interpreted the meaning of the chain rule in relation to the real context (S5). This also supported student S5's interpretation and justification of the chain rule from the real context of modeling task. She was able to interpret and explain the relationship among three variables via the contextual meanings of each variable.

## DISCUSSION AND CONCLUSION

In this study, we aimed at facilitating students' inquiries into the chain rule by supporting generalization from modeling activities. We especially focused on the use of analogy and diagrammatic reasoning to foster students' construction of abduction. As a result, we determined that the students used abduction based on both analogy and diagrammatic reasoning and built and generalized mathematical models to derive the chain rule. Though we could not show the entire mechanism of model generalization by abduction, we revealed sub-mechanisms of model generalization and the use of abduction supported by analogy, diagrammatic reasoning, and a real context.

First, students' use of diagrammatic reasoning and analogy were closely related to construction of abductions, as other researchers have theoretically claimed (Hoffmann, 2004; Otte, 2006; Prawat, 1999). The students' uses of analogy were in the form of directly applying existing mathematical rules to a new context, so their uses of analogy resulted in construction of undercoded abduction. In other words, students rarely modified existing rules when applying them to a new context. For example, students directly applied "relationship between area and length of edge" and "differentiation rules of a polynomial" to a given problem situation (Episode 1). Although analogy is known to play a key role in knowledge construction, as Lee and Sriraman (2011) have emphasized, the students could not use analogy productively at the beginning of their inquiries. The students' uses of diagrammatic reasoning were also in the form of directly applying conventional rules to diagrams. As Hoffmann (2004) pointed out, the rationality inherent in diagrammatic reasoning is guaranteed by conventional rules which are applied to transform or experiment on diagrams. Given that, students were convinced of the results of diagrammatic reasoning even though they did not properly utilize diagrammatic reasoning (Episode 3).

Although students applied existing mathematical rules to a new context in inadequate ways, their use of analogy and diagrammatic reasoning supported their inquiry into the chain rule. To be more specific, the results of analogy and diagrammatic reasoning supported the existence of a generalized rule that explains models and the modeling process in a given real situation. It also contributed to forming the initial hypothesis based on the problem situation. That is, the students were able to apply existing rules to a new context, and the results of this were partially in accord with their modeling activities. Given that, they were able to determine that there was some regularity or generalized mathematical rule similar to existing rules that explained the given situation.

On the other hand, mathematical modeling activities functioned as evidence that could be used to verify the validity and consistency of their use of analogy and diagrammatic reasoning based on conventional rules, which fostered students' use of

creative abduction to modify existing rules. To be more specific, students reexamined their use of conventional rules when the results of analogy and diagrammatic reasoning were not consistent with their modeling and tried to reinterpret and modify their existing rules (Episode 3). As Sfard (2008) pointed out, one of keys to knowledge construction is using mathematical objects or rules in an approximate way in a new context and then testing and revising their usage. In this respect, while the use of analogy and diagrammatic reasoning was a starting point for forming a plausible hypothesis by approximating existing rules to a new context, mathematical modeling of a given situation supported students' verification and validation of their own reasoning and revision of those rules in constructing a creative abduction. As Cottrill (1999) noted, it was difficult for students to deductively derive and justify the chain rule. As Lee (2011) claimed, when deductive reasoning for specific mathematical contents is too challenging it is necessary to consider a modeling approach, which fosters students' conjecturing, revising, and validating via modeling activities, as an alternative way to support students' mathematical inquiry as a complement to the Lakatosian perspective.

Second, we determined that the use of abduction is closely related to model construction and generalization. From a Peircean semiotic perspective, abduction is closely related to generalization (Otte, 2006), which we empirically confirmed. The students' generalizations of models were mainly based on their use of creative abduction (Episodes 2 and 3). We summarize students' constructed models and their generalized versions in Table 3.

**Table 3.** Students' model generalizations

Type	Episode 1	Episode 2	Episode 3
Model construction	$a(t) = 10 - \sqrt{3t}$ $a'(t) = \frac{1}{2}(100 - 3t)^{-\frac{1}{2}}$	$a'(7) = \frac{-3}{2\sqrt{79}} = \frac{-3}{2\sqrt{a(7)}}$	$\frac{da}{dt} = \frac{1}{2a} \frac{dS}{dt}$
Model generalization		$a'(t) = \frac{1}{2a(t)} \frac{dS}{dt}$	$\frac{dS}{dt} = \frac{dS}{da} \frac{da}{dt}$

It is important for a generalization to make conjectures (Carraher, Martinez, & Schliemann, 2008; Lee, Chen, & Chang, 2014). That is, a generalization progresses first by conjecturing generalized rules for observed results then verifying and revising a provisional generality. In this study, generalization of mathematical models occurred by use of abduction to set explanatory hypotheses on observed results and validating them rather than inductively accumulating many cases. As we have seen in the Results chapter, generalization of mathematical models was also a dynamic process rather than an inductive process. The students used abduction in an approximate way to build provisional generalities based on observation of a particular example situation and then verified them via other cases.

Third, the real context was also one of the main factors that supported model generalization. In Episode 2, students observed patterns and generalized these patterns by considering the contextual meanings of each value in the pattern. In other words, the students were able to generalize a model of a particular time to every moment since their observed pattern held for every moment which shared the same context. We also determined that a real context supported students' justification of their use of abduction. In Episode 3, S5 informally justified her conjecture using contextual interpretations of a formula. The role of context in justification is important since algebraic or formal proof of the chain rule was regarded as an algebraic trick, as Cottrill (1999) noted.

Fourth, Leibnizian notations of calculus supported students' conjectures about and justification of the chain rule. Leibnizian notation is similar to fractions (Tall, 1992) and has a contextual and relational aspect in the infinitesimal perspective (Cottrill,

1999). Although it is usually a conventional symbol (Presmeg, 2005), it also involves intuitive and physical representations of fractions of two quantities (Tall, 1992). With the assistance of these two diagrammatic aspects of Leibnizian notation, students were able to determine that their conjectured formula represented a compound ratio among variables (Episode 3, S5). As Tall (1992) pointed out, issues related to notation are sensitive. However, considering only the symbolic aspect of Leibnizian notation loses much of its potential. Further research on Leibnizian notation in the context of learning calculus is encouraged.

Students' difficulties in the learning of calculus were reported, and so the necessity of finding an alternative way to facilitate students' inquiry into key ideas of calculus has been discussed (Haciomeroglu, 2015; Hashemi, Abu, Kashefi, Mokhtar, & Rahimi, 2015; Sahin, Yumez, & Erbas, 2015). In this study, we confirmed that mathematical modeling supports students' inquiries into the chain rule and facilitation of the use of abduction promotes generalization of models. One of the key issues of debate in mathematics teaching and learning is the conflict of opinions on the role of real contexts. The results of this study indicate the existence of a positive role for real contexts when conventional rules are used along with analogy and diagrammatic reasoning. Although we partially confirmed the synergic relation between real context and use of conventional rules in the first two points of discussion, it is still debatable. Also, the adaptability of some content areas to real contexts may be variable. Further studies involving a variety of modeling tasks in different content areas are encouraged in order to verify the possibility of including mathematical modeling in teaching and learning mathematics.

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