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# Identifying and Fostering Higher Levels of Geometric Thinking

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Pierre M. Van Hiele created five levels of geometric thinking. We decided to identify the level of geometric thinking in the students in Slovenia, aged 9 to 11 years. The majority of students (60.7 %) are at the transition between the zero (visual) level and the first (descriptive) level of geometric thinking. Nearly a third (31.7%) of students is at the first level whereas 4.3 % of students are at the zero level. Only 1.4 % of students reached the second level of geometric thinking. Students had the most difficulty with the use of appropriate geometric language, so a teaching approach to improve mathematical language was created, in an attempt to accelerate the transition to a higher level of geometric thinking. The teaching approach that proved successful was based on the use of different materials, concrete experiences, promoting the use of appropriate geometric language and motivation of students.

*Keywords*: geometry, levels of geometric thinking, second triad, the promotion of geometric thinking

## THEORETICAL BACKGROUND

Geometry is an abstract science, where the relations between concepts are managed through the appropriate use of language. Research carried out around the world shows that students have many difficulties in topics that relate to geometry. The connection between language and geometric concepts was dealt by Pierre M. Van Hiele, who based on solving geometric problems developed a theory that advocates the levels of geometric thinking.

Van Hiele designed and described in detail five levels of geometric thinking (Van Hiele, 1999; Van Hiele, 1957, as cited in Fuys, Geddes & Tischer, 1984). These levels are as follows:

- *Visual level (Level 0) - students identify figures by their appearance;*
- *Descriptive level (Level 1) - figures are no longer identified by their appearance, but by certain properties;*
- *Informal deduction level (Level 2) - properties are logically arranged;*
- *Formal deduction level (Level 3) - an individual uses deduction in establishing theorems and creates original proofs, thereby addressing axioms and the necessary and sufficient conditions;*
- *Rigor (Level 4) - different geometric systems based on axioms are compared (Van Hiele, 1999; Van Hiel, 1957, as cited in Fuys, Geddes & Tischer, 1984).*

Van Hiele believes that the level of an individual is influenced by learning rather than by age, attended grade or biological maturity (Van Hiele, 1957, as cited in Fuys, Geddes & Tischer, 1984; Burger & Shaughnessy, 1986; Wu & Ma, 2006). The teaching approach can thus accelerate or inhibit the development of geometric thought. Among other things, Van Hiele emphasized the importance of experience; he stated that students cannot operate properly on some level, if they have no experience, allowing them to think at this level (Van Hiele, 1959, as cited in Mayberry, 1983).

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***State of the literature***

- In the 50-ies of the last century Van Hiele developed a theory and formed levels of geometric thinking.
- A lot of attention was paid to levels of geometric thinking in the 80-ies of the last century.
- The subject matter is still relevant today.

***Contribution of this paper to the literature***

- In our research it was established that the majority of students are at the transition between the zero level and the first level of geometric thinking.
- Developed teaching approach proved to be successful in achieving higher levels of geometric thinking. It was determined that the use of terminology in geometry improved.
- It was also established, that students at certain tasks, did not progress following their exposure to teaching.

Van Hiele (1957, as cited in Fuys, Geddes & Tischer, 1984), Mayberry (1983) and Burger and Shaughnessy (1986) believe that an individual should proceed from one level to the other in a consecutive manner. In addition, they found the individuals with different concepts to be at different levels and that some students never reach the formal deduction level (Burger & Shaughnessy, 1986; Wu & Ma, 2006; Mayberry, 1983).

Many researchers, such as Burger and Shaughnessy (1986), as well as Fuys, Geddes and Tischer (1988), Gutiérrez, Jaime and Fortuny (1991) and Usiskin (1983) had difficulty determining the level of an individual. Usiskin (1983) did not classify 19 % of students on any of the levels. Among other things he was identifying the level of geometric thinking in 2699 students aged 14 to 17 years. Prior to their exposure to teaching the geometry content he established half of the students to be on the level 0, 14 % of them to be on the level 1, 5% on the second level, and 28 % below the level 0.

Extensive research was also carried out by Wu and Ma (2006), who wanted to identify the level of 5581 students from the first to the sixth grade of elementary school. They found that all of the first and the second grade students were below or on the level of zero (Wu & Ma, 2006). Likewise, most of the third-grade students were on the level zero, while most of the students of the fourth, fifth and sixth grade were at the first level of geometric thinking (Wu & Ma, 2006). In the fifth and sixth grade, around 20 % of students were on the second level (Wu & Ma, 2006).

Halat (2006) focused on the assessment of the influence of gender on the levels of geometric thinking as well as motivation of year six students for the learning of geometric content according to Van Hiele's approach. He found that the gender affects neither the progress to higher levels of geometric thinking nor the motivation for teaching according to Van Hiele's approach (Halat, 2006).

Van Hiele - Geldof was the first to address promoting higher levels of geometric thinking. The learning approach devised by her (Van Hiele - Geldof, 1957, as cited in Fuys, in Geddes & Tischer, 1984) is based on the experiences and applying numerous materials, on the basis of which the learner develops geometric thinking. Her sample consisted of 12-year-old students. Geometric concepts and symbols are introduced when students have gained sufficient experience with particular content. Only then the relationships between concepts may be established. She also took into account the learner's prior experience, which she often made use of in the discussion (Van Hiele - Geldof, 1957, as cited in Fuys, Geddes & Tischer, 1984).

Van Hiele - Geldof (1957, as cited in Fuys, Geddes & Tischer, 1984) and Van Hiele (1999) believes that in order to proceed from one level of thinking to another one it is necessary to follow the five stages of teaching, namely:

- Inquiry: students are introduced to the area of examination;
- Directed orientation: the tasks are presented in such a way that students gradually learn particular properties;
- Explanation: a teacher connects the lessons learned with the correct terms;
- Free orientation: a teacher presents tasks that can be completed in different ways and enables children to become more proficient with what they already know.
- Integration: students are given the opportunity to connect the acquired knowledge. The activities are summarized, which allows students to connect the old and the new knowledge.

The teacher has different roles in various stages: task planning, directing a student's attention to geometric properties of shapes, introducing the terminology, fostering students to use appropriate terminology, and promoting students' explanations and problem solving (Van Hiele, 1999). Van Hiele - Geldof (1958, as cited in Fuys, Geddes & Tischer, 1984) was successful in teaching, as after 25 hours of teaching and learning the students moved from the level zero to the first level of geometric thinking.

Also, Usiskin (1983) and Fuys, Geddes and Tischer (1988) addressed promoting higher levels of geometric thinking, yet they were not satisfied with the achieved levels of students after their exposure to teaching geometry content. Usiskin (1983) established that, after 140 hours of teaching, one third of students remained on the same level or even regressed, one third of students progressed for one level while one-third of students progressed for at least two levels. Fuys, Geddes and Tischer (1988) attributed bad students' performance to the lack of experience with geometry content and to a set of textbooks that did not promote higher levels of geometric thinking (Fuys, Geddes & Tischer, 1988). The latter was also confirmed in this research, as there are very few tasks that can be solved on the second level of geometric thinking in the textbooks, which also address teaching and learning of geometry in the second triad in Slovenia. The conclusion was reached that it would be sensible to devote more attention to promoting higher levels of geometric thought.

The teaching of geometry concepts starts as early as in the first grade of primary education (at the age of 6) and continues until the last year of elementary school (Učni načrt: Matematika, 2011). The curriculum does not contain any special recommendations for the teachers as to how to promote higher levels of geometric thinking in students. The prevailing aims of learning geometry are naming, drawing, using symbols in geometry and recognizing geometrical objects. It is worth considering to enrich the aims with the following competences in geometry: the ability to explain the differences and similarities among geometrical objects, the establishment of hierarchies among them, the investigation and solving problems in geometry (In our national curriculum, these activities are much more emphasized in arithmetic than in geometry.), the ability to apply knowledge to new situations and the ability to explain solutions of geometry tasks. We believe that most of the tasks in geometry in the teaching materials are predictable and can thus be solved in a particular, in many cases in a prescribed way, often lacking any meaningful challenge.

In the first two grades, geometry is introduced at the visual level; in the third and fourth grades, children are encouraged to progress to the descriptive level; in the fifth grade, pupils start to develop informal deductive level by learning about the relation between a square and a rectangle: a square is rectangle; each rectangle is not square.

### Empirical Part

Van Hiele's findings encouraged many researchers to deal with the issue of levels of geometric thought in students. In Slovenia no extensive research has been conducted thereto, so our team decided to establish the levels of geometric thinking according to Van Hiele in the students of the second triad. Based on the results obtained, a teaching approach was developed, with the aim to foster higher levels of geometric thinking.

## RESEARCH PROBLEM AND METHODOLOGY

In this study the focus was on two issues, namely, to determine the level of geometric thinking about the content of shapes in students who attend the fourth (nine years), fifth (ten years) and sixth (eleven years) grade of elementary school. The other research problem was to design a teaching approach that is focused on the development of appropriate terminology and the associated hierarchy of concepts in geometry and to promote higher levels of geometric thought.

### Research Questions

The posed research questions were as follows:

- On what level of geometric thinking according to Van Hiele are students aged nine to eleven years?
- Are there any differences in the levels of geometric thinking related to gender, grades and the final assessment score in mathematics?
- How effective is the teaching approach designed to promote higher levels of geometric thought?

### Methods of Data Collection

With a view to determine the level of geometric thinking in students of the second triad, 19 Slovenian schools were raffled off. From each school one division of the fourth, the fifth and the sixth grade participated in the research. The sample consisted of 782 students and is referred to as group A in the following text. Schools were provided with knowledge tests and detailed instructions for their solving. The tests were taken in May and June 2010, and the test time limit was 45 minutes.

In order to determine the effectiveness of the learning approach, six teachers from five Slovenian schools were presented with Van Hiele's theory of levels of geometric thinking and the designed teaching approach thereto. Teaching through the designed approach to learning took place during the regular hours of mathematics in November and December 2010. Teaching according to created teaching approach was carried out in 10 lessons, each 45 minutes long. The group of students who took part in the teaching approach referred to as group B) was assessed twice: one week before the teaching approach took place and again one week after the teaching approach. The test as well as the instruction for solving the test solved by group B after the teaching approach were the same as for the group A. The group A was tested only once.

Cronbach's alpha coefficient of reliability of the test is 0.773.

### Sample

The sample was divided into two groups. The A group consisted of 782 students from 19 Slovenian schools (approximately half of them were situated within urban areas, while the other half was located in rural surroundings) of which 385 were boys (49.2 %) and 385 girls (49.5%). The proportion of students from each class was almost the same. These students took the knowledge test, on the basis of which the level of geometric thinking in students of the second triad was identified. The A sample was a random one. The B group was composed of six sections of fourth grade 98 students of five Slovenian elementary schools. These students were taught the geometry content according to the developed teaching approach. Upon the completion of the teaching, they took the same knowledge test as was set to the students of the A group. In the examination after the completion of the teaching approach the total of 98 students, who were present in all lessons, participated. There were 86 such students, namely 42 boys and 44 girls. The B group sample was a convenience one.

### Research Tools

Two tests were used in our research. Test 1 (see appendix 1) consisted of 8 tasks (each divided in examples as a, b, c…) and was solved by the students in group A and also by the students in group B after the teaching approach was carried out. Before participating in the teaching approach, the students in group B also solved test 2, which consisted only of task 5 in test 1.

Table 1 represents the level of geometric thinking according to Van Hiele needed for solving a particular task. The categorization of the tasks was additionally confirmed by two experts in didactics of mathematics.

## INTERPRETATION OF RESULTS

### Identifying the Levels of Geometric Thinking

In order to identify the level of geometric thinking in the students of the second triad a test was compiled, consisting of eight tasks. Some of the tasks we created ourselves. The second and the last task which we designed on the bases of the task made by Lehrer, Fennema, Carpenter and Ansell (1993, as cited in Fox, 2000) and Geddes and Fortunato (1993, as cited in Fox, 2000). The fourth task created Burger and Shaughnessy (1986).

Individual tasks were composed of several parts, so 19 tasks were set a particular level of geometric thinking. The aim of six of these tasks was to test the second level of geometric thinking, whereas the other tasks were meant to test the first level. It should be noted that the tasks that were attributed to the second level were also possible to solve at the first level, or at the level of zero; such was for example the task where students had to draw the triangles, which differ from each other.

On the basis of the solutions, the answers were ranked in different levels. On the zero level of geometric thinking those answers were ranked which show that students decided on the basis of the appearance of the figures, as well as those answers that were are irrational from the point of view of mathematical correctness. The solutions of students, which attested to the students' knowledge of the properties of different figures, and proved correct from the mathematical point of view, were ranked on the first level. The second level of geometric thinking was assigned to students, the answers of which established both, their knowledge of the properties of geometric figures, and the logic organization of properties or the hierarchy of geometric concepts.

Like many other researchers (Burger & Shaughnessy, 1986; Fuys, Geddes & Tischer, 1988; Gutiérrez, Jaime & Fortuny, 1991; Usiskin, 1982) we also encountered problems when identifying the levels. Namely, many students were established to be in transition between the zero level and the first level, and in transition between the first and second level. For this reason, it was decided to introduce further levels of 0.5 and 1.5. On the level of 0.5 the students were classified who did not consider figures as a whole, but took into account the particular properties of the figures, thereby making many mistakes. The most common such mistake was that instead of using the word "side" they used the word "edge." These students clearly demonstrated the fact that they do not take the figures as a whole, and have not yet developed an adequate vocabulary to properly verbalize their thinking about the figures. The level of 1.5 was attributed to the students who were very much familiar with the properties of

**Table 1.** Tasks of Test 1 in Relation to the Levels of Geometric Thinking

| Tasks of test 1 | 2a | 2b | 2c | 3 | 4a | 4b | 5a | 5b | 5c | 5d | 5e | 6a | 6b | 6c | 7 | 8a | 8b | 8c | 8d |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Level of geometric thinking | 1 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |

different figures, but these qualities lacked logical arrangement. An example of this was for example the 5d task where one had to come up with the solution of a rectangle to be a square, also because the square has right angles. Those students who enumerated as many as four properties of squares and rectangles, but did not mention right angles, were ranked on the level 1.5.

In the first task, the students were asked to name the geometric shapes. A rectangle, a square, a hexagon, an ' upright ' and a ' narrow ' triangle and an ' upright ' and a ' narrow ' rectangle were in the figure. The students did not have any difficulty naming the shapes. The level of geometric thinking had not been determined for this task and was not the issue of our analysis. The second task was composed of three parts. In each part the students had to find the odd one out, namely a shape that did not relate to the other ones, and justify their choice, which did not pose any significant problems to them. In the third task, more than half of the students correctly drew a shape that has one set of parallel sides. The fourth task was to draw different triangles. 41.7% of students were assigned the second level, as they drew different types of triangles. Among these, 24 % students responded that it was possible to draw an infinite number of them.

The students fared worse in the 5a task, in which different quadrilaterals were drawn. The students had to provide the letter K (kvadrat in Slovene language) in the squares, and the letter P (pravokotnik in Slovene language) in the rectangles, which was accomplished only by a quarter of the respondents. We have not been able to fully explain the results, but it is assumed that the students believed they had to write a letter on each shape.

They fared slightly better in the listing of differences (58.9 % of the students were assigned the first level) and similarities (68.7 % of the students achieved the first level) between a rectangle and a square. 15.7 % of the students appropriately justified why each rectangle is not a square, while 31.7% of them adequately justified why the square is a rectangle.

In the sixth task, the students were asked to circle among different quadrilaterals those which were considered to have a common property, and to justify their decision. Then, these shapes were drawn twice more and each time the students were asked again to circle the figures with a common property, but the latter should be distinguished from the previous ones. The students had difficulty with this task, as only 36.7 % of the students correctly circled the figures and justified their decision for the first time, 19.8% students did the same for the second time, and 20.8 % of the students for the third time.

In task no. 7, the students were asked to name the figure with two sets of parallel sides, which did not pose any significant problem to them, as 88 % of the students were assigned to the first level. In task no. 8 rhombuses were drawn, which were named ‘Purps’ followed by figures other than ‘Purps’. We used the unusual name ‘Purp’ with the aim to encourage students to focus on the features of the shapes and rather than trying to connect the shapes with some they already knew. The students had to circle 'Purps' first; 61.2 % of the students were successful in doing this. Then they had to draw three different ‘Purps’ and describe the characteristics of the ‘Purps’. 62.3 % of the students drew them correctly and 22.4 % students were attributed the second level of geometric thinking in describing the 'Purp'. At last they had to write if a square is also a ‘Purp’ and why. 21.9 % of the students adequately reasoned the square being also the 'Purp.'

As mentioned before, the students fared worst in the first part of the task no. 5, where they had to identify rectangles and squares among various quadrilaterals. The level of zero was assigned to almost half of the respondents. Great difficulties were encountered also in the interpretation of the reason of each rectangle not being a square, and, that the least a square is a rectangle. In the task no. 6 the students found it difficult to identify the figures with similar properties, and even more problems with the justification of their choice of particular figures. The students experienced the least difficulties in taking out a shape and providing the explanation thereto.

The average level of geometric thinking was calculated in three ways. First, the level achieved in relation to the particular tasks of the test was calculated. In doing so, all A group students were taken into account, including those that did not solve the tasks. The level was calculated by adding up the percentages of the achieved level at individual tasks, and dividing the sum by the number of tasks. The aim was also to establish the average level of the whole examination, not only in relation to particular tasks, so each level was numerically evaluated. The students, who reached the first level, were attributed one point, whereas those who reached the second level were assigned two points. If a student was assigned a level 0.5, he was credited 0.5 points. The student was given 1.5 points if he was ranked on the level of 1.5. The corresponding assessment scale was also produced. The average level of the entire examination was calculated in two ways. First, the students, who did not solve the tasks, were not attributed a level zero, but the information about the non-completed task was excluded - this information was not taken into account. In the second case, the students, who did not solve the tasks, were attributed the level of zero. It seems that the latter method of calculating the average level best reflects the level of the students in the second triad. It was established that when the students, who did not solve the tasks, were assigned the level of zero, the majority of the students (60.7 %) were in the transition from the level of zero to the first level of geometric thinking. Almost one third of the students, i.e. 31.7 %, were ranked on the first level, 4.3 % on the level of zero, and only 1.4 % of students in the second triad were ranked on the second level. From these data, it can be inferred that students in the second triad know certain properties of geometric figures, but they do not logically arrange their properties; in addition, students are not able to use this knowledge in new circumstances.

In the conducted research the statistically significant difference between the achieved level and the grade was established, namely upper-class students were more successful in on 15 out of 19 tasks. Statistically, a significant difference between the grades was also confirmed in the calculation of the average level in relation to the entire test, taking into account the solutions of all the students ($F = 36.32$, $p < 0.01$), and excluding the solutions of students, who did not complete the task ($F = 52.34$, $p < 0.01$). In both cases, the students in higher grades performed better than pupils from the lower grades.

Thus, it was confirmed in the research that students in higher grades were at a higher level of geometric thinking, which was also established by Wu and Ma (2006). However, one also agrees with the finding by Burger and Shaughnessy (1986), namely that the level of geometric thought is not strictly determined by the age of students or the grade that students attend. It was namely found in the research that the younger students performed better at specific contents than the older ones.

A statistically significant difference was also observed between the achieved level and the final assessment in the past school year (which was confirmed for 16 times), and between the level and the assessment of the current school year (which was confirmed for 13 times). A statistically significant difference between the final assessment of the current ($F = 41.89$, $p < 0.01$) and past ($F = 21.74$, $p < 0.01$) school year, and the reached levels were also confirmed in determining the average level achieved at the entire test. The students who were awarded a higher final grade in mathematics are at the higher level of geometric thinking. The gender proved to be statistically significant only in three tasks. Girls were more successful than boys two times, whereas boys were more successful once.

Also, the findings of Burger and Shaughnessy (1986), Wu and Ma (2006) and Mayberry (1983) were confirmed, namely that students are at different levels of geometric thinking at different tasks.

When comparing the obtained results with those acquired by Wu and Ma (2006), it can be concluded that the results of the Slovenian students are comparable with the results of Wu and Ma (2006). At the same time the Slovenian students in the group fared better than the students who participated in the survey, conducted by Usiskin (1982).

From the results, it can be concluded that the majority of students in the second triad are familiar with the properties of geometric figures, but they did not logically arrange them. It was further established, that students have problems with the use of appropriate terminology in geometry and with applying the knowledge in new circumstances. The students' great difficulty with the wording of geometric concepts should also be highlighted, as being one of the key reasons for the poor results; the students did not use proper geometric expressions, so their level of geometric thinking was a lower one. Among other things they were still confusing the fundamental geometric concepts, such as the edge and the side, or they did not apply the fundamental concepts, suggesting their level of zero. Therefore, it seems important in the teaching of geometry content to give particular emphasis on developing, consolidating and using geometric terms, which are fundamental for appropriate wording of thoughts and geometric knowledge.

## Promoting Higher Levels of Geometric Thought

For the aforementioned reasons, a teaching approach was created, which aims to develop appropriate expressing of geometric terms, the so called hierarchy of concepts that is built with the assistance of language. As already mentioned, we developed a teaching approach which was carried out in 10 45-minute lessons. When addressing new learning materials, the teaching phases were considered as formulated by Van Hiele - Geldof (1958, as cited in Fuys, Geddes & Tischer, 1984) and Van Hiele (1999); the topic was introduced with inquiry, followed by directed orientation, explanation, free orientation and integration. The aim of the developed teaching approach was to contribute to the students' experiences, in order to promote and develop higher levels of geometric thinking. In the course of teaching, the students were exposed to various materials. Models of different shapes accompanied many activities. The students participated in several discussions, in which they had to use the appropriate geometric terminology. We considered different taxonomy levels of knowledge when teaching geometry concepts: basic conceptual knowledge, procedural knowledge and problem solving knowledge. Most of the time students were able to manipulate with the material, to talk, describe geometrical features of different shapes and were involved in different games specially prepared for this approach. All lesson plans and teaching analyses are published in the doctoral dissertation by Maja Škrbec (2013).

The students of the B group first learnt about parallelism and squareness. In the continuation they learnt that the square is a rectangle, too, and, that not each rectangle is a square. The next step was the consolidation of the lessons learned, which was carried out through a variety of activities, contributing to gaining experience in using or consolidating the appropriate geometric terminology. In the end, they integrated the learnt subject matter in a way that they created a poster, thereby availing themselves of the gained knowledge.

The lesson on parallelism began with inquiry. The students first observed the drawn lines, and then the teacher asked them to categorize the lines according to arbitrary properties. Then a directed orientation followed. The students were required to repeatedly measure the distance between the non-intersecting straight lines. They found the distance between some of the straight lines to be always the same. An explanation was provided thereto. Since the students did not know how the straight lines that never intersect are called, their teacher told them their naming was parallel lines and that the distance between them is maintained. Subsequently, a search for "sets of parallel lines" in the subjects in the classroom followed. The students also identified parallelism by measuring the distance between the lines of objects. A discussion took place about the usefulness or necessity of parallelism. Then the students learnt to draw parallel lines in their notebooks and wrote down a rule for parallel lines. The last phase consisted of free orientation, in which the students were asked to draw or name in block capitals the figures, which consisted of parallel lines, and those who lacked them. They had to draw a figure with no parallel lines, and a figure with one or two sets of parallel lines. The students experienced no difficulties in solving these tasks.

As already mentioned, the learning of parallelism was followed by the learning about the squareness, which also started with inquiry. First the students observed the drawn lines and then sorted the lines according to optional properties.

The subsequent activity was related to the directed orientation, and the students were asked to have a good look at the intersecting lines. They found out that the lines intersected in a variety of ways. They were instructed by the teacher to classify the lines according to the manner of their intersection. In order to remove all the perpendicular lines, the teacher showed a part of the set square, which has a right angle, and showed the manner in which to measure the right angle in order to determine perpendicular lines. Then, by placing a set square, the students found that some of the lines were intersecting at right angles.

In the explanation phase the teacher explained that the chosen cards contained perpendicular lines, because the lines intersected at the right angles. Then, on the visual level, the students were familiarized with the concept of an angle. Subsequently, 'lines' were sought in the classroom, which intersected at right angles, and those that did not. The teacher encouraged the students to measure the size of angles with a set square. The students had no difficulty in finding the right angles. They also talked about where the right angles are used or needed. Afterwards, the students learned to properly draw perpendicular lines.

During the free orientation phase, the students had to draw or write the capital letters, where there were perpendicular lines and where there were not. They had to draw a shape which had one, two, four, three and five right angles. Drawing shapes with right angles posed more problems to the students than drawing shapes with parallel sides. The most difficult task for them was to draw a line with five right angles.

In the continuation of the created teaching approach the learning of the relationship between a rectangle and a square followed, namely that every square is a rectangle, whereas, at the same time, every rectangle is not a square. Teaching began with inquiry. Each student in the pair got 15 cards with a quadrilateral on each of them. One pupil in the pair had to take out the shapes that belong together, show the chosen shapes to his neighbor, and determine the criteria for choice. This activity was repeated several times.

In the case of the directed orientation, the teacher provided the instructions as to which characters should be chosen – e.g.: those with right angles, with two sets of parallel sides, with one set of parallel sides, with the sides of the same length, with the two sides of the same length, with no right angles, or with no parallel sides. Then the students took all the cards with squares and rectangles. The teacher posed them the following question: Are the squares also rectangles? Some students provided an affirmative answer, but they were not able to justify it.

The explanation of the posed questioned followed. The teacher produced a table on the blackboard, in which she entered the properties of squares, rectangles and optional quadrilaterals, with the assistance of students. Then she took the card with a drawn square, and put it above the column, in which squares were described. Together, they checked whether the drawn figure possessed all of the listed properties. Since it did, the teacher left the card above the squares. Then she took another card on which a square was drawn and put it above the column, presenting the properties of the rectangle. Together, they checked, whether it had all of the listed properties. As it was found that it did possess all such properties, the teacher left the square above the description of the properties of the rectangle, too. Following the same procedure it was established, that the square is a quadrilateral. Then the teacher took a card with a rectangle and attached it above the description of the properties of the square. Together, they found that the drawn rectangle did not have the sides of the same length; accordingly, the teacher removed it from the column, which contained the described properties of the square. She left it above the rectangles and quadrilaterals, after having determined that it had all of the properties of the quadrilateral. The optional quadrilateral did not have all the properties of a square and a rectangle, so it remained attached only to the description of the quadrilaterals. Figure 1 in the form of a table was produced on the blackboard.

By such an interpretation and concrete placing and removing of figures, the attempt was made to explain to the students the relationship between a rectangle and a square. The students then enumerated the properties that are not relevant in classifying the figure (its color, the manner of turning the figure, the size of the figure). The teacher also showed a colored square and put it between the squares. Then she turned the attached figure and the students agreed that its position had no bearing on the group of shapes, in which to classify it. The students also agreed that the size of the shape was not important either, because the teacher also showed a smaller rectangle and put it above the appropriate column in the table.

In their free orientation the students were asked to first pick out the cards with quadrilaterals, rectangles, and all the squares. After the completed task, the teacher drew their attention to the possible squares between the rectangles. Then the students played in groups of four. Each group was given a table, in which the figure was named in the top row. The figure was also drawn. In the left column, the observed properties were listed. Each student took out one card from the pile, on which one property of the figure was listed, and put it in the appropriate box. The teacher instructed the students that some of cards did not fall into any field and that there were too many of them, so they had to be especially careful. The groups were provided with different tables, which they changed after the game.

The integration phase followed, in which the students were presented with a similar table, which had to be filled in with the appropriate properties. In the table there was a trapezoid with two right angles, a rectangle, a rhombus and a square. The students were asked to enter the number of sides, vertices and angles, and also, whether the sides were of equal length and parallel, and whether the figures had right angles.

The gained knowledge was consolidated and the appropriate terminology developed. In order to enable the participating students to correctly apply the relevant geometry terms, nine different activities were designed, performed by two, three or four students. It took 10 minutes to undertake each activity. The activities differentiated from each other; some were based on the rules of the games, such as Memory, Old Maid and Ludo. The mentioned games were changed, so as to relate to the knowledge of geometric concepts and terminology. The students were also throwing two dices, on which different figures were shown, and then enumerated the differences and similarities between the figures. Accordingly, they had to classify the figures and their properties in the table, the Carroll Diagram, and configure it themselves. They classified, guessed and removed the shapes according to the descriptions of their classmates.

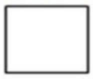
At the very end of the designed teaching approach, the students integrated the acquired knowledge in such a way that they made posters with the figures and the lessons learned; they created posters in groups of four or five students, which they presented to their classmates.

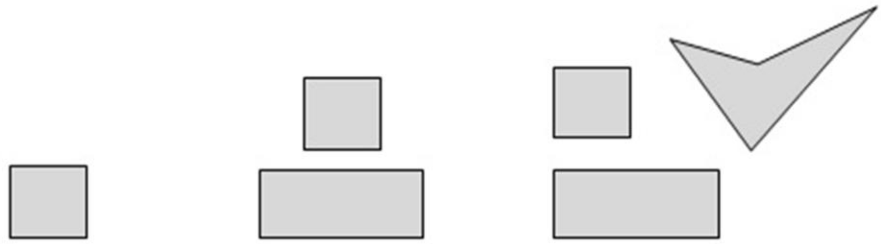
The aim of the generated teaching approach was to attempt to teach students certain new geometric content; they solved geometric problems, thereby using the appropriate teaching aids, and developed as well as used appropriate geometric terminology. At the end of the teaching process, the students applied that approach in solving the knowledge test, the same as the one set to the A group of students. At this point, it should be mentioned that no statistically significant differences were established after the knowledge test taking in relation to the teachers, who taught by the planned approach in different classes.

The students of the B group solved the task no. 5 of the test before and after the exposure to teaching. When taking into account the first four parts of the fifth task, it was established that there were no statistically significant differences between the B group students before teaching and the students of the fourth grade, pertaining to the A group ($p<0.05$). On the mentioned task, the students of the fifth grade of the A group performed statistically better than the B group students prior to their exposure to teaching ($p<0.05$).

The differences between the achievements of the A group students and the B group students were established also after their exposure to teaching by the developed teaching approach. The A group students of the fifth grade ($F=21.279$) were still statistically more efficient at the no. 5a task than the B group students ($p<0.01$). At the no. 5b task the B group students ($F = 23.042$) were statistically more efficient than the fourth

Completed Table

| | SQUARE  | RECTANGLE  | QUADRILATERAL  | QUADRILATERAL  |
|---|---|---|---|---|
| NUMBER OF THE SIDES AND VERTICES | 4 SIDES, 4 VERTICES | 4 SIDES, 4 VERTICES | 4 SIDES, 4 VERTICES | 4 SIDES, 4 VERTICES |
| LENGTH OF THE SIDES | ALL SIDES ARE OF EQUAL LENGTH | THE OPPOSITE SIDES ARE OF EQUAL LENGTH | THE OPPOSITE SIDES ARE OF EQUAL LENGTH | THE OPPOSITE SIDES ARE OF EQUAL LENGTH |
| PARALLEL SIDES | TWO SETS OF PARALLEL SIDES | TWO SETS OF PARALLEL SIDES | ONE SET OF PARALLEL SIDES | TWO SETS OF PARALLEL SIDES |
| NUMBER OF THE ANGLES | 4 ANGLES | 4 ANGLES | 4 ANGLES | 4 ANGLES |
| RIGHT ANGLES | HAS 4 RIGHT ANGLES | HAS 4 RIGHT ANGLES | DOES NOT HAVE ANY RIGHT ANGLES | DOES NOT HAVE ANY RIGHT ANGLES |



| | SQUARE | RECTANGLE | QUADRILATERAL |
|---|---|---|---|
| NUMBER OF SIDES | 4 sides | 4 sides | 4 sides |
| NUMBER OF VERTICES | 4 vertices | 4 vertices | 4 vertices |
| NUMBER OF ANGLES | 4 angles | 4 angles | 4 angles |
| LENGTH OF SIDES | All sides of equal length | Opposite sides of equal length | Sides are not necessary of equal length |
| PARALLELISM | Two sets of parallel sides | Two sets of parallel sides | Sides are not necessary parallel |
| RIGHT ANGLES | All 4 right angles | All 4 right angles | Not necessary right angles |

**Figure 1.** Table in the form of a figure, underpinning the explanation that each square is a rectangle

graders of the A group ($p < 0.01$). At the task no. 5c the B group students ($F = 4.576$) were statistically more efficient than the fourth ($p < 0.01$) and the fifth ($p < 0.04$) graders of the A group. At the no. 5d task the B group students ($F = 24.29$) were statistically more efficient than the fourth graders ($p < 0.01$), the fifth graders ($p < 0.01$) and the sixth graders ($p < 0.01$) of the A group. At the no. 5d task the B group students fared better than the fourth graders of the A group ($F = 15.776$, $p < 0.01$).

When comparing the solutions of the task no. 5, provided by the B group students before and after their exposure to teaching, statistically significant differences were established in all parts of the fifth task, with the exception of the no. 5a. The students fared better in the other four parts of the task no. 5 after their exposure to teaching. If only the task no. 5 is highlighted, it can be stated that that the teaching approach was not successful only in identifying a rectangle and a square between the various quadrilaterals; namely, in the 5a task 60.5 % of the B group students were attributed the level of 0. In the entire test, the aforementioned task should be pointed out, as the highest percentage of achieved level 0.

Our main interest was to determine whether students progress to a higher level of geometric thinking, especially relating some concepts of plane geometry, if they are exposed to geometry instruction in accordance with the five stages of teaching by Van Hiele.

The B group students solved the entire second and third task very well, while the percentage of the achieved first level varies between 68 and 78 %. The aim of these tasks was to test the first level. The students performed best on the 4a task, in which they had to draw different triangles. At the mentioned task, 70.2 % of the students exhibited the second level of geometric thinking.

As in the A group, in the B group the mean level of geometric thinking, taking into account the entire test, was calculated, too. Each level was assigned a numerical value. The results of all the A group students and the B group students are presented, who solved the test, including the results of all those students that did not solve individual tasks. In this case, the student was attributed the level of zero, so he was assigned zero points. The Figure 2 presents the percentage of students in the fourth, fifth, and the sixth grade, constituting the A group and the B group students (students in the fourth grade), considering the entire test.

From the Figure 2 it is evident that between the A group and the B group of students as well as between the grades of the A group of students there are no significant differences in the achieved levels, as to the sum of the points achieved at the level of zero, the level of 1.5, and the second level of geometric thinking. Thus, for example, only 1.2 % of the B group students and 1.4 % of the A group students, are on the second level respectively. The differences are most noticeable at the level of 0.5 and the first level of geometric thinking. Most of the students (73.3 %) on the level of 0.5 are from the fourth grade of the A group, whereas the least of them are from the sixth grade of the A group (50.5 %) and the B group (51.4 %). At the first level, the B group is most notable, as there are 44.3 % of students on the mentioned level, while there are as many as half fewer fourth graders of the A group on the same level. Among other things a statistically significant difference between the fourth graders of the A group and B group ($F = 3.148$, $p < 0.01$), in favor of the B group, was established, which attests to the successfulness of the

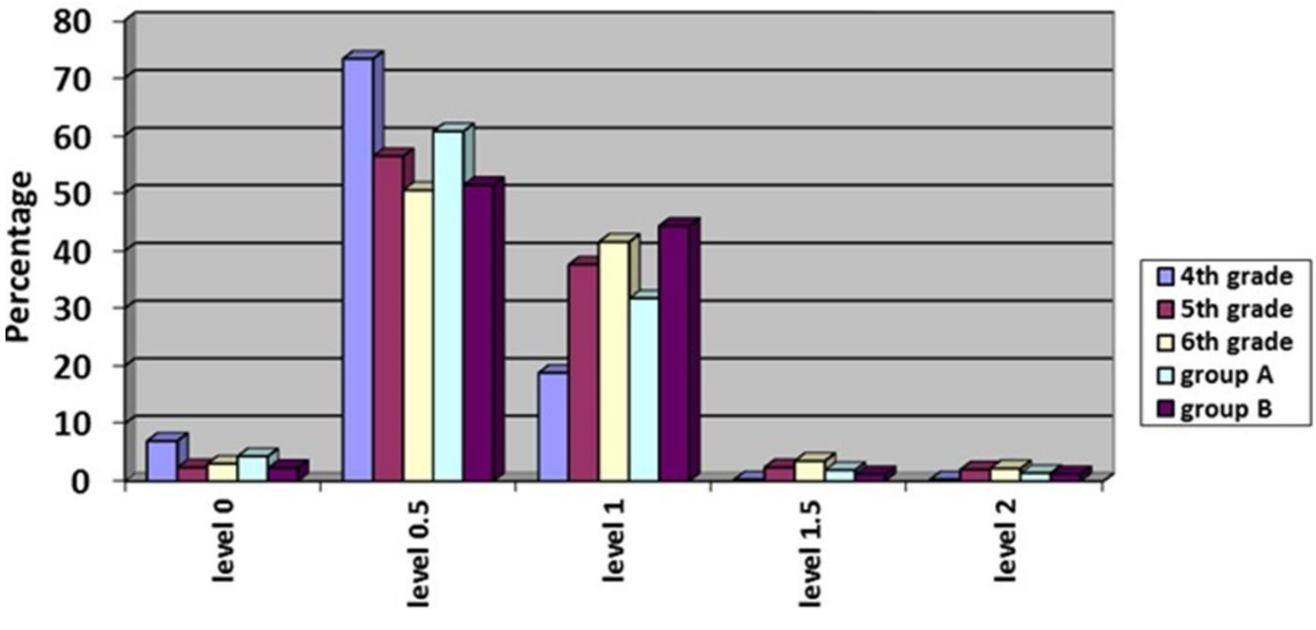



**Figure 2.** The levels reached with regard to the amount of points scored (of all the students of the A & B groups)

designed teaching approach. The results of the B group students can, therefore, be compared with the results of one or two year older students (the results of the fifth and sixth graders of the A group); namely, the results do not indicate that the students of the mentioned grades scored statistically better than the B group students.

Like Halat (2006) in this research no statistically significant differences between the genders were identified, considering the entire test. However, a statistically significant gender difference was found in three separate tasks, in which boys performed better than girls in each of them.

The research question about the effectiveness of the learning approach, can be answered in an affirmative manner, as at 12 out of 19 tasks the B group students performed statistically better than fourth graders of the A group; at seven tasks they outperformed the fifth graders of the A group and at three tasks they fared better than the sixth graders of the A group. The fourth graders of the A group were never statistically more efficient than B group students; the fifth graders were better at one task, and the sixth graders at two tasks. In addition, the B group students were statistically more efficient than the fourth graders throughout the test in the case when those students who did not solve the tasks, were ranked at the level zero. In the event that those students, who did not solve the tasks, were excluded, the B group students performed statistically better than the students in the fourth and fifth grades of the A group. Thus, it is possible to compare their result with the results of the sixth graders, because they were not statistically more efficient than the B group students, who attend the fourth grade.

It can be concluded that the developed teaching approach, the aim of which was to accelerate the transition of students to a higher level of geometric thinking, was efficient. However, it was also established, that students at certain tasks, e.g. at the removal of squares and rectangles of different quadrilaterals (no. 5a task), did not progress following their exposure to teaching. According to the established facts, greater emphasis should be put on teaching of various quadrilaterals, possibly also on their naming.

## CONCLUSION

Based on solving geometrical problems the Dutch mathematician Pierre M. Van Hiele developed a theory, formulated and precisely described five levels of geometric thinking. His findings encouraged a number of researchers to delve into the issue, and influenced, inter alia, the change of the curricula around the world.

The aim of the study was to establish the level of geometric thinking in students of the second triad in Slovenia, aged 9 to 11 years. 782 students of the fourth, fifth and sixth grade (the A group) took the test. It was found that the majority of students were at the transition between the zero level and the first level of geometric thinking and that students had problems using appropriate terminology in geometry. Therefore, in the developed teaching approach the emphasis was put on eliminating these problems and on acquiring the knowledge on a hierarchy between certain geometrical concepts. The designed learning approach was based on the approach developed by Dina Van Hiele - Geldof and Pierre Van Hiele, as the learning steps, the using of different materials, concrete experiences and discussions were taken into account.

The results showed, among other things, that the students who underwent the teaching approach performed better on 12 of 19 tasks than the students of the same age that were not provided with such teaching. They were statistically better on seven tasks than one year older students, and on three tasks they fared better than the students, who were two years older. The fifth graders who did not participate in teaching by the developed approach performed better than the fourth grade students who were exposed to the developed only on one task, while the sixth graders performed better on two tasks. These data, as well as the rest of the obtained data, show that the fourth graders who participated in the teaching by the designed approach solved the tasks at the level of the students of the fifth and sixth grades.

Although the teaching by the developed learning approach did not last for a long period (15 lessons), it was established that the use of terminology in geometry greatly improved, which was, inter alia, the aim of the teaching. It would be sensible to introduce the created teaching approach in the higher grades of elementary school, because it was found that also the students of the sixth grade experienced problems using appropriate terminology in geometry, as well as with the hierarchy of concepts.

The research results are an indicator for teachers, on which level of geometric thinking are students of the second triad, which content poses most problems to them and where they perform successfully. A teaching model to promote higher levels of geometric thinking is shown. Students should be encouraged to develop higher levels of geometric thinking by solving the tasks, at which specific content should be explained, described, understood, e.g. the hierarchy between concepts, further, the conclusions on the basis of the available data should be drawn, the new knowledge applied, but also their solutions justified. Also, such geometric problems should be addressed, to which there are either numerous possible solutions or no solutions at all. In view of the fact that, in the textbooks, there are virtually no tasks that can be solved on the second level, teachers should provide students with the opportunity to solve such tasks, as well. In addition, teachers should place greater emphasis on teaching a hierarchy between geometrical concepts.

Our research has shown that teachers in Slovenia should implement their teaching of geometry with the problems which demand thinking at the descriptive level, problems which have more than one solution and problems without solutions. We have also found out that students had problems with the terminology in geometry which caused difficulties when they were asked to explain their solutions. The overview of the textbooks in Slovenia has shown that simple geometrical tasks prevail without encouraging the teachers to use different material and discussion when teaching geometry. The units of arithmetic and algebra are more in favor of contemporary research findings in terms of problem solving, open ended questions, using different strategies and material for teaching. There is no special attention given to establishing hierarchies between geometrical concepts: the relations are expressed at the surface level with no consideration of students' deeper understanding. We believe that teachers can contribute greatly to this area of teaching by their own understanding of geometrical concepts.

We also believe that some other countries face the same problems in teaching geometry and might therefore find our teaching approach and research findings applicable to their situation.

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◈◇◈

**Appendix**: Test 1

## GEOMETRIC SHAPES

**Encircle.**

I'm a boy.  I'm a girl.

I'm in a:  4th grade.  5th grade.  6th grade.

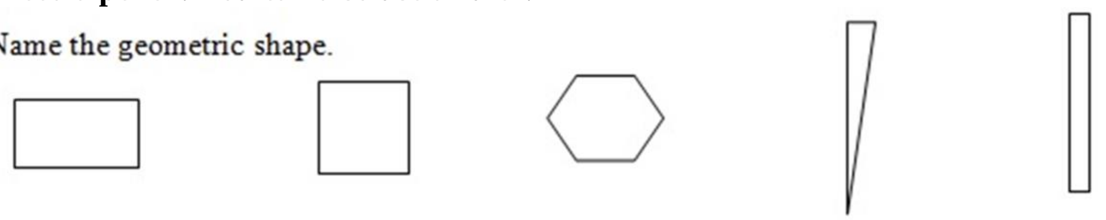
In the previous school year:

a) I was in a 3rd grade[1].
b) I was in a 4th grade and my final grade was 1 2 3 4 5 (encircle grade)[2].
c) I was in a 5th grade and my final grade was 1 2 3 4 5 (encircle grade).

In this school year I expect that my final grade in mathematics will be 1 2 3 4 5 (encircle grade).

**You need a pencil. You can also use a ruler.**

1. Name the geometric shape.



________ 1 a  ________ 1 b  ________ 1 c  ________ 1 d  ________ 1 e

2. Encircle a shape that in your opinion doesn't belong to the others in a row. Explain your decision.

2a  Why the shape doesn't belong to the others?

____________________________________

2b  Why the shape doesn't belong to the others?

____________________________________

2c  Why the shape doesn't belong to the others?

____________________________________

[1] There are no numerical grades in Slovenia till the 4th grade of primary school.
[2] Grade 5 is the best grade.

3. Draw a quadrilateral that has only one set of parallel sides.

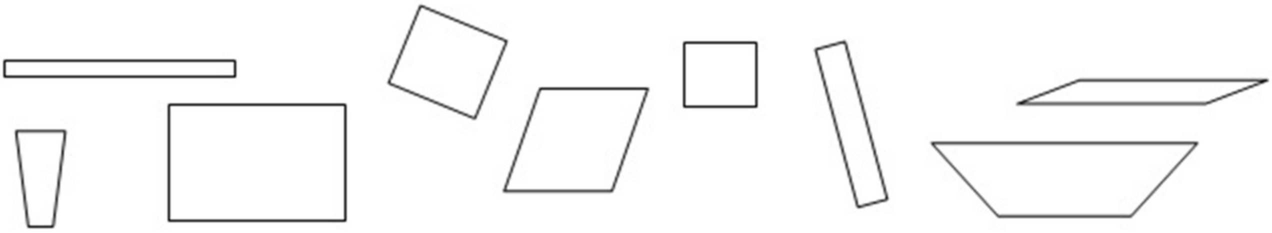
4a Draw a triangle.

Draw another triangle which is different from the previous one.

Draw another triangle which is different from the previous two.

4b How many different triangles could you draw? ________________________________

5a Write letter K in squares and letter P in rectangles.



5b List as many differences between a rectangle and a square as you can. _____________

___________________________________________________________________________

___________________________________________________________________________

5c List as many similarities between a rectangle and a square as you can. _______________

___________________________________________________________________________

___________________________________________________________________________

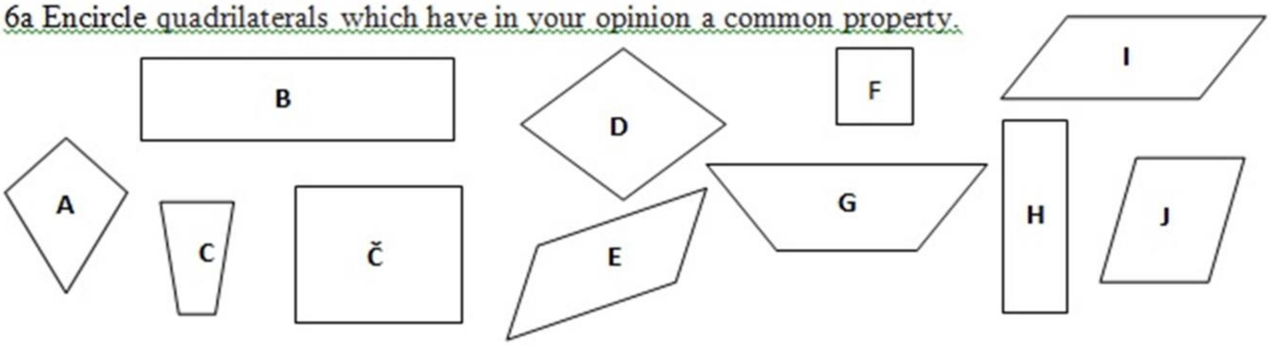
5d Is a square rectangle?_________________________

Explain why do you think so? ______________________________________________
____________________________________________________________________

5e Is a rectangle square? ____________________

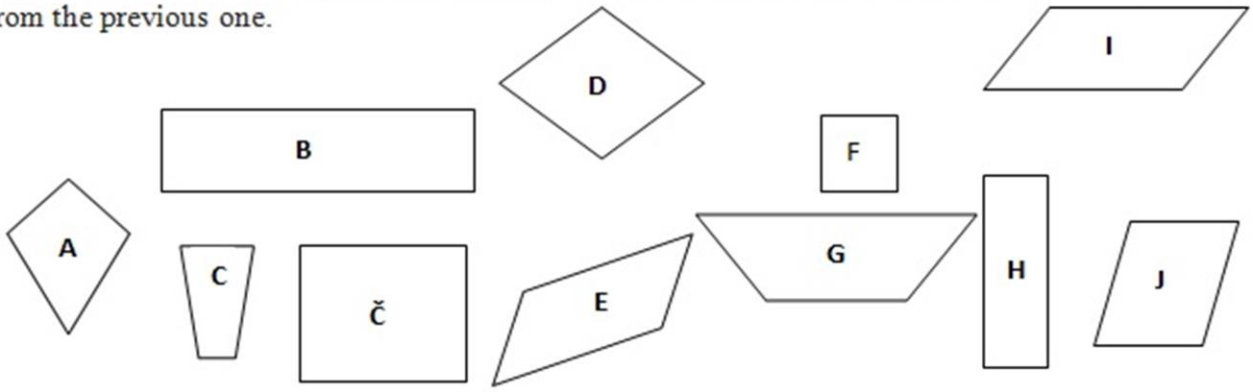
Explain why do you think so? ______________________________________________
____________________________________________________________________

6a Encircle quadrilaterals which have in your opinion a common property.



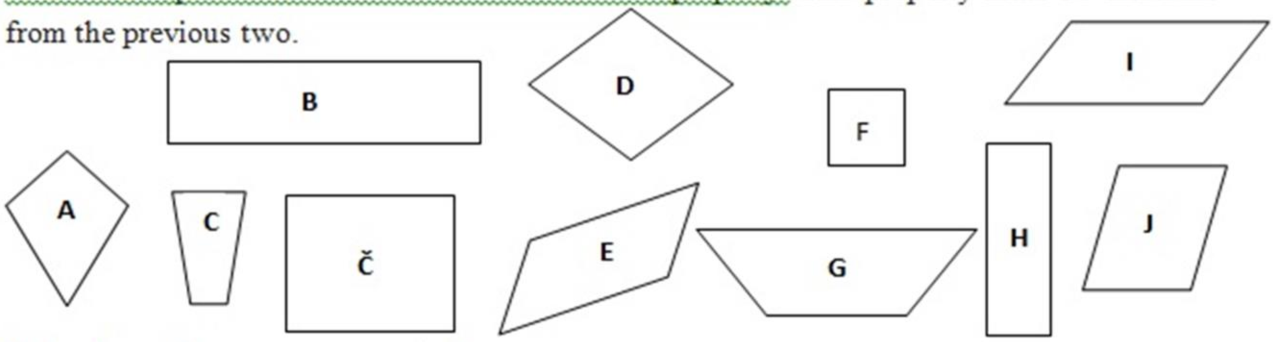

Write down this common property? ________________________________________

6b Encircle quadrilaterals which have a common property. This property must be different from the previous one.




Write down this common property? ________________________________________

6c Encircle quadrilaterals which have a common property. This property must be different from the previous two.

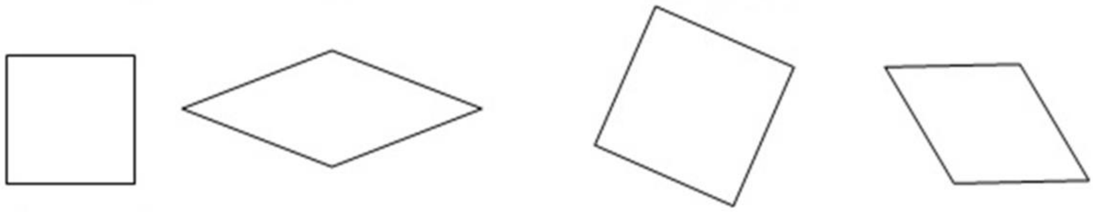



Write down this common property? ______________________________

7. Parallelogram[3] is quadrilateral that has two pairs of opposite sides parallel. Name one.

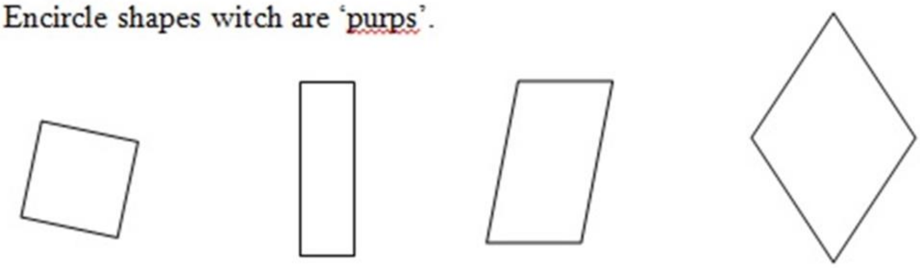
______________________________

8. For this shapes we came up with the name. These are 'purps'.



These aren't 'purps'.

8a Encircle shapes witch are 'purps'.



8b Draw three 'purps'.

[1] Children were not familiar with the name parallelogram. We wanted to know if they understand the description and write the names of the shapes they already knew (square, rectangle).

8. c Describe a purp'.________________________________________________________________
________________________________________________________________
________________________________________________________________

8d Is a 'purp' also square? __________________
Why do you think so? ________________________________________________________________
________________________________________________________________