MODESTUM
OPEN ACCESS
EURASIA Journal of Mathematics, Science and Technology Education, 2018, 14(6), 2067-2079
ISSN:1305-8223 (online) 1305-8215 (print)
Research Paper https://doi.org/10.29333/ejmste/86163

How do College Students Clarify Five Sample Spaces for Bertrand's Chord Problem?

Bongju Lee [1], Yong Sik Yun [2*]

[1] Kyungpook National University, Daegu, SOUTH KOREA
[2] Jeju National University, Jeju, SOUTH KOREA

Received 18 May 2017 ▪ Revised 17 December 2017 ▪ Accepted 7 February 2018

ABSTRACT

The concept of probability has a unique characteristic that causes confusion when a sample space is not clearly defined. This inherent nature of probability has been demonstrated in Bertrand's chord problem, which is well-known as the paradox of probability theory. This study demonstrated that a single probability in Bertrand's chord problem can be obtained by modifying it to clearly redescribe its sample space, and examined that how college students clarify the sample space. To this end, five modified questions were formed using Bertrand's chord problem to ensure that the sample space of each question was clearly expressed and were used to develop a survey questionnaire. The participants of the survey were 68 college students studying mathematics or mathematics education. The results of this study demonstrated that many college students have difficulty to seek out the sample spaces in some probability problems. Thus we suggested the importance of emphasizing to clarify the sample space in probability education.

Keywords: Bertrand's chord problem, paradox, college students, five modified Bertrand's questions, probability, sample space

INTRODUCTION

Probability is a relatively familiar term that is widely used in this information age. For instance, the question 'what is the probability of obtaining heads when flipping a coin?' is asked ordinarily and most people answer "$\frac{1}{2}$". This is because most people approach the question by thinking that a coin has a heads side and a tails side and the chance of the coin falling either heads or tails is the same.

However, it is debatable whether the heads and tails sides of a coin are indeed obtained under the same condition or not. The designs on the heads and tails sides of a coin are different and it is uncertain whether the specific gravity in the coin is evenly distributed. One aspect that must be understood in the experiment of flipping a coin is that the assessment "the probability of each side is the same because they exist under the same condition" is not an absolute truth, but a selected assumption. In other words, the $\frac{1}{2}$ probability of obtaining heads is not a transcendental and absolute truth, but it results from the assumption that each case has the same chance of occurring (Batanero, Henry, & Parzysz, 2005; Kim, 2008; Rubel, 2007; Woo, 1998).

Accordingly, the concept of transcendental and absolute probability does not exist. The classical definition of probability provided by Laplace in the 19th century, the so-called mathematical probability, is based on the fundamental premise that each element within a sample space has the same chance of occurring (Alexander & Kelly, 1999; Gauvrit & Morsanyi, 2014; Gillies, 2000; Woo, 1998). According to the classical interpretation, probability is defined as the ratio of the number of elements of a certain event to the number of the elements of the sample space, and geometrical probability typically fits this interpretation as well (Lee, 1997). Considering this classical interpretation of probability, the importance of clarifying whether each element within the sample space is under the same condition has been widely recognized and is highlighted in school education (Batanero & Borovcnik, 2016; Garfield & Ahlgren, 1988). Many studies reported important results on students' probabilistic

✉ leebj@knu.ac.kr ✉ yunys@jejunu.ac.kr **(*Correspondence)**

Contribution of this paper to the literature

- This study provides a new perspective on Bertrand's chord problem in that it approaches as a sample space.
- We highlight the importance of guiding to clarify sample space in probability education.
- This study shows some limits to the mathematical definition of probability can be overcome by clearly defining fundamental events with equiprobability.

thinking (e.g., Agus, Peró-Cebollero, Penna, & Guàrdia-Olmos, 2015; Fischbein & Gazit, 1984; Fischbein & Schnarch, 1997; Green, 1983; Hawkins & Kapadia, 1984; Konold, 1989; Kwon & Lee, 2015; Kwon, Kim, & Lee, 2014; Rubel, 2007; Shaughnessy, 1977). Some researchers emphasized and focused on sample space that plays an important role in probabilistic tasks and situations (e.g., Chernoff, 2009; Chernoff & Zazkis, 2011; Jones, Langrall, & Mooney, 2007; Jones, Langrall, Thornton, & Mogill, 1997, 1999; Shaughnessy, 1998).

Despite these researchers' emphasizing the sample space, recently, Choi, Yun, and Hwang (2014) reported that pre-service mathematics teachers still have the difficulty to understand the sample space. They proposed that students should be provided with the opportunity to think about the sample space in probability and statistics education. We suggest that teacher educators and teachers would use Bertrand's chord problem as one of these opportunities. That's because we saw both groups of pre-service and in-service mathematics teachers who didn't understand the solutions of Bertrand's chord problem in our classrooms. Many college students and mathematics teachers didn't understand why there exist three solutions for this problem and why it is called a paradox. We thought that's because they didn't have the chance to meet its solutions exposing the term sample space although each solution represents chance and inherent equiprobability. Like this, the mathematical definition of probability can cause further confusion unless clearly defining the fundamental events that have equiprobability in probability space. Lee (1997) and Woo (1998) noted that this paradox is the limits to the mathematical interpretation of probability with its classical definition. In this article, we would represent the term Bertrand's chord problem instead of Bertrand's paradox.

With this backdrop, this study used Bertrand's chord problem to demonstrate that the limits to the mathematical definition of probability in some probability problems can be overcome by clarifying the sample space, and to emphasize that it is important to teach students to clearly define the sample space in probability problem. To this end, each solution of Bertrand's chord problem was represented using the term sample space and this problem was modified in five ways to ensure the existence of only one sample space. These modified problems were given to college students studying mathematics and mathematics education to demonstrate that how well they could clarify the sample spaces and obtain a single probability.

BACKGROUND

Discussion of Bertrand's Chord Problem

Bertrand's chord problem has been discussed by many researchers (Aerts & Sassoli de Bianchi, 2014; Borovcnik, Bentz, & Kapadia, 1991; Drory, 2015; Gyenis & Rédei, 2014; Jaynes, 1973; Klyve, 2013; Marinoff, 1994; Porto, Crosignani, Ciattoni, & Liu, 2011; Rowbottom, 2013) since Bertrand introduced it in 1889. Bertrand's chord problem using Bertrand's words is as follows.

> ***Bertrand's chord problem.*** *We draw at random a chord onto a circle. What is the probability that it is longer than the side of the inscribed equilateral triangle? (Aerts & Sassoli de Bianchi, 2014, p. 1)*

Bertrand (1989) proposed this problem which leads to the different results $\mathrm{P(A)} = \frac{1}{2}$, $\mathrm{P(B)} = \frac{1}{3}$, and $\mathrm{P(C)} = \frac{1}{4}$. These probabilities are according to each of the three assignments for 'equally possible situations'; (A) the linear line between centers of chord and circle, (B) angles of intersections of the chord on the circumference, and (C) the center of the chord over the interior area of the circle (Jaynes, 1973). Even though Bertrand presented three answers for it, Jaynes (1973) concluded that Bertrand's chord problem is well posed and has the unique probability $\mathrm{P(A)} = \frac{1}{2}$. This conclusion was followed by his viewpoint toward probability theory that "the only valid basis for assigning probabilities is frequency in some random experiment (p. 2)". Using an analogy with a cylindrical cake cutting, Rowbottom (2013) also insisted that Bertrand's original problem is vague and all Bertrand's three solutions are not the effective potential ways.

Other researchers who had referenced Bertrand's problem supported Bertrand's three different solutions. Borovcnik, Bentz, and Kapadia (1991) described that each of the three solutions shows chance determined by the equiprobability of Laplace's definition through its individual random generator. Marinoff (1994) showed Bertrand's chord problem is equivocally brought up but the many versions of Bertrand's original problem by clearly stated

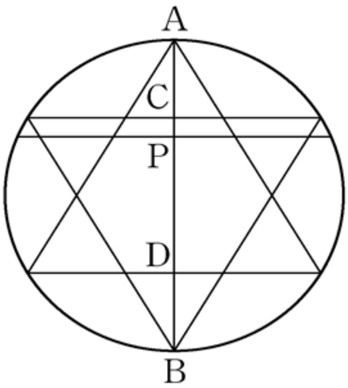


Figure 1. A chord vertical to diameter AB

variations lead to different solutions can be solved. Porto, Crosignani, Ciattoni, and Liu (2011) basically accepted that Bertrand's results are all correct as well as many other possible ones because a chord drawing randomly cannot be uniquely defined. They provided a realistic physical experiment associating with it as alternative to Bertrand's chord problem. Furthermore, Klyve (2013) defended Bertrand's intention against Rowbottom (2013)'s interpretation of Bertrand's chord problem. Klyve noted that Rowbottom misinterpreted what Bertrand said about the random chord although his conclusion is correct according to his own description. **Drory (2015) discussed that it has inherent ambiguity and depends on explicitly defining the selection procedure for the random chords.**

Meanwhile, Aerts and Sassoli de Bianchi (2014) showed that Bertrand's chord problem includes an easy problem and a hard problem. They reported that the easy problem is solvable by clarifying Bertrand's chord problem in precise terms. They presented one example for a specific physical realization of Bertrand's chord problem which has the same to three different answers proposed by Bertrand, and then insisted that Bertrand's three solutions should be more easily explained via three different conditional probabilities. Thus, Aerts and Sassoli de Bianchi concluded that Bertrand's chord problem should not anymore be considered as a paradox. Also, they remade the hard problem using modified Bertrand's chord problem in that two points instead of a straight line are randomized. This hard problem became solvable by calculating a uniform average, which they called a universal average, over all possible ways of selecting an interaction.

Unlike above researchers who focused on Bertrand's chord problem's solutions and answers, Gyenis and Rédei (2014) suggested a new interpretation of it and investigated the relation between Bertrand's chord problem and the classical interpretation of probability. They argued that this paradox is harmonized with how the science uses mathematical probability theory to model phenomena, without making any damage the principle of indifference and the classical interpretation of probability.

Solutions to Bertrand's Chord Problem using the Term Sample Space

In general, many researchers and practitioners cited three typical solution methods that lead to the results $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ (see, Aerts & Sassoli de Bianchi, 2014; Borovcnik, & Bentz, 1991, Drory, 2015; Lee, 1997; Marinoff, 1994; Porto et al., 2011; Woo, 1998). Kim (2008) introduced other two other solutions which lead to the results of impossibility and indefinite, respectively. However, they described the results and probabilities without using the term sample space in three different solutions for it. In this section, we tried below five solutions to help mathematics teachers and students understand the solutions of Bertrand's chord problem in terms of sample space. Also, these were supposed to be the basis of five modified Bertrand's chord questions given as testing tool in the next section.

Solution 1. Suppose that a line segment passes through midpoint D of the opposite side of the vertex A of an equilateral triangle inscribed in a circle, as shown in Figure 1 and that the point where the line segment meets with the circle is B. Assume that the set of all chords that are perpendicular to line segment AB is the sample space. The length of such perpendicular chords in this probability space is determined by the location of point P, which is defined by the intersection of line segment AB with a perpendicular chord.

Suppose that C is a point where line segment AB meets with a chord that has the same length as the triangle side within the sample space and that C is not located on the opposite side of vertex A. When a point Plies on line segment CD, the length of the chord is greater than the side of the equilateral triangle inscribed in the circle. In other words, the probability in this case is the same as the probability that point P is on line segment CD. Therefore, the length of line segment CD is $\frac{1}{2}$ of the length of diameter AB. Therefore, we obtain a probability of $\frac{1}{2}$.

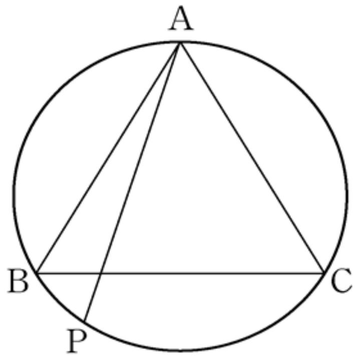


Figure 2. A chord fixed at point A

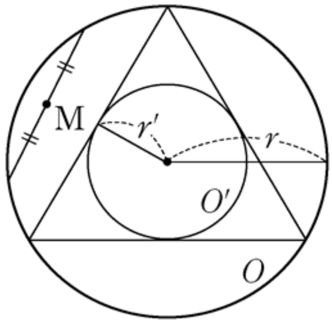


Figure 3. A chord with a midpoint M

Solution 2. As a chord is determined by two points on a circumference, consider the vertex A of an equilateral triangle inscribed in a circle as one endpoint of a chord, as shown in **Figure 2**. Suppose that the set of all chords that are drawn from vertex A is the sample space of a probability space. The chords included in this probability space are determined by the location of their other endpoints Ps.

Suppose that the other two vertices of the triangle inscribed in the circle are B and C. When the endpoint P of a chord is on arc BC, the length of the chord is greater than the side of the triangle inscribed in the circle. In other words, the probability in this problem is the same as the probability that point P is on arc BC. Therefore, the length of arc BC is $\frac{1}{3}$ of the length of the circumference. Therefore, the probability is $\frac{1}{3}$.

Solution 3. Suppose that the midpoint of a randomly drawn chord is M, as shown in **Figure 3**, and that the radius of a circle inscribed in an equilateral triangle is r'. A circle circumscribed about the equilateral triangle is denoted by O and the circle with radius r' by O'. The probability of the question is the same as the probability that the midpoint M is within the circle O' in a probability space of which a sample space is defined as a set of midpoints M of randomly drawn chords. Therefore, the area of circle O' is $\frac{1}{4}$ of the area of circle O. Therefore, this probability is $\frac{1}{4}$.

Solution 4. Suppose that a line segment passes through the midpoint of the opposite side of the vertex A of an equilateral triangle inscribed in a circle and that this line segment meets the circle at point B, as shown in **Figure 4**. Draw a tangent line l that passes through point B. Suppose that an extension of chord AP having A as one endpoint meets with tangent line l at the point P′. Each chord AP has a one-to-one correspondence with each point P′ on tangent line l. In other words, the statement "draw chord AP from vertex A" is equivalent with "select point P′ on tangent line l". Therefore, suppose that the set of all points defined by the intersections of the extensions of all chords drawn from vertex A of the triangle with tangent line l is the sample space of a probability space. The chords contained in this probability space are determined by the locations of points P′s.

Suppose an equilateral triangle ACD inscribed in a circle and that the contact points of the extensions of the two chords AC and AD with tangent line l are C′ and D′, respectively. When the contact point of the extension of a chord with tangent line l lies on line segment C′D′, the length of the chord is greater than the side of the equilateral triangle. In this case, however, the length of the line of the sample space is infinite; thus, if its probability is defined by the length, this leads to a contradiction. Therefore, it is impossible to solve the problem under this condition.

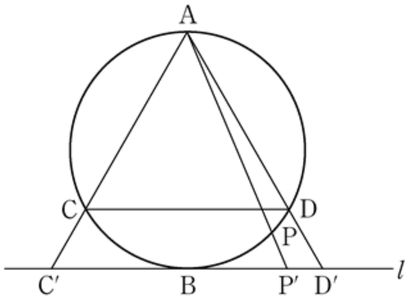


Figure 4. A point where the extension of a chord fixed at a point on a circle meets with a tangent line

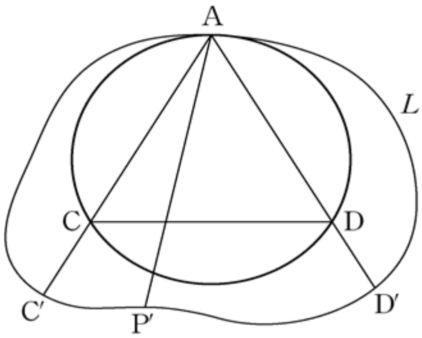


Figure 5. A point where an extended line of a chord fixed at a point on a circle meets with a closed curve

Solution 5. Suppose that a closed curve with length L is circumscribed in a circle and touches the circle at the vertex A of an equilateral triangle, as shown in **Figure 5**. As the extension of a chord has a one-to-one correspondence with the contact point on the closed curve, like in Solution 4, suppose that the sample space of a probability space is the set of all points defined by the intersections of the extensions of all chords drawn from A with the closed curve. The chords included in this probability space are determined by the locations of points P′s.

Consider an equilateral triangle ABC inscribed in a circle and that the contact points of the extensions of the two chords AC and ADwith the closed curve are C′ and D′, respectively. When the contact point of the extension of a chord with the closed curve is on curve segment C′D′, the length of the chord is greater than the side of △ ABC. Suppose that the length of curve segment C′D′ is l, the length of the closed curve is L, and the length of a partial curve is l. Therefore, the probability is $\frac{l}{L}$. However, considering that the closed curve circumscribed from point Acan be randomly drawn, the lengths L and l can vary. Therefore, the answer is indefinite.

As shown by the five solutions above, the cause of confusion in Bertrand's chord problem results from the ambiguous expression 'a chord randomly drawn', which does not clarify 'in which probability space its probability distribution is provided' or 'which fundamental events occur under the same condition'. In other words, this shows the importance of clarifying the premise of a sample space under the same condition in Bertrand's chord problem. Against this backdrop, we discussed below how well college students clarify each sample space in five modified Bertrand's chord questions asking a single answer.

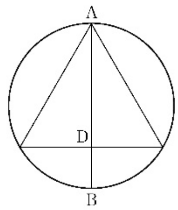
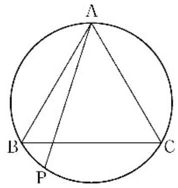
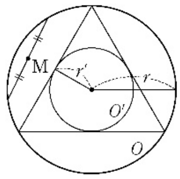
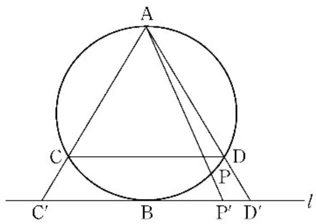
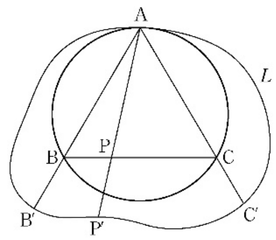
METHODS

Testing Tool

We modified Bertrand's chord problem to clearly show five fundamental events that have the same possibility to occur; in other words, to clearly find out the sample space which was intended in each question. Using the testing tools shown in **Table 1**, this study investigated how well the college students could correctly determine the sample space and the probability that satisfies the conditions of individual questions. We provided appropriate figure in each modified problem to help students find out all chords which were intended as the elements of the sample space.

The sample spaces designed in five modified questions in Table are as follows. In question No. 1, the sample space is defined as the set of all chords perpendicular to a fixed diameter of a circle. In question No. 2, the sample space is defined as the set of all chords drawn from one fixed vertex of a triangle. In question No. 3, the sample space is defined as a set collecting all midpoints Ms of randomly drawn chords. In question No. 4, the sample space

Table 1. Testing questions

Questions	
1. Suppose that a line segment passes through the midpoint D of the opposite side of the vertex A of an equilateral triangle inscribedin a circle and that the line segment meets the circle at point B, as shown in the figure on the right. What is the probability that the length of a chord whose extension is drawn perpendicularly to diameter AB is longer than the side of the triangle?	
2. Suppose a chord that defined by a random point P on the circumference of a circle and by the vertex A of an equilateral triangle inscribed in the circle, as shown in the figure. What is the probability that the length of a chord is greater than the side of the triangle inscribed in the circle?	
3. Suppose an equilateral triangle inscribed in circle O and another circle O' inscribed in the triangle. What is the probability that the midpoint M of a random chord of circle O, as shown in the figure, is located inside circle O'?	
4. Consider an equilateral triangle ACD inscribed in a circle, as shown in the figure, and the tangent line l that is parallel to side CD. Suppose that the contact points of sides AC and AD with tangent line l are C′ and D′, respectively. If a chord AP starts from vertex A and passes through a random point P on the circumference of the circle, what is the probability that the contact point P′ of line AP with tangent line l is located on line segment C′D′?	
5. Consider an equilateral triangle ACD inscribed in a circle, as shown in the figure, and a random closed curve L circumscribed in a circle that contacts the circle at vertex A. Suppose that the contact points of lines AC and AD with curve L are C′ and D′, respectively. If line AP starts from vertex A and passes through a random point P on the circumference of the circle, what is the probability that the contact point P′ of line AP with the closed curve is located on curve segment C′D′?	

is presented as the set of all intersection points where the extensions of all chords drawn from vertex A of the triangle meet with tangent line l. In question No. 5, the sample space is the set of all intersection points where the extensions of all chords drawn from the fixed vertex A of a triangle meet with the closed curve.

Participants

An offline survey using the above testing tools was planned to be conducted for the 3rd- and 4th-year students who major mathematics or mathematics education at the J national university. We recruited 68 volunteered students to participate in this survey using two department bulletin boards during one week. Table 2 shows the information of the survey participants, including their major subject, grade, and gender. Among the participants, the numbers of the 3rd- and 4th-year college students were 45 and 23, respectively. Three of the 68 participants responded that they had studied Bertrand's chord problem before and the remaining 65 students that they had not studied it yet.

Table 2. Participants

Major	Year		Gender		Total
	3rd	4th	Male	Female	
Mathematics	28	23	28	23	51
Mathematics Education	17	0	7	10	17
Total	45	23	35	33	68

Table 3. Distribution of responses to modified Question No. 1

Answers	1/2	1/3	1/4	Other incorrect answers	Non Response	Total
Frequency	49	4	4	7	4	68
Percentage	72	6	6	10	6	100

Data Collection and Analysis

The survey was conducted in their department lecture rooms in October 2014. The survey date depended on grades and major. The participants were supplied with the survey questionnaire and were instructed to solve the questions in the given order and not to change their solution process and answers; this method was intended to clarify how well they approach individual questions. The total response time for the five questions was unlimited to ensure that the students could provide their answers without restrictions.

To analyze how well the students responded on the five modified Bertrand's chord questions, we input their all answers in Microsoft Office Excel and calculated the frequency and the response percentage on each answer. We also qualitatively analyzed their incorrect answers to interpret how they approached to each question and to infer whether they had errors in selecting a sample space or in the concept of probability itself. We presented the students' wrong solutions as real examples.

RESULTS AND DISCUSSION

Responses to Modified Question No. 1

Question No. 1 was modified to define a set containing all chords perpendicular to a fixed diameter of a circle as the sample space. The distribution of the students' responses was as shown in **Table 3**. Approximately 72% of the surveyed students answered that the probability was $\frac{1}{2}$. In addition, two of the three students who responded that they had studied Bertrand's chord problem before answered "$\frac{1}{2}$" and the rest did not respond to Question No. 1. This indicated that most of the surveyed students could solve the probability question without much confusion when the sample space was defined as the set of all chords perpendicular to a fixed diameter of the circle.

Because all incorrect answers, except one incorrect answer of 0, were presented without any information on the solution process, it was difficult to determine the causes of errors. We just could infer that they would have any difficulty to solve some geometrical probability problems or to understand some mathematical concepts involved in this question. The student who answered "0" considered a right-angled triangle inscribed in the circle instead of an equilateral triangle, owing to a lack of understanding of the question.

Modified Question No. 2

Question No. 2 was modified to define the sample space as a set of all chords drawn from one vertex of a triangle. The participants' response distribution was as shown in **Table 4**. Approximately 72% of the surveyed students answered that the probability was $\frac{1}{3}$. In addition, all three students who had studied Bertrand's chord problem before answered "$\frac{1}{3}$". Among the 49 students who answered correctly to Question No. 1, 40 students answered correctly again. In other words, approximately 59% of the total surveyed students answered correctly both to Questions Nos. 1 and 2. This indicated that the surveyed students could solve the probability question without much confusion when its sample space was defined as the set of all chords drawn from one vertex of the triangle.

Table 4. Distribution of responses to modified Question No. 2

Answers	$1/2$	$1/3$	$1/4$	Other incorrect answers	Non Response	Total
Frequency	3	49	2	9	5	68
Percentage	5	72	3	13	7	100

Table 5. The case of a student who answered correctly to Question No. 1 and incorrectly to Question No. 2

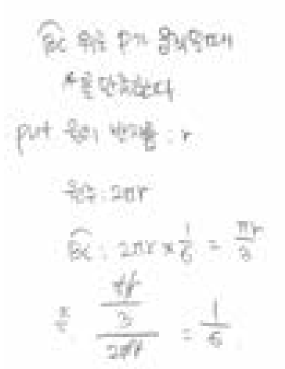
Correct answer to No.1		Incorrect answer to No.2	
	$\overline{\mathrm{AB}} = 1$ $\overline{\mathrm{CD}} = \frac{1}{2}$ When P moves on $\overline{\mathrm{CD}}$, * is satisfied. Therefore, the correct answer is $\frac{1}{2}$		When P is moving on arc BC, * is satisfied. Put radius: r, the circumference: $2\pi r$. Arc BC: $2\pi r \times \frac{1}{6} = \frac{\pi r}{3}$ $\frac{\frac{\pi r}{3}}{2\pi r} = \frac{1}{6}$

Table 6. Distribution of responses to modified Question No. 3

Answers	$1/2$	$1/3$	$1/4$	Other incorrect answers	Non Response	Total
Frequency	30	7	13	13	5	68
Percentage	44	10	19	19	7	100

To determine the causes of the incorrect answers to Question No. 2 of those who answered correctly to Question No. 1, their processes of solving the two questions were reviewed. **Table 5** shows the answers of one student who answered correctly to Question No. 1 but incorrectly to Question No. 2. This shows that some students made an error in calculating the length of the partial curve that meets the conditions, not in selecting a sample space or one related to the concept of probability.

Modified Question No. 3

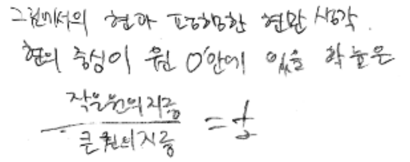
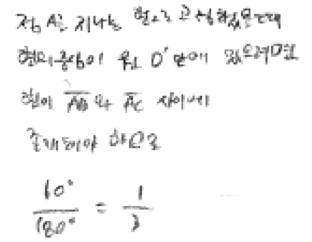
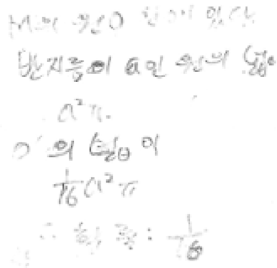
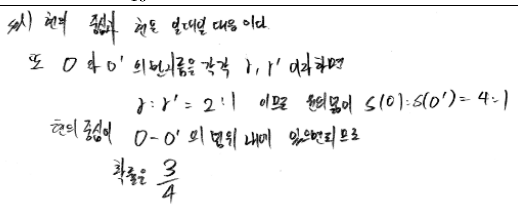
Question No. 3 was modified to define a set containing the midpoints of all random chords as the sample space; the response distribution is shown in **Table 6**. The number of students who correctly answered "$\frac{1}{4}$" to Question No. 3 was a mere 19% of the total sample. In addition, only one of the three students who had studied Bertrand's chord problem before answered "$\frac{1}{4}$" and the other two answered "$\frac{1}{2}$" and "$\frac{1}{3}$". In particular, only seven of the 13 participants who answered correctly to Question No. 3 also answered correctly both to Questions No. 1 and No. 2. This indicates that more than 80% of the surveyed students calculated the probability in Question No. 3 incorrectly.

Unlike the two previous questions, Question No. 3 showed a relatively high incorrect answer rate. To determine the cause of this observation, some notable incorrect cases were reviewed, as shown in **Table 7**.

The cases (A) and (B) were the most common as participants' answers of this question. It was possible to assume that most of the students who incorrectly answered had a difficulty in finding the sample space of this question. These students revealed the error of recognizing only a part of the sample space intended in this question because they did not consider all random chords and limited it to the set of some chords. In other words, these students could understand the concept of probability but had difficulty to clarify the sample space satisfying the condition. From this result, we could reconfirm the reason why many pre-service and in-service mathematics teachers had difficulty to focus on the position of midpoint M which uniquely determines the chord among original three solutions of Bertrand's chord problem. It can be inferred that they had difficulty to seek out the set to all midpoints because of not considering all random chords.

In the cases (C) and (D), only one student answered incorrectly, as shown in the **Table 7**; the student succeeded in defining the sample space but made errors in calculating probability and in determining the figure of the event.

Table 7. Cases of incorrect answers to Question No. 3

	Cases	Error analysis
(A)	 Consider chords that are parallel with the chord shown in the figure. The probability that the midpoints of these chords are located within circle O': $\frac{\text{diameter of the small circle}}{\text{diameter of the large circle}} = \frac{1}{2}$	It shows error in determining the sample space and event. The participant failed to consider random chords and defined a set of the midpoints of chords that are parallel with the chord shown in the figure as the sample space.
(B)	 When defining chords that pass through point A, chords should be located between $\overline{\mathrm{AB}}$ and $\overline{\mathrm{AC}}$ to have their midpoints located within circle O'. Therefore, $\frac{60°}{180°} = \frac{1}{3}$	It shows error in determining the sample space and events like the case (A). The participant failed to consider random chords and defined the sample space as a set of the midpoints of the chords that had one endpoint fixed at one vertex of the triangle.
(C)	 M is located within circle O'. The area of a circle with radius $a$is $a\pi^2$. The area of circle O' is $\frac{1}{16}a^2\pi$. ∴ Probability: $\frac{1}{16}$	Participant successfully determined the sample space and event but made an error in the process of calculating the area of the circle inscribed in the triangle.
(D)	 The midpoints of the chords correspond one-to-one with the chords. If the radius of circles O and O' are r, r', respectively, then $r:r' = 2:1$ Thus, the areas of the circles are $S(O):S(O') = 4:1$. The probability that the midpoints of the chords are located within the $O - O'$ range is $\frac{3}{4}$	The participant succeeded in defining the sample space but made an error in the process of calculating the figure of the event.

Considering these results, we conclude that most of the students who answered incorrectly had difficulties in defining the sample space and event in this question, which is essential to calculate geometrical probability.

Modified Question No. 4

Regarding modified Question No. 4, it is impossible to define the probability owing to the infinite length of a line of the sample space. The response distribution of the surveyed students was as shown in **Table 8**. Approximately 74% of the participants answered "$\frac{1}{3}$" incorrectly. No student answered that it was impossible to

Table 8. Distribution of responses to modified Question No. 4

Answers	1/2	1/3	1/4	Other incorrect answers	Non Response	Total
Frequency	1	50	3	7	7	68
Percentage	2	74	4	10	10	100

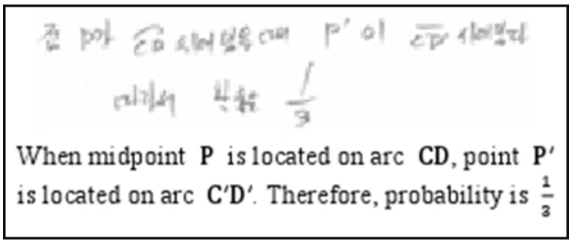


Figure 6. Case of incorrect answer Question No. 4

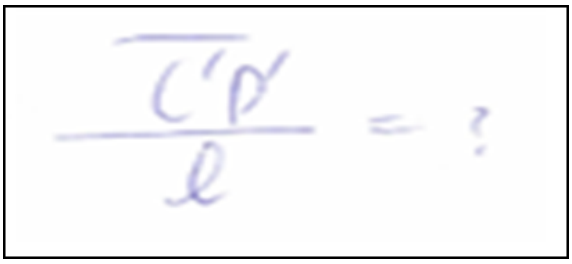


Figure 7. Case of correctly finding the sample space Question No. 4

Table 9. Distribution of responses to modified Question No. 5

Answers	1/3	Similar answers	Other incorrect answers	Non Response	Total
Frequency	45	1	10	12	68
Percentage	66	2	14	18	100

define a sample space satisfying the axiom of probability in this question. This result seems to be attributable to the fact that cases with infinite length and the area of a sample space had not been addressed in the process of teaching the concept of probability.

The case shown in **Figure 6** indicates the reason why most of the surveyed students answered "$\frac{1}{3}$" to this question. This case shows that most of participants failed to recognize the set of the contact points P′ between lines AP and tangent lines l as the sample space of this problem. In other words, they made an error in recognizing the set of the endpoints of all the chords drawn from the vertex A of the triangle as the sample space, like in modified Question No. 2.

However, one student determined the sample space and event correctly, as shown in **Figure 7**. The student understood that the sample space in this question was the line segment C′D′ on the tangent line but failed to calculate the probability because the length of the line starting from vertex A was ∞ and the length of the line segment was finite.

Modified Question No.5

In terms of modified Question No. 5, the probability is indefinite because the closed curve, the sample space in this case, is circumscribed at point A and it can be randomly drawn. The response distribution of the surveyed students was as shown in **Table 9**. Approximately 66% of the participants answered "$\frac{1}{3}$" incorrectly and most of them failed to correctly determine the sample space and event, like in modified Question No. 4. Unlike in Question No. 4, some students correctly described the probability as a generalized ratio of the length of the figure of the sample space to the figure of the event.

The reason why most of the surveyed students answered "$\frac{1}{3}$" to this question can be assumed from the case shown in **Figure 8**. This case shows that most of the participants failed to recognize the set of the contact points P′s between lines AP and the closed curve as the sample space in this case, like in Question No. 4. In other words, they

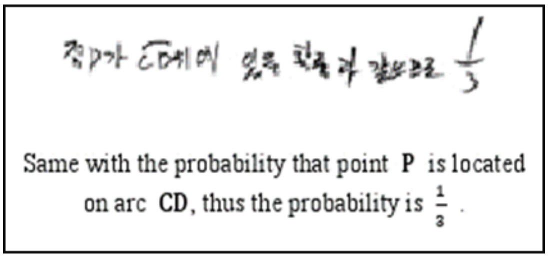


Figure 8. Case of incorrect answer Question No. 5

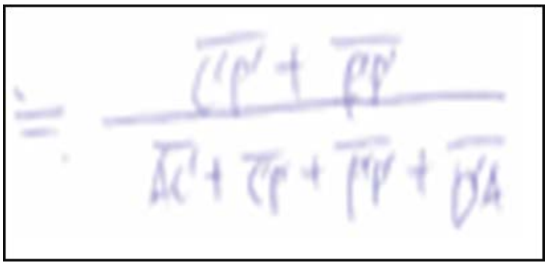


Figure 9. Case of correctly finding the sample space Question No. 5

made the same error as in Question No. 4. However, some students determined the sample space and event correctly and generalized the probability, as shown in **Figure 9**.

In **Figure 9**, the student correctly understood that the sample space was a random closed curve and that the event was the union of partial curves of this closed curve. This particular student also correctly determined the sample space in Question No. 4 and had not studied Bertrand's chord problem before. These results show that most of the students had difficulties in seeking out the sample space which was intended in the question, which is essential for defining the probability in some cases.

CONCLUSION AND IMPLICATIONS

The concept of probability has a unique characteristic that causes confusion when no clear definition is provided for the sample space, which is a universal set of elements with the same chance to occur. We can identify this inherent nature of probability via Bertrand's chord problem, which is known as a probability paradox. This problem has three solutions because it doesn't clarify sample space which is intended. **So, we modified Bertrand's chord problem to have the single probability, and we examined how** college students studying mathematics and mathematics education seek out the sample spaces represented in five modified Bertrand's chord questions. Additionally the study highlighted the importance of clarifying the sample space in defining the probability space. From the results of the survey, the following implications for probability education were obtained.

First, in some modified Bertrand's chord questions (No. 1 & No. 2) in which the sample space is clearly defined, approximately 70% of the surveyed college students correctly determined the sample space and event and calculated its probability. These results imply that many students can get a single probability by clarifying the sample spaces in some questions without any confusion. We suggest that it is necessary to clearly expose the sample space in probability questions unless we does not use Bertrand's chord problem as an open-ended question that can generate multiple answers with various approaches.

Second, in questions with clearly defined sample spaces (No. 3, No. 4, & No. 5), approximately 80% of the surveyed students failed to determine the sample spaces and events correctly. This means that many students still have difficulty in determining all possible outcomes that could occur in some probabilistic tasks even though they understand the concept of probability. It is in agreement with previous studies which suggested that many students have difficulty in seeking out the sample space (e.g., Borovcnik & Bentz, 1991; Borovcnik, & Kapadia, 2009; Konold, 1989; Speiser & Walter, 1998). Probability causes many controversies every moment when students accept it (Freudenthal, 1973) and students' probabilistic thinking improves by education (Fischbein, 1975; Fischbein & Schnarch, 1997; Kwon, Kim, & Lee, 2014; Rubel, 2007). With these reports, the result of this study indicates that the importance of the sample space should be highlighted in probability education and that the process of clearly recognizing sample spaces should be introduced. Under the current secondary school curriculum, probability questions are limited only to sample spaces with finite numbers of elements; however, it is necessary to highlight the importance of clearly recognizing sample spaces when teaching the concept of probability. At this stage, it is recommended to use Simpson's paradox (Borovcnik et al., 1991) when considering sample spaces with finite numbers of elements.

Finally, we are looking forward that this study helps mathematics teachers and students to understand Bertrand's chord problem and its solutions in terms of sample space. In particular, we suggest that the modified question No. 4 and No. 5 may be used to help a deeper understanding of the probability and its sample space because most of students had difficulties to clarify the sample spaces and to calculate probability in these questions. Also we agreed with the conclusion of Aerts and Sassoli de Bianchi (2004) that Bertrand's chord problem is not anymore a paradox.

REFERENCES

Aerts, D., & Sassoli de Bianchi, M. (2014). Solving the hard problem of Bertrand's paradox. *Journal of Mathematical Physics, 55*(8), 1-15. https://doi.org/10.1063/1.4890291

Agus, M., Peró-Cebollero, M., Penna, M. P., & Guàrdia-Olmos, J. (2015). Comparing psychology undergraduates' performance in probabilistic reasoning under verbal-numerical and graphical-pictorial problem presentation format: what is the role of individual and contextual dimensions? *Eurasia Journal of Mathematics, Science, & Technology Education, 11*(4), 735-750. https://doi.org/10.12973/Eurasia.2015.1382a

Alexander, R., & Kelly, B. (1999). *Mathematics 12: Western Canadian edition*. Don Mills, Canada: Addison-Wesley.

Batanero, C., & Borovcnik, M. (2016). *Statistics and probability in high school*. Rotterdam, The Netherlands: Sense Publishers.

Batanero, C., Henry, M., & Parzysz, B. (2005). The nature of chance and probability. In G. A. Jones (Ed.), *Exploring probability in schools: challenges for teaching and learning* (pp. 15-37). New York, NY: Springer.

Bertrand, J. (1889). *Calcul des probabilities* [Calculation of probability]. Paris, France: Gauthier-Villars.

Borovcnik, M., & Bentz, H. J. (1991). Sample space and symmetry view. In R. Kapadia & M. Borovcnik (Eds.), *Chance encounter: Probability in education* (pp. 77-82). Dordecht, The Netherlands: Kluwer.

Borovcnik, M., & Kapadia, R. (2009). Research and developments in probability education. *International Electronic Journal of Mathematics Education, 4*(3), 111-129. https://doi.org/10.1007/978-3-319-31625-3_1

Borovcnik, M., Bentz, H. J., & Kapadia, R. (1991). A probabilistic perspective. In R. Kapadia & M. Borovcnik (Eds.), *Chance encounter: Probability in education* (pp.50-62), Dordecht, The Netherlands: Kluwer.

Chernoff, E. J. (2009). Sample space partitions: An investigative lens. *The Journal of Mathematical Behavior, 28*(1), 19-29. https://doi.org/10.1016/j.jmathb.2009.03.002

Chernoff, E. J., & Zazkis, R. (2011). From personal to conventional probabilities: From sample set to sample space. *Educational Studies in Mathematics, 77*(1), 15-33. https://doi.org/10.1007/s10649-010-9288-8

Choi, J., Yun, Y., & Hwang, H. (2014). A study on pre-service teachers' understanding of random variable. *Journal of Korea Society Educational Studies in Mathematics School Mathematics, 16*(1), 19-37.

Drory, A. (2015). Failure and uses of Jaynes' principle of transformation groups. *Foundations of Physics, 45*(4), 439-460. https://doi.org/10.1007/s10701-015-9876-7

Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children*. Dordrecht, The Netherlands: Reidel.

Fischbein, E., & Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions? *Educational Studies in Mathematics, 15*(1), 1-24. https://doi.org/10.1007/BF00380436

Fischbein, E., & Schnarch, D. (1997). The evolution with age of probabilistic, intuitively based misconceptions. *Journal for Research in Mathematics Education, 28*(1), 96-105. https://doi.org/10.2307/749665

Freudenthal, H. (1973). *Mathematics as an educational task*, Dordrecht, Holland: Reidel.

Garfield, J., & Ahlgren, A. (1988). Difficulties in learning basic concepts in probability and statistics: implications for research. *Journal for Research in Mathematics Education, 19*(1), 44-63. https://doi.org/10.2307/749110

Gauvrit, N., & Morsanyi, K. (2014). The equiprobability bias from a mathematical and psychological perspective. *Advances in Cognitive Psychology, 10*(4), 119-130. https://doi.org/10.5709/acp-0163-9

Gillies, D. (2000). *Philosophical theories of probability*. New York, NY: Routledge.

Green, D. R. (1983). A survey of probability concepts in 3000 pupils aged 11-16 years. In D. R. Grey, P. Holmes, V. Barnett, & G. M. Constable (Eds.), *Proceeding of the first international conference on teaching statistics* (pp. 766-783). Sheffield, UK: Teaching Statistics Trust.

Gyenis, Z., & Rédei, M. (2015). Defusing Bertrand's paradox. *The British Journal for the Philosophy of Science, 66*(2), 349-373. https://doi.org/10.1093/bjps/axt036

Hawkins, A., & Kapadia, R. (1984). Children's conceptions of probability: a psychological and pedagogical review. *Educational Studies in Mathematics, 15*(4), 349-377. https://doi.org/10.1007/BF00311112

Jaynes, E. T. (1973). The well-posed problem. *Foundations Physics, 3*(4), 477-492. https://doi.org/10.1007/BF00709116

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realties. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 909-955). New York, NY: Macmillan.

Jones, G., Langrall, C. W., Thornton, C., & Mogill, A. (1997). A framework for assessing and nurturing young children's thinking in probability. *Educational Studies in Mathematics, 32*(2), 101-125. https://doi.org/10.1023/A:1002981520728

Jones, G., Langrall, C. W., Thornton, C., & Mogill, A. (1999). Students' probabilistic thinking in instruction. *Journal for Research in Mathematics Education, 30*(5), 487-519. https://doi.org/10.2307/749771

Kim, J. H. (2008). *Probability Theory for insurance mathematics.* Seoul, Korea: Kyo Woo Sa.

Klyve, D. (2013). In defense of Bertrand: The non-restrictiveness of reasoning by example. *Philosophia Mathematics, 21*(3), 365-370. https://doi.org/10.1093/philmat/nkt028

Konold, C. (1989). Informal conceptions of probability. *Cognition and Instruction, 6*(1), 59-98. https://doi.org/10.1207/s1532690xci0601_3

Kwon, J. G., & Lee, B. (2015). Probabilistic misconceptions of elementary pre-service teachers. *Advanced Studies in Contemporary Mathematics, 25*(1), 179-188.

Kwon, J. G., Kim, H. J., & Lee, B. (2014). The differences of probabilistic thinking between ordinary students and students talented in mathematics and science across grade levels. *Proceedings of the Jangjeon Mathematical Society, 17*(2), 259-271.

Lee, K. H. (1997). A Study on the Probability. *The Science Education Research Collection of Cheongju National University of Education, 18*, 21-35.

Marinoff, L. (1994). A resolution of Bertrand's paradox. *Philosophy of Science, 61*(1), 1-24. https://doi.org/10.1086/289777

Porto, P. D., Crosignant, B., Ciattoni, A., & Liu, H. C. (2011). Bertrand's paradox: A physical way out along the lines of buffon's needle throwing experiment. *European Journal of Physics, 32*(3), 819-826. https://doi.org/10.1088/0143-0807/32/3/017

Rowbottom, D. P. (2013). Bertrand's paradox revisited: Why Bertrand's solutions' are all inapplicable. *Philosophia Mathematics, 21*(1), 110-114. https://doi.org/10.1093/phimat/nks028

Rubel, L. H. (2007). Middle school and high school students' probabilistic reasoning on coin tasks. *Journal for Research in Mathematics Education, 38*(5), 531-556. http://www.jstor.org/stable/30034964

Shaughnessy, J. M. (1977). Misconceptions of probability: an experiment with a small-group, activity-based, model building approach to introductory probability at the college level. *Educational Studies in Mathematics, 8*(3), 295-316. https://doi.org/10.1007/BF00385927

Shaughnessy, J. M. (1998). Research in probability and statistics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465-494). New York, NY: Macmillan.

Speiser, B., & Walter, C. (1998). Two dice, two sample spaces. In L. Pereira-Mendoza (Ed.), *Proceedings of the fifth international conference on teaching statistics* (pp. 61-66). Singapore: International Statistical Institute.

Woo, J. H. (1998). *Educational basis for school mathematics.* Seoul, Korea: Seoul National University Press.