

## How do prospective primary teachers approach area conservation? An analysis based on intra-mathematical connections

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### Abstract

This study examines the intra-mathematical connections evidenced in groups of prospective primary school teachers' written productions when solving and justifying an area-conservation task comprising an initial task and follow-up questions. A mixed-methods design combined qualitative content analysis with exploratory co-occurrence and clustering analyses. Procedure and justification were among the most frequent connections identified, whereas equivalent procedure and alternate representation appeared less frequently, particularly in the follow-up questions. A cluster map and cross-specification sensitivity analysis showed that some similarities among groups recurred across retained alternative specifications, whereas broader grouping boundaries were more sensitive to methodological choices. Overall, the findings indicate that area conservation was expressed through different combinations of mathematical objects and connection types, highlighting the complexity of justification and the potential of such tasks for teacher education.

**Keywords:** intra-mathematical connections, area conservation, prospective primary school teachers, mixed methods, initial teacher education

## INTRODUCTION

Area is not only a measurable quantity but also a mathematical object whose meaning depends on being compared, transformed, and justified (Freudenthal, 1983). However, school teaching often reduces it to the direct application of formulas, so that "finding area" becomes a matter of selecting an expression and substituting values, with limited attention to the geometric conditions that justify its use or to the meaning of the quantities involved (Ekawati et al., 2019; Huang, 2017; Runnalls & Hong, 2020; Zacharos, 2006). This procedural orientation may be functional in prototypical configurations, but it weakens when figures are non-standard, differently oriented, or lack explicit measurements, situations in which geometric reasoning should prevail over numerical substitution (Caviedes et al., 2023a; Lehmann, 2023).

From this perspective, decomposition and recomposition are especially important because they

make it possible to justify the equivalence of surfaces by transforming one figure into another without overlaps or gaps, thereby giving geometric meaning to formulas rather than treating them as isolated rules (Caviedes et al., 2024; Sari & Ng, 2022). Such reasoning is central to area conservation, understood as the invariance of area under changes in the spatial arrangement of a figure's parts, and has been recognized as crucial for understanding the geometric meaning of area beyond mechanical procedures (Kospentaris et al., 2011; Piaget et al., 1981; Sari et al., 2023). In this sense, understanding that a parallelogram can be transformed into a rectangle without altering its area provides a geometric justification for the corresponding formula, and similar reasoning can be extended to other figures such as triangles and trapezoids (Caviedes et al., 2023a, 2024; D'Amore & Fandiño Pinilla, 2007).

This geometric justification, however, does not arise automatically from carrying out the transformation, it requires students to organize the figures involved in terms of parts, wholes, and spatial relations, a process

### Contribution to the literature

- This study contributes to current research on area and area conservation by examining how groups of prospective primary teachers (PPTs) articulate intra-mathematical connections when solving and justifying area conservation tasks.
- Whereas previous work has mainly documented difficulties associated with area conservation (e.g., Hong & Runnalls, 2020; Kospentaris et al., 2011; Zacharos, 2006), this study approaches the issue through a relational lens, combining qualitative content analysis with exploratory co-occurrence analysis to explore both the connections identified and their articulation.
- In doing so, it offers a relational perspective on area conservation in which justification (J) connections appears to be associated with the coordination of procedures and representations, rather than with the isolated presence of any one of these elements.

referred to in the literature as spatial structuring (Battista et al., 1998). Recognizing invariance depends on seeing a transformed figure not merely as a different visual configuration, but as a reorganization of the same surface. Thus, difficulties with area conservation may arise not only from limited procedural knowledge, but also from insufficient coordination between the spatial organization of the figure and the measurement relationships that justify area equivalence (Battista, 1999; Battista et al., 1998; Caviedes et al., 2023a; Lehmann, 2023).

Research has consistently shown that area conservation is difficult for both students and PPTs. Learners often fail to recognize invariance when figures are transformed, rely on perceptual features or associated linear quantities such as side length or perimeter, and struggle to justify area equivalence through geometric relationships (Kamii & Kysh, 2006; Kordaki, 2003; Kospentaris et al., 2011; Caviedes et al., 2023a). Similar difficulties are reported for PPTs, who often depend on prototypical figures and have trouble interpreting irregular or unfamiliar configurations, especially when explicit measurements are not provided (Caviedes et al., 2022; Hong & Runnalls, 2020; Murphy, 2012). These findings suggest that the difficulty is not merely procedural, but also concerns the coordination of representations, transformations, and validity criteria when reasoning about area.

In light of this, it is important to examine not only whether PPTs recognize area conservation, but also how they coordinate the mathematical objects that give this property meaning when solving and justifying such tasks. Although previous research has documented difficulties associated with area, area conservation, and non-prototypical figures, less is known about how PPTs relate procedures, representations, properties, and justifications within the same mathematical activity. Because the coordination of these mathematical objects is inherently a relational phenomenon, it can be productively approached through the lens of intra-mathematical connections (Caviedes et al., 2023b; de Gamboa et al., 2024). Accordingly, the present study has a twofold aim: first, to characterize the intra-

mathematical connections established by PPTs when addressing area conservation tasks; and second, to identify the way these connections relate to one another.

## THEORETICAL FRAMEWORK

### Mathematical Connections

The notion of mathematical connection has gained increasing prominence in mathematics education research because of its potential to describe how students organize, coordinate, and make sense of different mathematical objects when solving tasks (Businskas, 2008; García-García & Dolores-Flores, 2018). Although the construct has been developed from diverse perspectives, there is broad agreement in understanding connections as a phenomenon of a cognitive nature, since they arise when the learner establishes meaningful relationships among mathematical ideas, concepts, procedures, representations, or propositions (Businskas, 2008; Dolores-Flores & García-García, 2017; Evitts, 2004; García-García & Dolores-Flores, 2018, 2021). This conception places connections at the core of mathematical understanding, insofar as they make it possible to explain not only what a student knows, but also how that knowledge is articulated when interpreting, solving, and justifying a task.

A particularly relevant distinction in the literature is that between intra-mathematical connections and extra-mathematical connections (de Gamboa et al., 2023). The former are established among mathematical objects belonging to the disciplinary domain itself, whereas the latter link mathematics to contexts external to it, such as phenomena from everyday life or other disciplines. In this study, the focus is placed on intra-mathematical connections. Within these, de Gamboa et al. (2023) distinguish between conceptual connections and process-related connections. The former refers to relationships among mathematical objects associated with the same concept, property, or problem; the latter emphasize transversal dimensions of mathematical activity, such as heuristic exploration or decision-making during problem-solving. Here, conceptual connections are of particular interest, since the aim is to

explore how PPTs coordinate the mathematical objects involved around a specific geometric idea: area conservation. In this context, it is not enough to identify whether a group of PPTs obtains a correct answer; it is necessary to analyze what relationships the group establishes in order to show, for example, that two non-congruent figures may nevertheless have the same area, or that a transformation through cutting and recomposition preserves surface area, even though it alters the overall appearance of the figure.

Different authors have developed classifications of mathematical connections which, although they differ in denomination and scope, show an implicit consensus around a set of recurring types (Businskas, 2008; Eli et al., 2011; García-García & Dolores-Flores, 2018; Hatisaru, 2023; Rodríguez-Nieto et al., 2023). In this study, we adopt an analytical framework developed inductively from the observation of mathematics classrooms (de Gamboa et al., 2023), which incorporates the connection types showing the greatest convergence in the literature and organizes them according to the emphasis they place on different aspects of the mathematical object. Within conceptual intra-mathematical connections, a distinction is made among connections that emphasize representations, connections that emphasize procedures, and J connections.

The equivalent representation (ER) connection is established between two representations of the same mathematical object within the same register (Businskas, 2008; García-García & Dolores-Flores, 2018). In area conservation tasks, this connection appears when a group recognizes that two different graphical configurations represent the same surface area; for example, a parallelogram and the rectangle obtained by cutting and recomposing it.

The alternate representation (AR) connection links two representations of the same mathematical object in different registers, in Duval's (2006) sense. In this study, it is manifested when a group translates a geometric recomposition into a symbolic or verbal register, making explicit the relationship between the figural transformation and the invariance of area.

The procedure (P) connection relates a concept to a procedure associated with it (Dolores-Flores & García-García, 2017). In area conservation, it is observed when a group links the invariance of area to actions such as cutting, translating, recomposing, or compensating parts of a figure.

The equivalent procedure (EP) connection is established between two different valid procedures for solving the same task (de Gamboa et al., 2023). In area conservation tasks, this connection appears when a group addresses the equivalence in area between two figures through more than one procedure, for example, through geometric recomposition and through the equality of base and height.

Finally, the J connection refers to a premise-conclusion relationship through which a mathematical claim is supported by explicit mathematical reasons (de Gamboa et al., 2023). Accordingly, the J connection is not restricted to relations between two objects of the same type but is identified whenever one mathematical object functions as the reason supporting another. In these tasks, it appears when a group explains why area is conserved; for example, by stating that two figures have the same area because they are formed by the same parts, without overlaps or gaps.

## METHODOLOGY

### Study Design

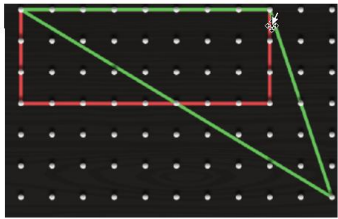
A mixed-methods convergent design was adopted (Creswell & Plano Clark, 2018; Tashakkori & Creswell, 2007) to identify the intra-mathematical connections established by PPTs when solving and justifying an area conservation task. The qualitative component made it possible to characterize the connections and mathematical objects present in the responses, whereas the quantitative component explored regularities in how those connections co-occurred. Integration of both strands is achieved through merged visualizations, in which the qualitative coding decisions and the quantitative analyses are brought together within a single representational format.

This study is additionally framed within design-based research (Cobb et al., 2003), which aims to develop mathematical and didactic knowledge for the establishment of mathematical connections. From this perspective, validity is constructed iteratively through successive cycles of design, implementation, analysis, and redesign. Accordingly, the results presented here, corresponding to the first implementation cycle, are not conclusive; rather, they contribute to the progressive refinement of the instructional proposal.

### Participants and Context

The participants were third-year PPTs enrolled in the primary education degree program at Universitat Autònoma de Barcelona, organized into 18 groups (four of three members and fourteen of four members). Three groups did not provide informed consent and were therefore excluded, leaving groups G1 and G15. From these, G6 was excluded from the analysis due to incomplete written productions, leaving 14 groups whose data are reported in the Results section, comprising 55 participants (one group of three members and thirteen groups of four members). The participants had prior instruction in isometries and the invariance of geometric properties, which provided a relevant background for the proposed task, as it required them to coordinate relationships among transformations, equivalences, and the invariance of surface area. Data

**Table 1.** Task proposed to the PPTs in session 1

| Task statement                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      | Task representation                                                                |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------|
| Part 1: The grid squares represent equal units of area. How might the area of the triangle compare to that of the rectangle? Justify your answer using two different procedures, one involving geometric transformations; one involving calculation.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |  |
| Part 2:<br>Q1. Which of your two procedures seems more visual and accessible to children? Why?<br>Q2. How could you connect both procedures in instruction to help students move from intuitive reasoning (seeing, cutting, moving) to a formal understanding of area and the formula for triangles?<br>Q3. Suppose we stretch the triangle horizontally so that its base becomes twice as long while keeping the same height.<br>(a) Predict how the area would change.<br>(b) Explain your reasoning visually (e.g., using the grid paper) and in writing.<br>(c) How could you help primary students reason about this change without relying only on the formula for area?<br>Q4. Now, what happens if we double both the base and the height?<br>(a) How will the area grow compared to the original triangle?<br>Q5. Using the above ideas, how many rectangles can you draw that have the same area as the original one? What relationship do you notice between the length of the base and the height across those figures? | (adapted from Mamona-Downs & Papadopoulos, 2006)                                   |
| Note. Numerical values were restricted to positive integers, consistent with the primary-school area-measurement context of the task                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |                                                                                    |

were collected during the 2025-2026 academic year over two class sessions, each consisting of one hour of lecture and two hours of practical work. In the first session, groups analyzed curricular connections and solved a task on area equivalence (Table 1); in the second, they analyzed students' responses and proposed instructional strategies. Participation was voluntary and exclusively for research purposes. The study was approved by the university's Ethics Committee, and the data were handled in accordance with the ethical principles applicable to educational research (British Educational Research Association, 2018).

Instrument and Data Collection

The written task consisted of two parts (Table 1). The first part required establishing and justifying area equivalence through two different procedures: one based on geometric transformation, and one based on calculation. The second part included follow-up questions that extended the mathematical activity to the comparison of procedures, their connection for teaching purposes, the effects of changes in base and height on area, and the generalization of relationships among quantities.

Analysis Procedure

The analysis was organized into two complementary phases: a qualitative phase and a quantitative phase. In the qualitative phase, a content analysis was conducted through a deductive-inductive coding process (Cohen et al., 2017), comprising two analytically distinct, though interconnected, stages. In the first stage, the mathematical objects mobilized by the groups in their

written productions were identified, drawing on the indicators proposed by Caviedes et al. (2023b) (Table 2). At this stage, justification refers to a category of mathematical object that may support a J connection once established, rather than to the connection itself, which is defined independently in the theoretical framework as a relationship between two mathematical objects (de Gamboa, 2023). The purpose of this stage was not to record every object present in the written productions, but to identify those involved in the groups' responses to the task.

In the second stage, based on this prior identification, the intra-mathematical connections were classified according to the typology proposed by de Gamboa et al. (2023). This sequence was followed because connections are manifested through the articulation among mathematical objects (de Gamboa et al., 2024), making their prior identification a necessary step. The units of analysis were the written productions generated by each group for each question.

Coding was carried out collaboratively by the four authors. The first author, who also conducted the teaching sessions, led an initial reading of the written productions. Coding decisions were then made jointly through dedicated coding meetings, in which the system of categories (Table 2) was applied to identify mathematical objects and the typology proposed by de Gamboa et al. (2023) was applied to classify the resulting connections. Instances of initial disagreement were resolved through discussion, returning to the original productions and operational definitions until consensus was reached. This consensus-based approach constitutes a recognized procedure for establishing trustworthiness

**Table 2.** System of categories and codes used to identify mathematical objects mobilized in groups' written productions

| Code | Category       | Operational definition                                 |
|------|----------------|--------------------------------------------------------|
| R2   | Representation | Use of materials or physical manipulation.             |
| R3   | Representation | Visual geometric configuration or decomposition.       |
| R4   | Representation | Use of numbers, expressions, or formulas.              |
| P1   | Procedure      | Comparison by direct superposition.                    |
| P2   | Procedure      | Cutting and rearranging parts.                         |
| P3   | Procedure      | Convenient decomposition of figures.                   |
| P4   | Procedure      | Geometric transformations.                             |
| P5   | Procedure      | Decomposition into congruent units.                    |
| P6   | Procedure      | Counting area units.                                   |
| P7   | Procedure      | Calculation using linear dimensions and formulas.      |
| J2   | Justification  | Justification through partition into equivalent areas. |
| J3   | Justification  | Conservation of area under transformation.             |
| J5   | Justification  | Justification of $b \times h$ for the rectangle.       |
| J6   | Justification  | Justification of $([b \times h]/2)$ for the triangle.  |
| Pp2  | Property       | Additivity of area.                                    |
| Pr3  | Proposition    | Congruence as a basis for equivalence.                 |
| C1   | Concept        | Structuring rows and columns                           |
| C2   | Concept        | Area as a surface quantity.                            |
| C3   | Concept        | Functional relationship between quantities.            |
| C4   | Concept        | Direct or inverse proportionality.                     |

Note. Adapted from Caviedes (2023b)

in collaborative qualitative content analysis (Hill et al., 2005; McDonald et al., 2019). Importantly, an initial version of the coding was reported in Caviedes et al. (2026), where feedback from the research community provided an additional opportunity to review and refine the coding decisions before finalizing the analysis.

Both the mathematical objects identified and the connection types inferred were coded in binary terms (presence or absence). For each group and question, this yielded a matrix recording the mathematical objects identified, the connections inferred from them, and the textual evidence supporting each coding decision. The question-level matrices were then used to construct three further matrices, corresponding to the three levels of analysis presented below. First, for the first part of the task, the question-level matrices were used to compute, across all groups, the co-occurrence between each pair of connection types. Second, for the full task, the question-level matrices were used in two ways: summed for each group into a group-level matrix recording the total frequency of each connection type, and used to compute, across all groups, the co-occurrence between each connection type and each coded mathematical object. Third, the question-level matrices from both parts of the task were summed for each group into a separate group-level matrix recording the total frequency of each connection type across the full task.

In the quantitative phase, these matrices were used to conduct a co-occurrence analysis, understood as the identification of relationships among categories that appear together within the same unit of analysis (Gómez, 2009). The data were processed using Python (v3.x). The quantitative analysis was applied at three differentiated levels. At the first level (first part of the

task), co-occurrence among connections was examined through a graph generated using the NetworkX, NumPy, and Matplotlib libraries, providing a visual exploration of the relational structure among connections in the groups' written responses to the first part of the task. At the second level the co-occurrence between connections and mathematical objects across the full task was represented through a heatmap generated using the pandas, Seaborn, and Matplotlib libraries, to identify which objects most frequently co-occurred with each connection type. Finally, at the third level, the connections identified across the full task were integrated into a cluster map. Average-linkage hierarchical clustering with correlation distance was applied using the pandas, Seaborn, and Matplotlib libraries, with the aim of exploring how connections were relatively distributed across groups' written productions.

## Qualitative Analysis

### Part 1 of the task

G7's solution (**Figure 1**) is based on a geometric reorganization of the original figure using its diagonal as an axis of symmetry. Although the group refers to this axis incorrectly as the hypotenuse, this terminological inaccuracy, while relevant, does not affect the mathematical demand of the task. By working exclusively with visual configurations (e.g., triangle, parallelogram, and rectangle) the group constructs an equivalence among figures within the same geometric register.

This coordination of different configurations representing the same surface quantity suggests the

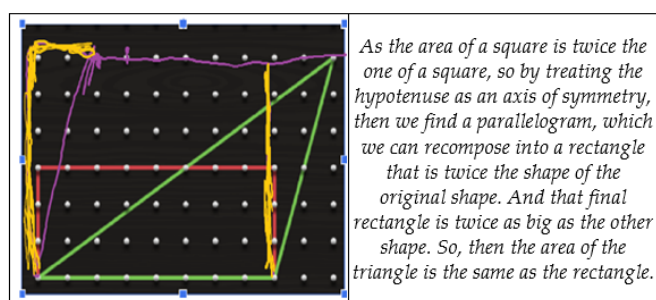


Figure 1. G7 solution process (the authors' own work)

establishment of an ER connection, since area is preserved when moving from one shape to another without leaving the visual register. This coordination may itself relate to how the group spatially structured the figure into parts that could be tracked across the transformation. At the same time, the equivalence is presented as the result of a sequence of actions performed on the figure: taking the diagonal and obtaining a parallelogram, which is then recomposed into a rectangle. Here, the group relates a geometric configuration to a transformation-and-recomposition procedure that makes the equivalence operative. This articulation between configuration and action suggests the establishment of a P connection, given that the meaning of area equivalence is constructed through the geometric procedure itself. Finally, when the group concludes that "the area of the triangle is the same as [that of] the rectangle" as a consequence of the transformation performed, it establishes an explicit relationship between a geometric premise, the recomposition into an equivalent figure, and a conclusion about area conservation. This makes it possible to infer the establishment of a J connection, since the equivalence is supported through the logical chaining of the transformations carried out. Taken together, the group's solution is grounded in objects within the geometric register and in decomposition-and-recomposition procedures, supported by the idea of area conservation under transformation.

Another group (G15) constructs its solution by decomposing the triangle into several parts, which are subsequently reorganized to complete the rectangle. By describing how one piece is translated and another rotated  $180^\circ$  to occupy the remaining region, the group

relates the idea of area equivalence to the geometric procedure of decomposing and recomposing figures through isometric transformations (Figure 2).

This coordination between a quantity and a sequence of actions that makes it possible to preserve and compare it suggests the establishment of a P connection, since the meaning of the equivalence is not stated declaratively but is instead constructed through the procedure itself. Next, when the group indicates that the recomposed pieces fit exactly into the remaining part of the rectangle, it coordinates two different geometric configurations representing the same surface area. This coordination within the same register suggests the establishment of an ER connection, since these are equivalent geometric representations of the same object. As with G7, this coordination may itself relate to how the figure's parts were spatially structured and tracked across the decomposition and recomposition performed.

Subsequently, by introducing specific measurements and relating them to the pieces obtained, the group establishes a correspondence between the geometric configuration and its numerical description. Here, the object is no longer presented solely in the geometric register, but also in the symbolic register. This makes it possible to infer the establishment of an AR connection, since representations of the same object are being coordinated across different systems. The group also shows that the region formed by recomposing the two pieces can be described using the same dimensions as the part of the rectangle that remains to be completed, so that the area is obtained both through geometric reorganization and through calculation based on these dimensions.

The coordination of these two solution paths suggests the establishment of an EP connection, since these are different procedures leading to the same result. Finally, when the group states that the dimensions "match exactly" and that, for this reason, the pieces complete the rectangle, a relationship is established between a geometric condition and a conclusion about the soundness of the result. This premise-conclusion relationship makes it possible to infer the establishment of a J connection, since the procedure is supported by the coherence between the configurations obtained and their dimensions.

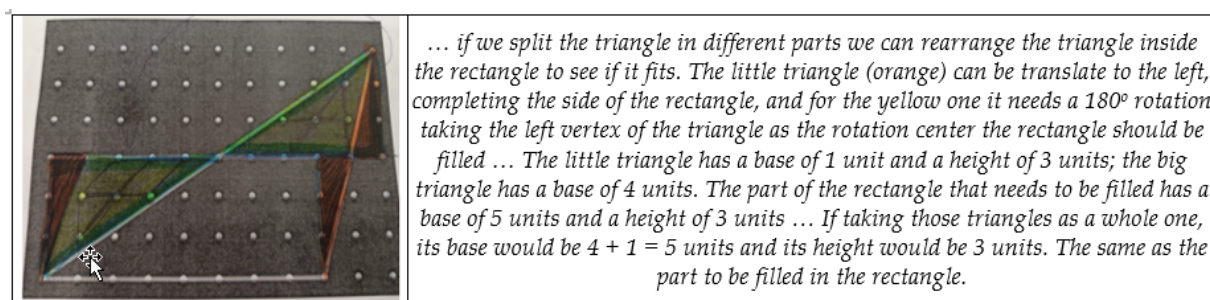


Figure 2. G15 solution process (the authors' own work)

Q1: "The geometric transformation approach (doubling, reflecting, forming a parallelogram) is more visual and accessible because children can see and manipulate shapes."  
 Q2: "We could start with hands-on activities, cutting triangles and forming rectangles, and finally introduce a visual representation of the formulas."  
 Q3: "Doubling the base doubles the parallelogram; since the triangle is half of that, the area becomes  $\times 2$ ."  
 Q3c: "Use grid paper, count squares before and after, and compare the figures."  
 Q4: "It becomes 4 times bigger, because both the base and the height are doubled ( $\times 2$  and  $\times 2 = 4$  times bigger)."  
 Q5: "... if the original rectangle has an area of  $A = \text{base} \times \text{height}$ , then any pair of numbers (b, h) that satisfies  $b \times h = A$  will create a rectangle with the same area. You can increase the base and decrease the height proportionally or vice versa, and still keep the product equal to A. Therefore, we can see an inverse relationship, as when the base increases, the height decreases so that the product stays constant."

Figure 3. G14's responses to the second part of the task (the authors' own work)

## Part 2 of the Task

Figure 3 illustrates G14's responses to the five follow-up questions in part 2 of the task (Q1 through Q5; see Table 1). Q1 and Q2 were coded as a single unit of analysis, as both questions address the same pedagogical dimension of the task, the identification, evaluation, and connection of the two procedures established in part 1, and groups' written responses to these questions constituted a continuous elaboration rather than two independent answers.

When the group identifies, in response to Q1-Q2, "doubling, reflecting, [and] forming a parallelogram" as the more visual procedure, it establishes an equivalence between different geometric configurations within the same register, which makes it possible to infer an ER connection. At the same time, this equivalence is constructed through the actions of doubling, reflecting, and recomposing, so a P connection can also be inferred, since the geometric representation is linked to the procedure that transforms it. When the group proposes "cutting triangles and forming rectangles ... [to] introduce a visual representation of the formula," it coordinates the geometric procedure with the formal procedure, suggesting the establishment of an EP connection. In addition, the shift from geometric manipulation to symbolic formulation makes it possible to infer an AR connection. At this point, the formula appears as the formalization of a geometric relationship previously observed.

In response to Q3, concerning the effect of doubling the base, the group states that "doubling the base doubles the parallelogram  $\rightarrow$  triangle is half of that  $\rightarrow$  area  $\times 2$ ." Here, a functional relationship among base, height, and area is established, making it possible to infer both a P connection and a J connection, since the result is supported through an explicit chain of premises and a conclusion. The explanation is also supported by the equivalence between the triangle and the parallelogram within the same register, reinforcing an ER connection. When the group proposes using a grid and counting squares, it again incorporates a P connection together with a visual comparison strategy.

In response to Q4, concerning the simultaneous doubling of base and height, the statement " $\times 2$  and  $\times 2 =$

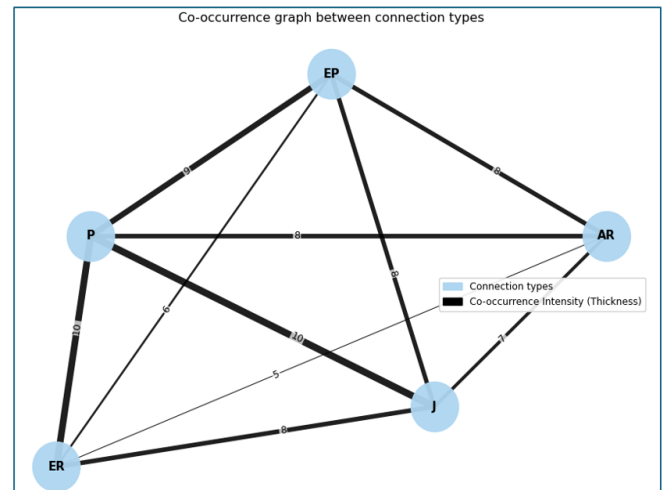


Figure 4. Co-occurrence graph of connections for the first part of the task (the authors' own work)

4 times bigger" suggests the establishment of both a P connection and a J connection, since the group relates the simultaneous variation of two linear quantities to the change in area and correctly identifies its quadratic growth. Here, the explanation is grounded in the functional relationship among base, height, and area, and in an implicit recognition of the two-dimensional nature of this quantity.

Finally, in response to Q5, concerning rectangles with equal area, the group states that any pair satisfying  $b \times h = A$  preserves area, and that base and height maintain an inverse relationship; in doing so, it establishes a P connection and a J connection by using the functional condition as the basis for equivalence. Here, the functional relationship among quantities and the inverse proportionality between base and height under a constant product are made explicit.

## RESULTS

### Part 1 of the Task: Intra-Mathematical Connections in Groups' Written Solutions

Figure 4 represents the co-occurrence relationships among the five types of intra-mathematical connections: EP, AR, ER, P, and J, identified in the groups' written productions for the first part of the task. Each node

**Table 3.** Frequency of connections established by groups in part 2 of the task

| Group | EP | AR | ER | P | J |
|-------|----|----|----|---|---|
| G1    | 0  | 0  | 0  | 5 | 5 |
| G2    | 1  | 2  | 2  | 6 | 3 |
| G3    | 1  | 1  | 3  | 6 | 4 |
| G4    | 1  | 0  | 2  | 6 | 6 |
| G5    | 1  | 0  | 2  | 6 | 6 |
| G7    | 1  | 1  | 1  | 6 | 5 |
| G8    | 1  | 1  | 2  | 6 | 5 |
| G9    | 1  | 1  | 2  | 6 | 4 |
| G10   | 1  | 1  | 0  | 3 | 3 |
| G11   | 1  | 1  | 2  | 6 | 4 |
| G12   | 1  | 1  | 2  | 6 | 5 |
| G13   | 1  | 1  | 2  | 6 | 5 |
| G14   | 1  | 1  | 2  | 6 | 3 |
| G15   | 1  | 1  | 0  | 6 | 5 |

Note. Total: EP = 13; AR = 12; ER = 22; P = 80; &, J = 63

corresponds to one connection type, and each edge indicates the number of groups in which two connection types co-occurred. Edge thickness is proportional to the frequency of that co-occurrence. Read in conjunction with the qualitative analysis presented above, the graph therefore provides a complementary view of which connections tended to appear together and how these co-occurrences were expressed in the groups' initial written solutions.

The graph shows the co-occurrence of each pair of connection types in the groups' written productions for the first part of the task. The two highest co-occurrence values in the dataset are P-J = 10 and P-ER = 10, suggesting that both justification and ER connections tend to appear alongside procedure comparatively often in part 1.

Considered together with the qualitative analysis presented above, these co-occurrences help clarify how the connections were articulated in the written productions. The high P-J value may suggest that justificatory work in these productions tended to be grounded in the actions performed on the figure rather than in abstract statements about area conservation. The equally high P-ER value indicates that within-register geometric equivalences also tend to co-occur with procedural work, suggesting that visual recognition of equivalence and procedural action were frequently articulated together. Moreover, P-EP = 9 is also comparatively high, suggesting that when groups used two different procedures, these tended to be established in close relation to each other. P-AR = 8, EP-AR = 8, and EP-J = 8 are comparable, suggesting that EPs, ARs, and justification all tended to appear alongside procedural work at similar rates.

For its part, ER-J = 8 suggests that within-register geometric equivalences also tended to co-occur with justification at a comparable frequency. AR-J = 7 is somewhat lower, suggesting that register shifts were

slightly less consistently accompanied by justification than were geometric equivalences or EPs. The two lowest co-occurrence values are AR-ER = 5 and EP-ER = 6, suggesting that ER co-occurred less consistently with AR and EP than with procedure or justification.

Overall, the graph aligned with the qualitative analysis suggest a configuration in which procedure co-occurs comparatively often with all other connection types, particularly with justification and ER, while representational connections appear more articulated with the procedural and justificatory connections than with each other. It should be noted that part 1 explicitly required two procedures and a justification, which means that the comparatively high co-occurrence values involving P, EP, and J may partly reflect task demands.

### Part 2 of the Task: Intra-Mathematical Connections Established in Groups' Written Responses

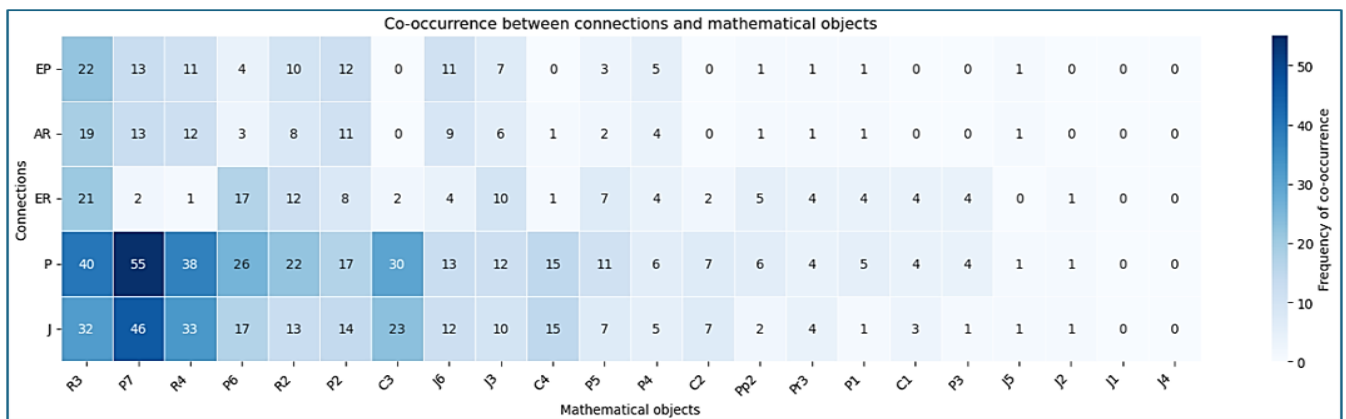
Across groups, the frequency of connections in the follow-up questions (Q1-Q5) shows a clear predominance of P, followed by J, with ER occupying an intermediate position and EP and AR appearing less frequently (Table 3).

This distribution suggests that, once the main part of the task had been solved, groups tended to continue establishing connections primarily through procedure and justification, and to a lesser extent through shifts in representation or explicit comparison between procedures. In this sense, the follow-up questions appear to deepen the connections already established during the initial solution of the task, rather than substantially broadening their diversity.

The high frequency of procedure, along with the qualitative analysis, connections indicate that responses were consistently grounded in making explicit how formulas, counting strategies, and transformations relate to area. In this way, the P connection appears to be mobilized to interpret changes in the figure, compare configurations, or identify regularities. The high frequency of J connections likewise suggests that the follow-up questions encouraged a shift from stating a result to justifying it, that is, toward making explicit why area is preserved or changes under certain conditions. By contrast, the comparatively lower frequency of EP and AR connections suggests that, beyond the initial part of the task, groups appeared to focus less on contrasting methods or shifting between registers, and more on developing explanations directly linked to the mathematical structure of the situation.

### Full Task: Relationships Between Intra-Mathematical Connections and Mathematical Objects in Groups' Responses

The heatmap (Figure 5) illustrates the frequency of co-occurrence between the five types of connections and the mathematical objects identified in the groups'



**Figure 5.** Co-occurrence frequencies between connection types and mathematical objects across the full task (the authors' own work)

written productions across the full task. Rows correspond to connection types and columns to the mathematical objects coded in the responses. Each cell indicates the frequency with which a connection and an object were identified together, with darker shades representing higher co-occurrence frequencies.

Interpretation of these frequencies requires two qualifications. First, some co-occurrences are partly constitutive of the connection definitions themselves; for example, P connections are expected to co-occur frequently with procedural objects. Second, several frequencies are also partly shaped by the task design. part 1 explicitly requested procedures and justification, while later questions elicited work involving relationships among quantities. High frequencies involving objects such as P7, R3, R4, and C3 are therefore interpreted descriptively and not as evidence of independent associations between those objects and particular connection types.

Within these constraints, three descriptive patterns are noteworthy. First, within the ER row, conservation of area under transformation (J3) shows the highest co-occurrence frequency among the justification-category objects (J3 = 10; J6 = 4; J2 = 1; J5 = 0). This indicates that, when ER and justification-category objects appeared together, J3 was the justification object most frequently identified alongside ER. Second, within the justification (J) row, the highest co-occurrence frequencies occur with procedural and quantitative objects, including calculation using linear dimensions and formulas (P7 = 46), numbers, expressions, or formulas (R4 = 33), and functional relationships between quantities (C3 = 23). These frequencies are higher than those observed for justification-category objects such as J6 = 12, J3 = 10, J2 = 1, and J5 = 1. Given the explicit demands of the task, the individual frequencies should not be interpreted independently of task elicitation. Descriptively, however, the J row shows that J connections were more often identified alongside procedural and quantitative objects than alongside the justification-category objects themselves. Third, within the EP and AR rows, co-

occurrence with justification-category objects is comparatively limited relative to several procedural and representational objects. For EP, for example, the highest frequencies occur with R3 = 22, P7 = 13, P2 = 12, and R4 = 11, whereas co-occurrence with J3 and J6 is 7 and 11, respectively. A similar pattern appears for AR, with R3 = 19, P7 = 13, R4 = 12, and P2 = 11, compared with J3 = 6 and J6 = 9. These within-row differences indicate that EP and AR were more frequently identified together with several procedural and representational objects than with justification-category objects. Because these are raw co-occurrence frequencies and the overall frequency of each connection type differs, no cross-connection comparison of relative prevalence is inferred from these values.

### Full Task: Overall Group Tendencies

The cluster map (Figure 6) combines a heatmap with hierarchical clustering to explore similarities in the relative distribution of connections across groups' written productions. Rows represent the groups (G1 to G15, excluding G6), and columns represent the five connection types. Cell color indicates the relative intensity of each connection within a group, with darker shades reflecting greater relative weight. For the row clustering, average linkage with correlation distance was applied. The cophenetic correlation coefficient was 0.77, indicating a reasonably strong correspondence between the original pairwise distances and the distances represented by the resulting hierarchy. The cluster map is therefore interpreted as a descriptive and exploratory representation rather than as evidence of a definitive classification.

To facilitate visual comparison among groups with different total connection frequencies, row-max normalization was applied: within each group, the frequency of each connection was divided by the maximum frequency in that row. The dominant connection within each group therefore takes the value of 1.00, while the remaining connections are rescaled to the interval [0, 1] according to their relative weight. This

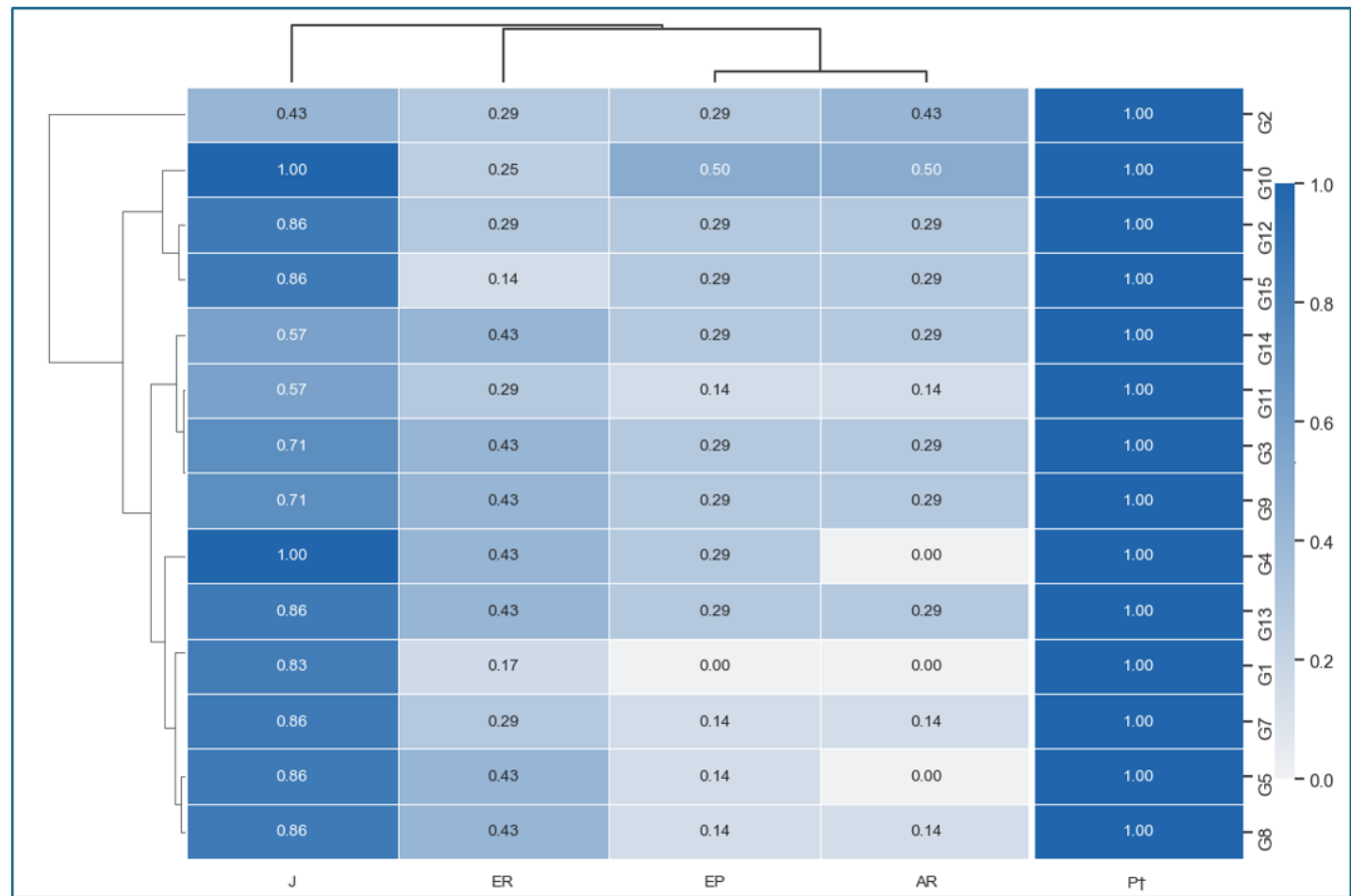


Figure 6. Cluster map of connection frequencies across groups' written productions (the authors' own work)

normalization serves a visualization purpose and does not alter the row-clustering result, since correlation distance is invariant to positive rescaling of individual rows. The groupings discussed below are therefore interpreted in terms of the relative distribution of connections within each group, rather than absolute differences in connection frequency.

Under this normalization, P takes the value 1.00 in all groups. P was consequently excluded from the separate clustering of columns and was displayed at the right of the heatmap as a fixed reference column. The remaining four connections (J, ER, EP, and AR) were clustered separately using average linkage with correlation distance. Within this exploratory column clustering, J and ER appeared as the closest pair, whereas EP and AR were positioned further from this pair. This column structure is interpreted descriptively as an indication of similarity in how these four non-constant connection types vary across groups; it does not establish an independent relational structure among the connections.

An initial reading of the row dendrogram was examined through alternative cuts at  $k = 2$ ,  $k = 3$ , and  $k = 4$ . These cuts are used to describe the structure of the selected hierarchical solution and are not treated as a formal stability analysis. The fusion heights help contextualize these partitions. The largest interval between successive fusion heights extends from

approximately 0.102 to 0.222. Across this interval, the two-cluster partition is retained, with G2 separated from the remaining thirteen groups. The next largest interval extends from approximately 0.058 to 0.102 and corresponds to the three-cluster partition, in which G10, G12, and G15 form one grouping, G2 remains separate, and the remaining ten groups form a third cluster. The interval corresponding to the four-cluster partition is smaller, extending from approximately 0.039 to 0.058. At this level, the ten-group cluster divides into G3, G9, G11, and G14 on the one hand, and G1, G4, G5, G7, G8, and G13 on the other. These differences in fusion height are interpreted as descriptive features of the selected hierarchical solution rather than as evidence of an optimal number of clusters.

Because membership obtained from a single hierarchical specification may depend on methodological choices, a cross-specification sensitivity analysis was conducted. Seventeen combinations of linkage method (average, Ward, complete, and single), distance metric (correlation, Euclidean, and cosine), and data representation (row-normalized values and raw counts) were examined. Five specifications produced cophenetic correlation coefficients of at least 0.70 and were retained for comparative analysis. The specification displayed in Figure 6 (average linkage with correlation distance) was one of these five. For each

retained specification, exact partitions at  $k = 2$ ,  $k = 3$ , and  $k = 4$  were obtained, and pairwise co-membership was compared across specifications. The broader cluster boundaries varied across specifications, whereas several smaller subsets showed recurrent co-membership. The descriptions below therefore distinguish between recurrent cross-specification tendencies and groupings specific to the selected hierarchical solution.

### G2: A Recurrently Isolated Case

In the selected specification, G2 is separated from all other groups at  $k = 2$  and remains isolated at  $k = 3$  and  $k = 4$ . Across the five retained specifications, however, this pattern is not universal: G2 appears as a singleton in two of the five specifications at  $k = 2$  and in four of the five specifications at both  $k = 3$  and  $k = 4$ . Its isolation is therefore better described as a recurrent tendency, particularly at the finer partitions.

In the selected specification, G2 shows AR at  $AR = 0.43$  and the lowest relative justification value in the dataset ( $J = 0.43$ ), while EP and ER both take intermediate values ( $EP = 0.29$ ;  $ER = 0.29$ ). This configuration provides a descriptive contrast with the multi-group configurations identified in the selected hierarchy, while the sensitivity analysis shows that its isolation becomes more recurrent at the finer cuts.

### A Recurrent G12-G15 Pairing, with G10 in the Selected Specification

In the selected specification, G10, G12, and G15 form a three-group cluster at  $k = 3$ , which remains unchanged at  $k = 4$ . The cross-specification analysis indicates that the most consistent element of this grouping is the pairing of G12 and G15: these two groups co-cluster at  $k = 4$  in all five retained specifications. G10 is less consistently associated with them. At  $k = 4$ , G10 appears as an isolated case in four of the five retained specifications and is grouped with G12 and G15 only in the selected specification.

Within the selected hierarchical solution, G10 shows the highest relative values of EP and AR in the dataset ( $EP = 0.50$ ;  $AR = 0.50$ ), together with  $J = 1.00$ . In absolute terms, however, these values correspond to  $EP = 2$ ,  $AR = 2$ , and  $P = 4$  and therefore reflect, in part, G10's lower P frequency relative to most groups rather than unusually high absolute frequencies of EP or AR. G12 and G15 both show comparatively prominent justification ( $J = 0.86$ ) and moderate values of EP and AR (0.29 in both groups), while ER is 0.29 for G12 and 0.14 for G15. The recurrent G12, G15 pairing is therefore the more consistent cross-specification observation, whereas the inclusion of G10 is specific to the  $k = 4$  partition of the hierarchical solution shown in Figure 6.

### A Recurrent G3-G9-G14 Grouping, with G11 in the Selected Specification

At  $k = 4$  in the selected specification, G3, G9, G11, and G14 form one grouping. Across the five retained specifications, however, the recurrent component is the subset G3, G9, G14: all three pairwise relationships among these groups are preserved in all five specifications. G11 does not show the same level of cross-specification consistency and is therefore treated as part of this grouping only in the  $k = 4$  partition derived from the selected hierarchy.

Within this  $k = 4$  partition, justification lies between  $J = 0.57$  and  $0.71$ . ER ranges from  $ER = 0.29$  to  $0.43$ , and EP and AR range from  $EP = 0.14$ - $0.29$  and  $AR = 0.14$ - $0.29$ , respectively. Thus, within the selected specification, this grouping shows moderate relative justification together with moderate ER and comparatively lower EP and AR. The recurrent cross-specification observation is therefore the G3, G9, G14 subset, while the broader four-group configuration is specific to the selected specification.

### A Recurrent G4-G5-G7-G8 Grouping, with G1 and G13 in the Selected Specification

At  $k = 4$  in the selected specification, G1, G4, G5, G7, G8, and G13 form a six-group cluster. The cross-specification analysis identifies G4, G5, G7, G8 as the recurrent component: all six pairwise relationships among these four groups are preserved across all five retained specifications. G13 shows a less consistent placement. It co-clusters with each of G4, G5, G7, and G8 in four of the five retained specifications, but it also co-clusters with G12 and G15 in four of five. Its placement therefore varies across specifications rather than aligning consistently with a single recurrent subset. G1 likewise does not show recurrent co-membership with the four-group core across the retained specifications and is therefore treated as part of the six-group configuration only in the  $k = 4$  partition derived from the selected hierarchy.

Within the selected specification, justification is comparatively prominent ( $J = 0.83$ - $1.00$ ). EP and AR are more limited in relative terms ( $EP = 0.00$ - $0.29$ ;  $AR = 0.00$ - $0.29$ ), whereas ER ranges from  $ER = 0.17$  to  $0.43$ . For its part, G1 shows  $EP = 0.00$  and  $AR = 0.00$ , while the remaining groups show varying non-zero values in at least one of these connections. This grouping can therefore be described as showing comparatively high relative justification alongside more limited EP and AR in several cases. Cross-specification comparison, however, supports the four-group subset G4, G5, G7, G8 more consistently than the broader six-group boundary.

Overall, the cross-specification comparison identified several recurrent descriptive tendencies. At  $k = 4$ , the subsets G3, G9, G14 and G4, G5, G7, G8, together with the G12-G15 pair, showed complete pairwise co-membership across all five retained specifications. G2

appeared as a singleton in four of the five retained specifications at  $k = 3$  and  $k = 4$ , compared with two of five at  $k = 2$ . G10 likewise appeared as a singleton in four of the five retained specifications at  $k = 3$  and  $k = 4$ , compared with three of five at  $k = 2$ . By contrast, the placement of G1, G11, and G13 and the broader boundaries of the displayed groupings varied across specifications. The groupings shown in [Figure 6](#) are therefore treated as exploratory configurations within the selected hierarchical solution, while the recurrent subsets identified across specifications are interpreted as the more consistent descriptive tendencies in the data.

## DISCUSSION

A first point concerns the relationships among the different types of intra-mathematical connections evidenced in the groups' written productions. The results do not suggest that area conservation was expressed through a single type of connection. Rather, procedure, ER, AR, EP, and justification appeared in different combinations across the task. This is consistent with previous research showing that recognizing geometric equivalence or applying an area formula does not necessarily entail being able to justify why area is preserved under transformation (Caviedes et al., 2022; Hong & Runnalls, 2020; Kospentaris et al., 2011). In the present study, written responses in which conservation was made explicit frequently also contained procedures, representations, or relationships among quantities. Some of these co-occurrences are partly related to the explicit demands of the task; they are therefore interpreted descriptively.

From this perspective, justification deserves particular attention. Procedure and justification were among the most frequent connections identified, and they also appeared together frequently in the first part of the task. The task, however, explicitly requested both procedures and justifications, so their frequency and co-occurrence cannot, on their own, be interpreted as evidence of a privileged structural relationship. The heatmap provides a complementary observation: within the justification row, J connections were identified more frequently alongside procedural and quantitative objects, such as calculation using linear dimensions and formulas (P7), numbers, expressions, or formulas (R4), and functional relationships between quantities (C3), than alongside the justification-category objects themselves. Read together with the qualitative analysis, this pattern shows that justification in the written productions was frequently expressed in conjunction with procedural and quantitative elements.

The exploratory clustering adds a further descriptive dimension by showing that the relative prominence of justification was not uniform across groups. In the selected specification, justification was comparatively prominent in the G10, G12, G15 and G1, G4, G5, G7, G8,

G13 groupings, lower in G2, and intermediate in G3, G9, G11, G14. These broader groupings, however, were sensitive to the clustering specification. At  $k = 4$ , the cross-specification analysis identified three recurrent subsets whose members co-clustered across all five retained specifications: (G12, G15), (G3, G9, G14), and (G4, G5, G7, G8). The placement of G1, G11, and G13 varied across specifications, while G10 was grouped with G12 and G15 only in the selected specification. The clustering results therefore point to recurrent descriptive similarities among particular subsets of groups rather than to fixed group-level categories.

The cases of G2 and G10 further illustrate why the normalized values should be interpreted cautiously. Within the selected hierarchy, G2 shows the lowest relative justification value in the dataset ( $J = 0.43$ ) together with a comparatively high AR value ( $AR = 0.43$ ). It also appears as a singleton in four of the five retained specifications at  $k = 3$  and  $k = 4$ . This case illustrates that a comparatively prominent AR connection within a written production can coexist with relatively limited justification. G10 shows a different relative distribution, with  $J = 1.00$  and normalized EP and AR values of 0.50. These latter values correspond to raw frequencies of  $EP = 2$  and  $AR = 2$  and partly reflect G10's lower P frequency ( $P = 4$ ), rather than unusually high absolute frequencies of EP or AR. G10 also appears as a singleton in four of the five retained specifications at  $k = 3$  and  $k = 4$ . Its grouping with G12 and G15 is therefore specific to the partitions derived from the selected hierarchy rather than a recurrent cross-specification pattern.

More generally, the relative prominence of justification does not correspond to a single accompanying distribution of the other connection types. Groups with comparatively high normalized justification values differ in the relative presence of EP, AR, and ER. The descriptive results therefore show variation in the connections and mathematical objects that appear alongside justification across the written productions. This variability highlights the complexity of the role of justification in area-conservation tasks, since justification is present in conjunction with different mathematical resources rather than through a single recurring configuration. This interpretation is consistent with approaches that examine mathematical connections as relationships among mathematical objects and processes rather than as isolated elements (de Gamboa et al., 2023, 2024).

The follow-up questions in part 2 explicitly required explanation, comparison, prediction, and engagement with relationships among quantities. The visibility of particular mathematical objects and associated connections in this part of the task, including formulas, geometric equivalences, and quantitative relationships, therefore partly reflects these demands. This is consistent with research emphasizing that work on area

involves more than applying formulas and may involve making explicit relationships among geometric transformations, quantitative expressions, and justification (Caviedes et al., 2023b, 2024; Hong & Runnalls, 2020). Several written productions also included statements concerning dependence or proportional relationships among base, height, and area. The present data, however, cannot support conclusions about the development of formal proportional reasoning. What they do show is that relationships among quantities were used in some written justifications concerning conservation or variation of area. This extends the empirical description beyond visual comparison of figures to include quantitative relationships expressed in the written productions, aligned with studies showing that conservation tasks can involve mathematical activity extending beyond immediate visual perception (Sari & Ng, 2022; Sari et al., 2023).

Theoretically, these results support the usefulness of examining area conservation through intra-mathematical connections because this perspective makes it possible to describe not only which mathematical objects appear in written productions, but also which connections are evidenced among them (de Gamboa et al., 2023; García-García & Dolores-Flores, 2018; Hatisaru, 2023).

## CONCLUSIONS

The aim of this study was twofold: first, to characterize the intra-mathematical connections evidenced in groups of PPTs' written productions when addressing area-conservation tasks; and second, to examine how these connections appeared in relation to one another across the task.

With respect to the first aim, all five connection types considered in the study (procedure, justification, ER, AR, and EP) were identified in the groups' written productions, although their frequencies differed across the two parts of the task. Procedure and justification were among the most frequent connections identified, whereas EP and AR appeared less frequently, particularly in the follow-up questions of part 2. These differences should be considered in relation to the task design, since several questions explicitly required procedures, justification, comparison, or explanation and therefore provided different opportunities for particular connection types to be evidenced.

With respect to the second aim, the analyses indicate that the relationships among connections were expressed through different relative combinations across the groups' written productions. The co-occurrence analysis showed that connection types appeared alongside different mathematical objects, while the exploratory clustering revealed that some similarities among groups recurred across alternative analytical

specifications and others depended more strongly on the methodological choices used to construct the hierarchy. This suggests that intra-mathematical connections in area-conservation tasks are better described in terms of recurring relational tendencies than in terms of fixed categories or a single common configuration.

Taken together, these findings suggest that area conservation, as evidenced in the groups' written productions, is better characterized through relationships among multiple mathematical objects and connection types than through the presence of isolated elements (Caviedes et al., 2022; Hong & Runnalls, 2020). In particular, justification appeared alongside procedures, representations, EPs, and quantitative relationships in different combinations across groups, suggesting that its role in area-conservation tasks is complex and cannot be reduced to a single pattern of association (Caviedes et al., 2023b, 2024). These findings also have implications for teacher education, as tasks that require PPTs to compare transformations, use different representations and procedures, relate quantities, and make justification explicit may create opportunities for a broader range of intra-mathematical connections to become visible when working with area conservation.

The study has several limitations. First, it corresponds to the first implementation cycle of a design-based research study, so the findings are preliminary and primarily inform refinement of the instructional proposal. Second, the analysis focused on written group productions and therefore captures the connections evidenced in the final responses rather than the interactional processes through which they were constructed. Third, the small number of group productions ( $n = 14$ ), the use of a single task at one institution, and the exploratory nature of the quantitative analyses limit the scope of generalization. Fourth, the cross-specification sensitivity analysis showed that the boundaries of the broader exploratory groupings varied according to the clustering specification, although several smaller subsets recurred across retained specifications. Fifth, coding was conducted through structured team consensus rather than independent parallel coding; future cycles could incorporate independent coding of a subset of the data as an additional check on the coding process.

Future implementations could advance in several complementary directions: examining whether similar patterns of connections are evidenced in tasks with different degrees of openness; incorporating interviews or audiovisual recordings to investigate how connections are constructed and negotiated during group interaction; examining how the articulation of connections changes across successive cycles of teacher education; and investigating more directly how groups' spatial structuring of the figures may relate to the connections evidenced in area-conservation tasks. These

extensions could help identify task and instructional conditions associated with a broader or more explicit articulation of procedures, representations, EPs, quantitative relationships, and justification (de Gamboa, 2023, 2024).

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**Ethical statement:** Formal ethics committee approval was not required for this study. The project was assessed by the Comitè d'Ètica en la Recerca at the Universitat Autònoma de Barcelona, which determined that, under the applicable institutional guidelines, formal ethical approval was not necessary. Informed consent was signed on paper and stored securely, accessible only to the research team, per institutional data-protection requirements.

**AI statement:** AI tools were used to proofread the English text and to refine the Python code.

**Declaration of interest:** No conflict of interest is declared by the authors.

**Data sharing statement:** Data supporting the findings and conclusions are available upon request from the corresponding author.

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