

Implementing the Final Stage of Working with the Planimetric Problem While Teaching as a Means of Improving Geometry Knowledge of Schoolchildren

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ABSTRACT

The issue under study is urgent today because there is a necessity for students to develop skills in working with the mathematical problem at the final stage of its solution in order to get excellent results while learning geometry and when encouraging intellectual and personal development. The aim of the research is to develop the theory and methodology of the final stage of working with planimetric problems as a means of improving the quality of geometry knowledge of schoolchildren. The key research method of the issue is match making between the components of the final stage of working with the mathematical problem and their corresponding operations. The research has resulted in defining the structure of the final stage of working with mathematical problems. It allowed to perform a certain set of operations composing the skill of working with the mathematical problem at the final stage of its solution. The article shows the technique for composing special tasks in order to form operations corresponding to the final stage of working with mathematical problems. It is proved that students' ability to carry out the above described stage of solving a mathematical problem helps them to get excellent results while learning geometry. The author's technique of teaching students to work with the mathematical problem at the final stage of its solution, proposed in the article, can be used by mathematical teachers in their practical work, by the authors of resource books for students and teachers and by students of pedagogical universities while doing their special courses.

Keywords: mathematical problem, final stage of working with the mathematical problem, technique, results of teaching geometry, system of tasks

INTRODUCTION

The Relevance of the Research

Current stage of mathematical teaching aims at promoting student a) to eagerly get profound and deep knowledge and b) to gain skills of applying it in a conscious and creative way (Russiyskaya Gazeta, 2013). Geometry has considerable potential for that. However, teaching experience demonstrates a low level of Geometry knowledge and skills in middle school students. It can be due to a) the relative complexity of the subject compared to other subjects of the mathematical discipline and b) less time for Geometry learning. The following question is still urgent: how to ensure a high level of knowledge in students and increase it? Therefore, the issue of intensive Geometry teaching in middle school is being prioritized. It particularly concerns solving planimetric problems as a basic learning activity when students acquire basic geometric concepts and facts and develop logical thinking, heuristic

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Contribution of this paper to the literature

- The proposed technique of solving the mathematical problem at its final stage enables students to gain new knowledge of the subject, enhances a comprehensive knowledge of Geometry and is the basis for getting excellent educational results.
- For the first time the authors of the article defined the structure of the final stage of solving a mathematical problem, identified operations corresponding to each aspect of the stage, developed the technique forming these operations through the unique system of tasks and described principles of building the system of tasks.
- The technique described in the article makes it possible for teachers to use mathematical problems of school books instead of specifically selected tasks hardly ever mentioned in conventional methodology of teaching Mathematics. Thus, mathematical teachers can apply this technique on a regular basis.

and research skills and creative abilities. Hence, the final stage of solving the mathematical problem is of key importance because its implementation involves the acquisition, revision, systematization and compilation of the information learnt and the discovery of new knowledge by school students (Galushkin, 2018; Kryukova et al., 2017; Kvon et al., 2018; Mutavchi et al., 2018; Potapova et al., 2018). Various aspects of using the final stage of solving the problem in teaching mathematics are widely discussed in the scientific and methodological literature, the works of famous mathematicians, methodologists and teachers. Despite different aspects of the papers analyzed in the present research, they all have a common thesis, namely: the final stage of solving the mathematical problem is a necessary and significant part of the solution and has considerable potential for teaching, developing and educating students and for improving Mathematics teaching. At the same time studying the experience of mathematical teachers shows that the capacities of the final stage of solving the mathematical problem are not fully used in school teaching. Many teachers pay no attention to this stage or think that it is just enough to obtain the answer to the problem (Akishina et al., 2017). One of the reasons is the lack of the technique of solving a mathematical problem at its final stage in the theory and methodology of teaching problem solving. The need for such a technique was mentioned by Kolyagin (1977), Kanin and Nagibin (1982) and Sarantsev (1995, 2002), the leading specialists in mathematical teaching techniques. Thus, there is a contradiction between the considerable potential of the final stage of solving the mathematical problem to get excellent results in teaching students and the technique of its applying in the teaching and learning process. The aim of the research is to show that applying the teaching technique of solving the mathematical problem at its final stage while teaching Geometry (Zelenina, 1998, 2003, 2004, 2005) may increase deep, profound and conscious knowledge in schoolchildren raising the quality of Geometry teaching. The approach described in the article does not require developing specific content of mathematical teaching and specific teaching modes and methods, extra mathematical knowledge of students, and can be implemented at every Mathematics lesson, i.e. regularly.

Purposes and Objectives of the Research

The purpose of the research is to develop theoretical and methodological basis for applying the final stage of solving the mathematical problem as a means of getting excellent results of schoolchildren while learning Geometry. The main objectives are a) to define the structure of the final stage of solving the mathematical problem and to select its operations as components, b) to explore the possibility of applying the final stage of solving the mathematical problem as a means to raise the quality of Geometry knowledge, c) to work out the teaching and learning technique of solving the mathematical problem at its final stage, and d) to identify methodological recommendations for implementing research results.

Literature Review

The analysis of methodological literature of the present study indicates that one of the most significant ways of improving Mathematics teaching at school is result-oriented regular teaching of students to solve geometrical problems taking into account the implementation of the final stage of working with them. The efficiency of the tendency is determined by the following functions of the final stage: individual gaining of new knowledge and skills, revising, generalizing and classifying by Poya (1991), Kanin and Nagibin (1982), Sarantsev (2002), Tsukar (1982), Ivanova (1992), Domkina and Lapteva (1999), Bikić, Maričić, and Pikula (2016)); mathematical education of students, developing their flexibility, initiative and quick thinking (Collis, Watson and Campbell (1993), Lester and Cai (2016)); activating mental activity of schoolchildren and developing their cognitive abilities (Kanin and Nagibin (1982), Guba (1972), Yasinovy (1974), Sarantsev and Kalinkina (1994)); working out heuristic methods (Artemov (1973), Kuznetsova (1992), Semenov (1995)) and elements of research activity (Sarantsev (1995), Tokmazov (1994), Guba (1972), Demchenkova (2000), Baranova (1999, 2003), Pleskach (2014)); reaching aesthetic potential for

Mathematics activity (Sarantsev (2003)), and stimulating creativity (Cherepanova (1964), Evnin (2000)). Thus, many researchers recognize high potential of the final stage to solve the mathematical problem in order to gain conscious, deep and profound knowledge. It contributes to getting excellent learning results. Studying the issue of applying the final stage of solving the mathematical problem in Mathematics teaching has several directions in the methodology of teaching Mathematics. One of them is *determining the contents of the final stage of solving the mathematical problem*, described by Kanin and Nagibin (1982), Kolyagin (1977), Poya (1991) and Sarantsev (1995, 2002). The second direction is connected with teaching *variability of mathematical problems*. Guba (1972), Ivanova (1992), Cherepanova (1964), Tsukar (1982), Karpushina (2006), Huang and Gu (2017), Gu, Huang and Marton (2004) select variability ways in order to compose new problems based on the solved one, give examples of such work with the problem. Gotman (1991), Olbinsky (1998, 2002) identify the ways of problem development in order to analyze the solved problem and compose new problems. Izaak (1987), Evnin (2000) consider generalization, concretization and dealing with special limiting cases to be the main research methods of the problem. The next research direction is connected with finding certain *techniques of working with the problem at its final solution stage*. Teaching analogy to students at the final stage of solving planimetric problems is described in the thesis research by Yudina (2011). Izaak (1983) devoted his works to using generalizations at the final stage of solving geometric problems. Ivanova et al. (2000) refer to the final stage of solving geometric problems when evolving the technology of developmental mathematical teaching. The manual for graduate students of pedagogical universities contains clear examples of the final stage of working with word and geometric problems updated to the lesson level. One more direction in the studies of applying the final stage of solving the mathematical problem while teaching and learning is *constructing the blocks of interconnected problems*. Sarantsev and Kalinkina (1994) establish the principles of sorting geometric problems by using the methods of obtaining knowledge. Tokmazov (1994), Nedogarok (1989), Skrabich (2005) describe the method of constructing the block of related problems. Likhota (1983), Melnik (1986), Georgiev (1988), Rukshin (1981), Goldman and Zvanich (1990) give clear examples of cycles of interconnected geometric problems for extra-curricular activity in Mathematics. Sullivan and Clarke (1992), Collis, Watson and Campbell (1993), Bahar and Maker (2015) solve so-called 'open' problems in their papers. Such a component of the final stage of solving the mathematical problem as *searching for different solution paths* is more often considered. Gotman and Skopets (2000), Mostov and Nakonechny (1976), Prasolov (1988), Ponarin (1992), Henríquez-Rivas and Montoya-Delgadillo (2016) and others studied various aspects of solving the problem in different ways, developed new types of lessons and gave a great number of specific problems solved in different ways. Sarantsev (2003), Roshchina (1996, 1998) Czarnocha and Baker (2016) expressed a completely new viewpoint on the final stage of working with the problem, namely: understanding the final stage of solving the mathematical problem as *a means of implementing aesthetic mathematical potential* and involving students into creative activity. It should be noted that the researchers study separate aspects of applying the final stage of solving the mathematical problem when teaching Mathematics and do not consider this stage as an integral phenomenon. It results in the lack of working techniques for a problem at its final solution stage in the theory and methodology of problem solving teaching. But these techniques might ensure the regular application of capabilities of this stage at Mathematics lessons.

MATERIALS AND METHODS

The technique for the final stage of solving the mathematical problem is based on the correspondence, established during the research, between the operations of the given stage and the tasks developing these operations. The operations, corresponding to the final stage of solving the problem, are included into its structure which determines the principle of the task composition developing these operations. When solving the problem at its final stage students' activity has two phases: reflexive and transforming. The reflexive phase returns students to the implemented stages of a problem solution and their understanding. The transforming phase directs students' activity to further problem development. Each of the phases, selected in the structure, has their corresponding components of the problem solution at its final solution stage. Each component comprises a block of corresponding operations. The flow-charts given below depict a complete description of the operations in each block (Figure 1, 2).

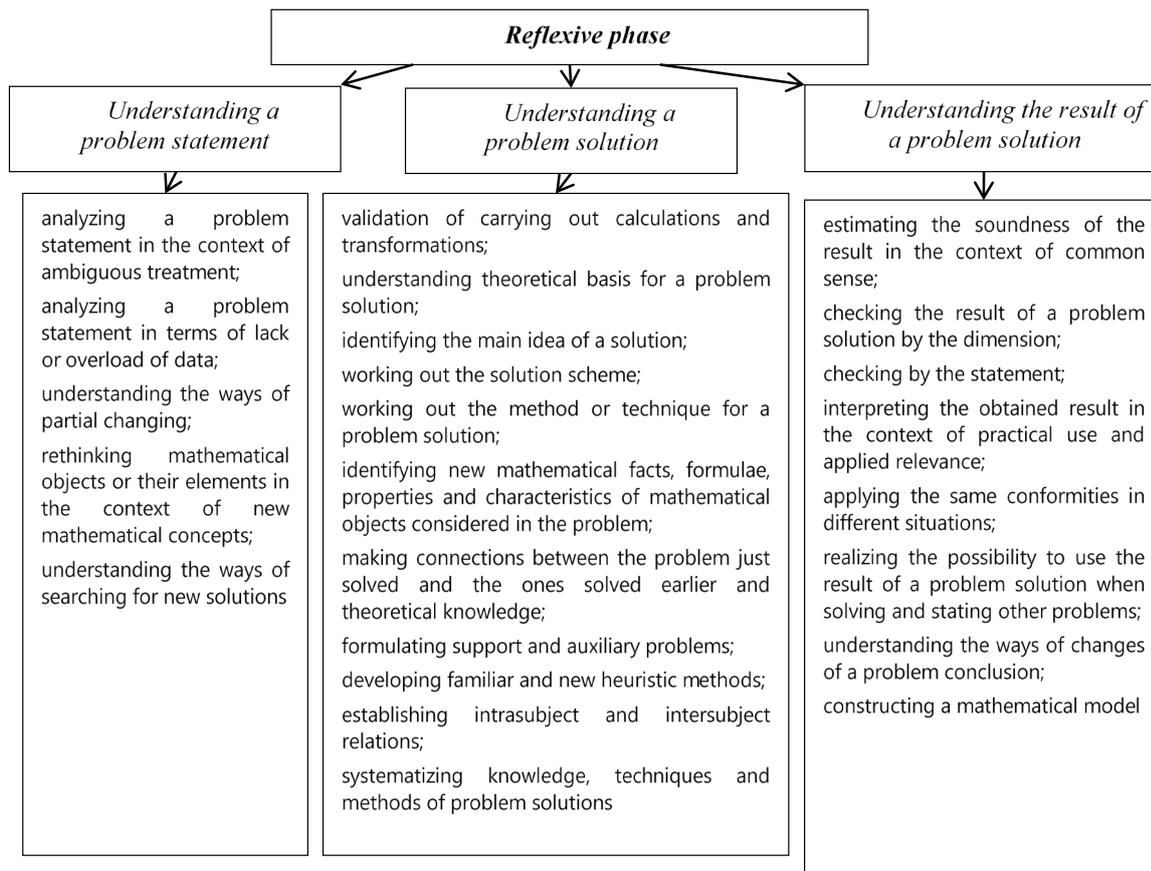


Figure 1. Reflexive phase of the final stage of working with the mathematical problem

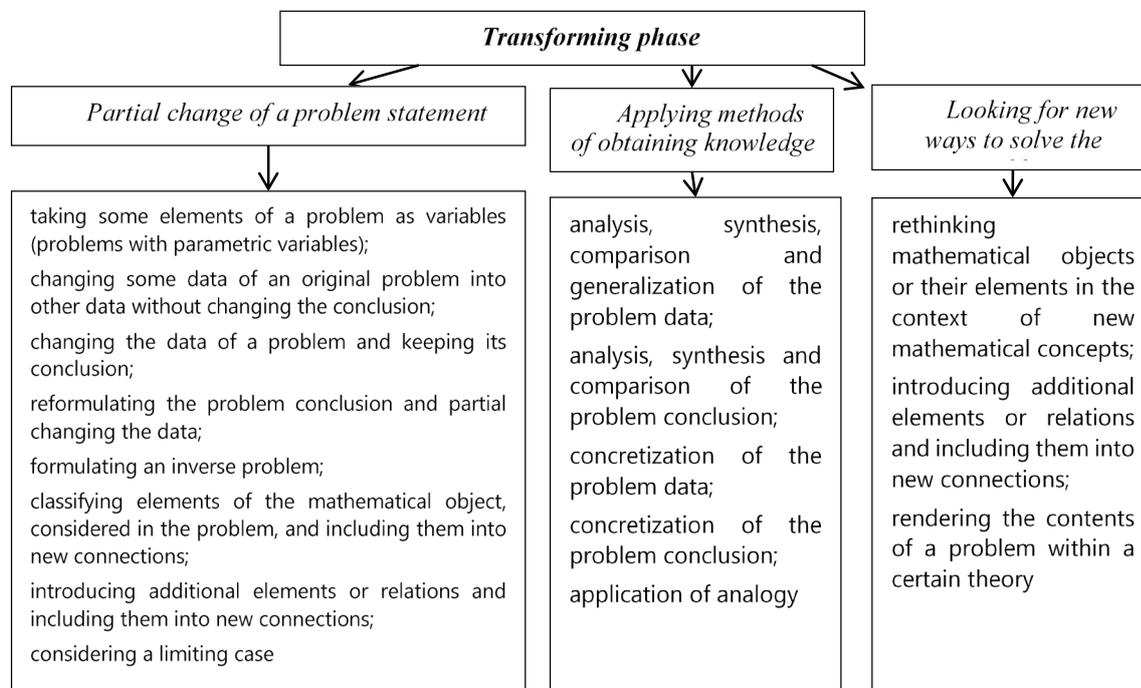


Figure 2. Transforming phase of the final stage of working with the mathematical problem

Thus, school teaching and learning technique for solving a planimetric problem at its final solution stage involves designing a special system of tasks aimed to select and form the operations corresponding to the final

solution stage. The peculiarity of such tasks is that the problem statement has not only such familiar phrases as 'Solve the task', 'Find out', 'Prove', etc. but sets students for further work, i.e. has instructions to fulfill some operation corresponding to the activity implementing the final stage of a problem solution. The contents of these tasks can coincide with the contents of students' school books partially or completely. The difference is in providing the information and in the teaching and learning technique respectively. The technique for arranging these tasks is described below. The arrangement is some modification of the tasks from school books and problem books for schoolchildren; therefore, they naturally fit into the system of tasks from school books and can be prepared by teachers after necessary methodological training. Let us show you the task groups corresponding to the operation blocks defining the structure of the final stage of solving the mathematical problem. The system of the tasks, methodological recommendations and patterns of arranging the tasks are described in other papers (Zelenina, 2004).

Tasks to Understand a Problem Statement

The aim of fulfilling these tasks is to teach students to use the problem statement for its further development. Achieving the aim means a) developing the ability to identify links between the objects given in the problem statement and between the problem data and requirements and b) developing the skill to gain the information from the problem statement and conclusion. Therefore, fulfilling the following tasks can help students: a) to devise the problem structure, b) to make conclusions from the tasks given, c) to reformulate the problem requirement, d) to make an assertion opposite the given one, e) to select the conclusion (data) corresponding to the given requirement and f) to analyze the problem statement in the context of ambiguous treatment, lack or overload of data and possibilities to set new or different links between the objects.

Tasks to Understand a Problem Solution

The main aim of these tasks is to develop skills of making, confirming and applying the conclusions while solving the problem. Mathematical problems help students to achieve this aim. When solving the problems students: a) form the theoretical basis, algorithm and main idea of problem solving, b) recognize and fix new mathematical facts, formulae and properties of mathematical objects considered in the problem, c) find the technique or way for solving a problem, d) formulate the heuristics appearing while solving the problem and support and auxiliary problems, e) establish intrasubject and intersubject relations and make connections between the solved problem and the ones solved earlier and theoretical knowledge.

Tasks to Understand the Result of a Problem Solution

The main aim of these tasks is to develop skills of checking the result of the solution, its further applying for developing a task situation and including it into the knowledge system to further use for solving other problems or practical tasks.

Therefore, the following tasks are to be included into the task system: a) estimating the soundness of the result in the context of common sense and checking by the dimension and the statement, b) interpreting the obtained result in the context of practical use and applied relevance, c) applying the same conformities in different situations, d) realizing possibilities to use the result of a problem solution when solving and stating other problems.

Tasks to Partially Change a Problem Statement

The main aim of these tasks is to develop skills of creating a new problem based on the solved ones by partial changing of its statement. The following tasks can help to achieve this aim. When doing these tasks students: a) completely or partially change problem data (requirements) keeping its requirements (data), b) identify or introduce new elements of the objects considered in the problem and include them into new connections, c) take some elements of a problem as variables, and d) formulate and solve inverse problems. The tasks of this type are closely connected or often flow organically from the tasks of first three groups. This is due to the fact that the transforming phase results from the reflexive one, as understanding the statement, search, process and result of the problem solution ensures a deep understanding of ways and methods of transformation and further development of the problem. All the above-said can be also referred to the tasks of the following two groups.

Tasks to Apply Methods of Obtaining Knowledge

The importance of acquiring the methods of obtaining knowledge for mathematical development of students is difficult to overestimate. Analysis, synthesis, abstraction, analogy execution, comparison and generalization are an integral part of creative mathematical activity. The skill of considering special and limiting cases and further

hypothesizing are of great significance. The tasks of this group aim at teaching school student to compose new problems based on the above described.

Let us take some problem as the generalization of some original problem if the last one is a special or limiting case of the first problem. The experience shows that time and effort spent on considering examples of generalizing mathematical conformities do not go to waste. Schoolchildren learn to see something general in quotients, acquire skills of independent expanding and extending their knowledge and get some ideas of methods of obtaining knowledge. When generalizing, the diverse analysis and arrangement of learning material are performed. It develops intellectual power, cognitive abilities and creative activity of students.

Tasks to Look for New Ways to Solve the Mathematical Problem

Doing the tasks of this group aims to develop students' skills and demand for solving a mathematical problem in different ways. Achieving the aim means a) to develop skills of rethink mathematical objects or their elements in the context of new mathematical concepts, b) to select, introduce additional elements or relations and include them into new connections, c) to render the contents of a problem within a certain theory. The tasks of this group must contain the comments directing another way of the problem solution. In this case doing these tasks may ensure developing a certain amount of ideas and ways to find different solutions.

Thus, the technique for working with a mathematical problem at its final solution stage is based on tasks of a particular type composed from ordinary problems through stimulating students to carry out some operation corresponding to the described stage of the problem solution.

RESULTS

The developed method of teaching students to work with the problem at the final stage of its solution in the process of studying geometry at the secondary school was experimentally tested in Kirov schools: secondary school No. 27 with in-depth study of individual subjects and secondary schools No. 48, 70. Approximately equal classes were selected in order to check the efficiency of the teaching and learning technique. Before selecting, their success level (the results of their tests and current and assessment grades) had been taken into account. For example, at the beginning of the school year the average quality level of knowledge in the experimental (EG) and control (CG) groups was almost the same. When moving up to the 9th year the range of grades in the experimental group was the following: "5" – 44 %, "4" – 48 %, "3" – 8 %, and in the control group it was: "5" – 42 %, "4" – 48 %, "3" – 9 %.

Comparison of the development level of the skill of conducting the final stage of working with a mathematical problem and the quality level of students' knowledge in both groups was based on the results of their final diagnostic tasks. Diagnostics was performed as a test and as a specific task to assess the level of using the operations corresponding to the final solution stage. The assessment time was the end of the school year after the curricular material had been learnt and a definite working period of developing skills of solving a mathematical problem at its final stage had ended. To study the effect of the developed teaching and learning technique for solving a mathematical problem at its final stage, students were given an 8-9-Grade Plane Geometry test. The work aimed at comparing the knowledge level in school students of experimental and control classes. The test had eight problems. Each problem was assessed with a particular amount of points (pointed in round brackets) depending on the complexity level. The time to complete the test was 1.5 hours. With correct and complete solutions to all eight problems, the student received 12 points, with all solutions were incorrect he got zero. Thus, the results of the work, expressed in the number of points, were distributed from 0 to 12.

- Problem 1.** Interval BK in triangle ABC is a bisector. A line passes through point K cutting side BC at point M so that BM=MK. Prove that KM and AB are parallel (1 point).
- Problem 2.** Two circles of equal radii are externally tangent at point S. Two secant lines a and b drawn through point S cut circle 1 at points A and B and circle 2 at points C and D. Prove that intervals AB and CD are equal (2 points).
- Problem 3.** There is a rhombus with the side equal to 1 and an acute angle at vertex equal to $\frac{\pi}{6}$. Point K is on side BC where BK=KC. Find the distance from vertex B to line AK (2 points).
- Problem 4.** Triangle ABC with acute angle C has altitudes AE and BD. Prove that $\triangle ABC \oslash \triangle EDC$ (1 point).
- Problem 5.** Prove that the midpoints of a convex quadrilateral are the vertices of a parallelogram (1 point).
- Problem 6.** There are point K and L on sides BC and DA of convex quadrilateral ABCD. Solve $S(BKDL):S(ABCD)$ if K and L are midpoints (1 point).
- Problem 7.** Equilateral triangles are constructed externally on the sides of a parallelogram. What is the type of a quadrilateral with the vertices not lying on the sides of the parallelogram? (2 points).

Table 1. The results of the test

| Points | | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
|---------------------------------------|---------------------|-----|-----|-----|-----|----|----|----|----|----|---|---|
| Number of students getting the points | in CG (81 students) | 4 | 4 | 7 | 9 | 11 | 12 | 14 | 10 | 9 | 1 | 0 |
| | in EG (80 students) | 7 | 8 | 8 | 15 | 16 | 9 | 7 | 6 | 4 | 0 | 0 |
| | total | 11 | 12 | 15 | 24 | 27 | 21 | 21 | 16 | 13 | 1 | 0 |
| Cumulative frequency | | 161 | 150 | 138 | 123 | 99 | 72 | 51 | 30 | 14 | 1 | - |

$n_1=81, n_2=80, N=161$

Table 2. Distribution of sample values

| | Sample 1 X[i] | Sample 2 Y[i] |
|----------------------|---------------|---------------|
| More than 8 | 38 (A) | 24 (B) |
| Less or equal than 8 | 42 (C) | 57 (D) |
| | 80 | 81 |

Problem 8. Squares ACMP, CBB1C1, ABNK are externally constructed on each leg of right equilateral triangle ABC (AC=CB) and on its hypotenuse. Solve the area of triangle OO1O2 (O1 is the center point of AKNB, O2 is the center point of CBB1C1, O is the center point of ACMP) (2 points). **Table 1** shows the results of the test.

Data Collection and Analysis of Results

To assess the differences in the results obtained of the control work in EC and QC groups (**Table 1**), we use the median criterion (Grabar & Krasnyanskaya, 1977). The median criterion is used to identify the differences in the state of a property in two sets based on a study of the members of two samples from these sets. In this case, the indicator of the tendency for a certain changing property is the median of the change in the property under study in each of the samples. In our case, the property under study is the quality of students' knowledge.

We have two sets: QC and EC. Let X characterizes the state of the studied property in EG group, and Y in CG group. Let $X[i], Y[i]$ be the results of measuring the properties of students in EG and CG respectively. In other words, $X[i], Y[i]$ is the number of points obtained by the i-th student. We take for $N=n_1+n_2$ the volume of the combined sample, where n_1 is the number of students of EG, n_2 is the number of students of CG. $N = 161$. Determine the median of the sample, that is, a value that is greater than 50% of the changes in $X[i], Y[i]$ and less than is also 50%. Thus, if all $X[i], Y[i]$ are arranged in ascending order, then the median will be equal to: $m=Z[(N+1)/2]$, if N is odd, or $m=(Z[N/2]+Z[N/2+1])/2$, if N is even, where $Z[i]= X[i]$ or $Y[i]$: КК и ЭК.

The number of students in both samples is 161 people - the number is odd. It means that the median is numerically equal to the value that stands at 81th place. In an ordered series of measurements based on the results of two samples, this value is 8. Using the data in the table, we will distribute the values of both samples into two categories: more median (> 8) and less than or equal to the median (≤ 8) (**Table 2**).

The tested hypothesis: H_0 (null hypothesis): the medians of the distribution of students in terms of the number of points received for the test, are the same in the aggregate of students of EG and CG, i.e. $m_1=m_2$.

H_1 (alternative hypothesis): $m_1 \neq m_2$.

We find the statistics value of the median criterion according to the formula (Grabar and Krasnyanskaya, 1977).

$$T_{\text{набл.}} = \frac{N(|AD - CB| - \frac{N}{2})^2}{(A + B)(B + D)(A + C)(C + D)}$$

In our case, $T_{\text{набл.}}=4,71$. For the level of significance $\alpha = 0.05$ and one degree of freedom, we find $T_{\text{крит.}}=3,84$. Since the inequality $T_{\text{набл.}} > T_{\text{крит.}}$ ($4,71 > 3,84$) is true, then in accordance with the decision rule when using the median criterion (Grabar and Krasnyanskaya, 1977), the null hypothesis is rejected and at the significance level $\alpha = 0.05$, the alternative hypothesis is accepted: students' distribution medians according to the number of points for the performance of a test are different in the aggregates of students in the experimental and control classes. At the same time, the results of students of EG have a tendency to be higher than the results of students of CG, that is, the quality of knowledge is higher in EG.

It should be noted that students' results in the experimental group tend to be higher than those of the students in the control group, i.e. the knowledge quality is higher in the experimental group.

During the experiment, when doing the test the students were given the problems solved by themselves or explained by the teacher to assess the level of acquiring the operations corresponding to the final stage of solving a mathematical problem. At the end of the problem solution the students were asked to analyze their solving process, namely: understand the statement, search, process and result of solving the problem and make corresponding

conclusions (reflexive phase). Then, with the results obtained at the reflexive phase they were asked to think of other problem solutions and to formulate new problems based on the solved one (transforming phase). This work aimed to assess the competency level of applying the operations corresponding to the reflexive and transforming phases of the final stage of problem solving. Let us demonstrate you the problems indicating the assessed operations implementing the final stage of solving a mathematical problem and give brief comments.

Problem 1. Interval BK in triangle ABC is a bisector. A line passes through point K cutting side BC at point M so that $BM=MK$. Prove that KM and AB are parallel (identifying the theoretical basis for its solution, selecting the heuristic rule, formulating an inverse problem). The analysis of searching the problem solution shows that to prove parallelism it may be useful to consider the interval, being the elements of a triangle, as direct and then to use the test for parallelism of straight lines. Understanding the problem statement reveals that it consists of two elements and the requirement consists of one element; therefore, one can interchange each element of the statement and the requirement. Thus, we shall obtain two inverse problems.

Problem 1'. In triangle ABC a straight line is drawn through point K of side AC. The line is parallel to side AB and cuts side BC so that $BM=MK$. Prove that BK is a bisector of the triangle.

Problem 1''. Interval BK in triangle ABC is a bisector. A line passes through point K. The line is parallel to side AB and cuts side BC at point M. Prove that $BM=MK$.

Problem 2. Two circles of equal radii are externally tangent at point S. Two secant lines a and b drawn through point S cut circle 1 at points A and B and circle 2 at points C and D. Prove that intervals AB and CD are equal (formulating new problems based on the change of a problem conclusion while retaining the data). The analysis of the statement, search and process of solving this problem reveals an opportunity to formulate new problems at its final solution stage based on the conclusion change while retaining the statement.

Problem 2'. Two circles of equal radii are externally tangent at point S. Two secant lines a and b drawn through point S cut circle 1 at points A and B and circle 2 at points C and D. Prove that straight lines AB and CD are parallel.

Problem 2''. Two circles of equal radii are externally tangent at point S. Two secant lines a and b drawn through point S cut circle 1 at points A and B and circle 2 at points C and D. Prove that quadrilateral ABCD is a parallelogram.

Problem 2'''. Two circles of equal radii are externally tangent at point S. Two secant lines a and b drawn through point S cut circle 1 at points A and B and circle 2 at points C and D. Prove that the midpoints of AB and CD and point S lie along a straight line.

Problem 3. There is a rhombus with the side equal to 1 and an acute angle at vertex equal to $\frac{\pi}{6}$. Point K is on side BC where $BK=KC$. Find the distance from vertex B to line AK (identifying the theoretical basis for the solution pattern, understanding the statement in the context of ambiguous treatment). The statement of the problem assigns us to vertex B. But at the same time we do not know whether angle B is acute or obtuse.

Problem 4. Triangle ABC with acute angle C has altitudes AE and BD. Prove that $\triangle ABC \sim \triangle EDC$ (based on understanding the statement of changing the problem data without changing the conclusion, getting new facts, formulae and object properties according to the problem solution process, selecting the elements of the object and including them into new connections, new ways of solving the problem, identifying the main idea of the problem solution, formulating heuristic prescription due to the search and process of the problem solution). Based on the similarity of triangles AEC and BDC (angle-angle) we can conclude that $\frac{EC}{DC} = \frac{AC}{BC}$ (*). In triangle ABC and EDC angle C is common and $\frac{EC}{AC} = \frac{DC}{BC}$. Hence, $\triangle ABC \sim \triangle EDC$ based on two proportional sides making equal angles. Let us check whether the case, being proved in problem 1, takes place if angle C is obtuse. Now we formulate the problem.

Problem 4'. In triangle ABC angle C is obtuse. AE and BD are altitudes. Is $\triangle ABC \sim \triangle EDC$ true? Problem 4' has the same solution as the stated.

Understanding the equality (*) achieved when solving the problem with the legs and hypotenuse of right triangles AEC and BDC makes it possible to conclude the following. Applying the main property of the proportion we have $\frac{AE}{AC} = \frac{BD}{BC} = \cos C = K$, where K is the coefficient of similarity. And we get a new case: the similarity coefficient of triangles AEC and EDC is equal to the cosine of their common angle C if angle C is acute, and $|\cos C|$ if angle C is obtuse. Quadrilateral ABED is circumscribed, as the circumcentre of right triangles AEB and ADB is the midpoint of side AB (point O). Hence, $\angle BAD + \angle BED = \angle ABE + \angle ADE = 180^\circ$ and another way of proving the similarity of triangles ABC and EDC: $\angle BAD + \angle BED = 180^\circ$ and $\angle BED + \angle DEC = 180^\circ$ (as supplementary angles). Thus, $\angle BAD = \angle DEC$. Similarly, $\angle ADE + \angle ABE = 180^\circ$ and $\angle ADE + \angle EDC = 180^\circ$ (as supplementary angles). Hence, $\angle ABE = \angle EDC$. So $\triangle ABC \sim \triangle EDC$ according to angle-angle similarity. Analyzing the search process and implementing the scheme of solving the problem in the second way we can conclude that the element determining the solution way is the circumcircle of quadrilateral ABED. Thus, at the final stage of solving this problem we have the following heuristic prescription: 'If while solving a problem you get the chance to consider a quadrilateral, think

whether it is possible to draw a circumcircle (or an incircle). It ensures comparing and assessing the angles and sides'.

Problem 5. Prove that the midpoints of a convex quadrilateral are the vertices of a parallelogram (making up and solving new problems based on concretization, identifying new properties of the problem object and including them into new connections, interpreting the result of the problem solution in the context of practical use). The analysis of the problem statement shows that we can study the type of the parallelogram obtained depending on the type of the original quadrilateral, namely: if diagonal lines of the quadrilateral are equal, the parallelogram obtained is a rhombus, and if diagonal lines of the quadrilateral are perpendicular, the parallelogram obtained is a rectangle. Besides, the area of the quadrilateral obtained is half of the area of the original one.

Problem 5'. Solve the area of the quadrilateral with its vertices being the midpoints of another quadrilateral. Practical interpretation of the results obtained at the final stage of solving the problem occurs, namely: the drawings reveal that the involute of an envelope can be constructed if only a rhombus is considered to be a quadrilateral.

Problem 6. There are point K and L on sides BC and DA of convex quadrilateral ABCD. Solve $S(BKDL):S(ABCD)$ if K and L are midpoints (making up a new task based on generalizing the stated one).

Problem 6'. There are point K and L on sides BC and DA of convex quadrilateral ABCD so that $BK:KC=LD:LA=m:n$. Solve $S(BKDL):S(ABCD)$.

Problem 7. Equilateral triangles are constructed externally on the sides of a parallelogram. What is the type of a quadrilateral with the vertices not lying on the sides of the parallelogram? (applying generalization to formulate new problems based on the solved one, making up an analogue problem). Concretization occurs in the line of considering a rectangle, a square and a rhombus as the stated parallelogram.

Problem 7' (similar to the stated above). Squares are constructed externally on the sides of a parallelogram. Study the type of a quadrilateral with the vertices being the centers of these squares.

Problem 8. Squares ACMP, CBB1C1, ABNK are externally constructed on each leg of right equilateral triangle ABC ($AC=CB$) and on its hypotenuse. Solve the area of triangle OO1O2 (O_1 is the center point of AKNB, O_2 is the center point of CBB1C1, O is the center point of ACMP) (solving the problem in different ways, generalizing the result of the problem solution).

The efficiency of the described technique is assessed by the operations carried out by students in the experimental and control groups and corresponds to the final stage of solving a problem. To make a careful analysis of the results of fulfilling the diagnostic work let us assign a number to each operation.

Reflexive phase:

- 1.1. Understanding theoretical basis for a problem solution.
- 1.2. Identifying the main idea of the solution.
- 1.3. Working out the solution.
- 1.4. Identifying new mathematical facts, formulae, properties and characteristics of mathematical objects considered in the problem.
- 1.5. Analyzing the problem statement in the context of ambiguous treatment.
- 1.6. Developing familiar and new heuristic methods.
- 1.7. Understanding the ways of partial changes of a problem statement.
- 1.8. Understanding the ways of changes of a problem conclusion.
- 1.9. Understanding the ways of searching for new solutions.

Transforming phase:

- 2.1. Changing the problem conclusion and keeping its data.
- 2.2. Changing the problem data and keeping its conclusion.
- 2.3. Formulating an inverse problem.
- 2.4. Classifying elements of the mathematical object, considered in the problem, and including them into new connections.
- 2.5. Rethinking mathematical objects or their elements in the context of new mathematical concepts.
- 2.6. Generalizing the problem data or conclusion.
- 2.7. Concretizing the problem data or conclusion.
- 2.8. Applying analogy.
- 2.9. Searching new ways of solving a problem.

The time to complete the test was 1.5 hours. Every student was given a sheet of paper with solutions to the test problems (solved on their own or written down after their teachers' explanations). Then they were asked to continue their work with the solved problem and to make appropriate notes on their sheets of paper. **Table 3** shows the results of the diagnostic work. The results show that in the control group the operations corresponding to the final stage of solving a mathematical problem are poorly developed. Schoolchildren of the experimental group demonstrate an increasing amount of some operations in the solved problem compared to those of the control group. We tend to believe this situation was caused by the lack of focused development of skills of working with the mathematical problem at its final solution stage. **Table 3** shows the results of the diagnostic work.

Table 3. The results of the diagnostic work

| | Operations corresponding to the final stage of the problem solution | CG (81 students) | | EG (80 students) | |
|--------------------|---|---|----|---|----|
| | | Number of students applying the operation | % | Number of students applying the operation | % |
| Reflexive phase | 1.1. | 73 | 90 | 76 | 95 |
| | 1.2. | 71 | 87 | 73 | 91 |
| | 1.3. | 72 | 88 | 75 | 93 |
| | 1.4. | 15 | 18 | 23 | 28 |
| | 1.5. | 11 | 14 | 29 | 36 |
| | 1.6. | 3 | 4 | 11 | 13 |
| | 1.7. | 9 | 11 | 15 | 19 |
| | 1.8. | 8 | 10 | 14 | 18 |
| | 1.9. | 11 | 14 | 35 | 43 |
| Transforming phase | 2.1. | 8 | 10 | 15 | 19 |
| | 2.2. | 8 | 10 | 14 | 18 |
| | 2.3. | 51 | 63 | 67 | 83 |
| | 2.4. | 5 | 6 | 9 | 11 |
| | 2.5. | 14 | 17 | 27 | 33 |
| | 2.6. | 2 | 2 | 4 | 5 |
| | 2.7. | 7 | 8 | 15 | 19 |
| | 2.8. | 7 | 9 | 15 | 19 |
| | 2.9. | 11 | 14 | 35 | 43 |

The diagnostics results show that students of the experimental group have a more profound knowledge of the subject and a higher development level of skills of working with the problem at its final solution stage. Thus, the experiment has proved our assumption about a positive effect of the teaching technique for working with the mathematical problem at the final solution stage on the quality of geometrical knowledge of students and their skills of solving problems. Also we have confirmed another assumption that while using conventional teaching in problem solving (without actualizing the final stage of solving the problem) a skill of working with the problem at its final solution stage is developed in some way or another, but the development level of the skill is lower compared to our focused teaching and learning technique. According to expert estimations of the teachers taking part in the experiment, the experimental students more eagerly than the control students seek individual problem solutions. Moreover, during the experiment the number of such students has increased. According to the survey questionnaire of the students, they take the approach to solve the problem and aim not just to solve the problem and get the result but arrange the final stage of working with it.

DISCUSSIONS AND CONCLUSIONS

The research has showed that the issues of implementing the final stage of solving the mathematical problem when teaching Mathematics are widely discussed in methodological literature. Many researchers recognize considerable potential of the final solution stage for increasing the quality of Geometry teaching because students at school solve problems at this stage in a comfortable and conscious way. But it is really important to regularly involve students into this activity, i.e. at every lesson. Applying the developed technique of teaching the operations of the final stage of working with the problem through doing special tasks ensures overcoming the difficulties. More important is the fact that these tasks can be based on the problems from school textbooks and they just need methodological recommendations described in the present research. Teachers must realize the importance of the final stage of solving the mathematical problem, its potential for a student's development and excellent educational results. To carry out this activity teachers need to do some theoretical course and to attend some training workshop on applying the teaching technique. These issues may be under discussion when preparing students of pedagogical universities (mathematical profile) studying the theory and methodology of teaching Mathematics, Elementary Mathematics and Geometry. Studying the developed technique may take place in the advanced training course.

Based on the analysis of methodological literature, the study of teachers' and our own teaching experience we have developed and implemented the technique for working with the mathematical problem at its final solution stage. The reflexive and transforming phases have been marked in the structure of the final stage or working with the mathematical problem. At the reflexive phase students return to the implemented stages of a problem solution and understand them. At the transforming phase students develop a problem solution. Each stage has particular components of working with the problem at the final solution stage. Each component has a set of operations corresponding to the final stage or working with the mathematical problem. Special tasks form these selected operations. Making such tasks is based on curricular problem material of school books, the instructions on doing

some operations corresponding to the final solution stage being included into the problem statement. Only small corrections of the problems from school books are to be made. The sample tasks given in the research may help teachers to compose different problems. The proposed technique needs no special teaching and learning arrangements, no changes in the Mathematics curriculum contents and no additional time for lessons. According to the experiment results, regular arrangement of focused work with the mathematical problem at its final solution stage has a significant effect on the quality of Geometry teaching and involves students into research and creative activity implementing educational, developing and bringing up potential for teaching and learning Mathematics.

RECOMMENDATIONS

The article material may be useful for mathematical teachers, teachers of additional education and university teachers being interested in increasing the level of their students' academic achievements and in stimulating their interests in the subject. The ways of enhancement of the technique involve composing sets of tasks on different topics of Plane Geometry having considerable potential for implementing their final solution stage.

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REFERENCES

- Akishina, E. M., Sudakova, Y. E., Prokopyev, A. I., Yakubenko, K. Y., Solovyeva, N. A., & Korzhuev, A. V. (2017). System of cultural experience development of humanities students in modern information media conditions. *Man in India*, 97(14), 115-127.
- Artemov, A. K. (1973). On heuristic methods for teaching geometry in school. *Mathematics in school*, 6, 25-29.
- Bahar, A., & Maker, C. J. (2015). Cognitive Backgrounds of Problem Solving: A Comparison of Open-ended vs. Closed Mathematics Problems. *EURASIA Journal of Mathematics, Science and Technology Education*, 11, 1531-1546.
- Baranova, E. V. (1999). *Methodology of applying academic studies when teaching Geometry in middle school*: PhD Thesis. Saransk: Mordovia State Pedagogical Institute named after M. E. Evsevev.
- Baranova, E. V. (2003). Special course of study «Academic studies in teaching Mathematics». *Mathematical bulletin of pedagogical institutes and universities of Volga-Vyatka region: periodic inter-university collection of scientific and methodological papers*, 7, 246-247.
- Bikić, N., Maričić, S. M., & Pikula, M. (2016). The effects of differentiation of content in problem-solving in learning geometry in secondary school *EURASIA Journal of Mathematics, Science and Technology Education*, 12(11), 2783-2795. <https://doi.org/10.12973/eurasia.2016.02304a>
- Cherepanova, T. P. (1964). Teaching the variability of a problem statement is a means of stimulating mental activity of students. *Mathematics in school*, 5, 36-39.
- Collis, K. F., Watson, J. M., & Campbell, K. J. (1993). Cognitive functioning in mathematical problem solving during early adolescence *Mathematics Education Research Journal*, 5(2), 107-123. <https://doi.org/10.1007/BF03217190>
- Czarnocha B., & Baker, W. (2016). *The Creative Enterprise of Mathematics Teaching Research: Elements of Methodology and Practice*. Rotterdam: Sense Publishers. <https://doi.org/10.1007/978-94-6300-549-4>
- Demchenkova, N. A. (2000). *Problem-searching tasks as a means of developing research skills in future mathematics teachers when learning Mathematics teaching techniques for pedagogical universities*: PhD thesis. Saransk: Mordovia State Pedagogical Institute named after M. E. Evsevev.
- Domkina, G. V., & Lapteva, T. A. (1999). One problem involves all Plane Geometry. *Mathematics*, 40, 28-30.
- Evnin, A. Y. (2000). *Studying a mathematical problem as a means of developing creative abilities of students*: PhD Thesis. Chelyabinsk: Chelyabinsk State Pedagogical University.
- Galushkin, A. A. (2018). The use of information and communications technology to foster the competitiveness of the entrepreneurial organizations within the Education and Science sector. *International Journal of Engineering and Technology (UAE)*, 7(3.14-Special Issue 14), 323-326.
- Georgiev, V. S. (1988). Experience of stimulating students' activity based on applying the cycles of problems. *Mathematics in school*, 1, 77-78.
- Goldman, A. M., & Zvavich, L. I. (1990). Academic series at mathematical lessons. *Mathematics in school*, 5, 19-22.

- Gotman, E. G. (1991). Variation of the problem about the square and its inscribed triangle. *Mathematics in school*, 1, 26-28.
- Gotman, E. G., & Skopets, Z. A. (2000). *One problem – different solutions*. Moscow: Prosveshcheniye.
- Grabar M. I., & Krasnyanskaya, K. A. (1977). *Application of mathematical statistics in pedagogical research*. Non-parametric methods. Moscow: Pedagogy.
- Gu, F., Huang, R., & Gu, L. (2017). Theory and Development of Teaching Through Variation in Mathematics in China. *Teaching and Learning Mathematics through Variation*, 2, 13-43. https://doi.org/10.1007/978-94-6300-782-5_2
- Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: A Chinese way of promoting effective mathematics learning. In L. Fan, N. Wong, J. Cai, & S. Li (Eds.). *How Chinese learn mathematics: Perspectives from insiders*. Mahwah: World Scientific, 309-347. https://doi.org/10.1142/9789812562241_0012
- Guba, S. G. (1972). *Variability of proof problems as a means of stimulating mathematical activity in students and generating their interest in the subject* (PhD Thesis). Yaroslavl: Yaroslavl State Pedagogical Institute.
- Henríquez-Rivas, C., & Montoya-Delgado, E. (2016). The Mathematical Work of Teachers in the Transition from the Synthetic Geometry to the Analytic at Secondary School *Bolema, Rio Claro*, 30(54), 45-66. <https://doi.org/10.1590/1980-4415v30n54a03>
- Ivanova, T. A. (1992). Variability of mathematical problems as a means of developing intellectual abilities in students. *Development of students when teaching and learning Mathematics: Inter-university collection of research papers*. N. Novgorod: NSPU named after M. Gorky, 6 – 22.
- Ivanova, T. A., Perevoshchikova, E. N., Grigoryeva, T. P., & Kuznetsova, L. I. (2000). *Theoretical basis for teaching Mathematics in middle school*. N. Novgorod: NSPU.
- Izaak, D. F. (1983). Generalizations of geometrical problems. *Mathematics in school*, 2, 55-57.
- Izaak, D. F. (1987). Occurrence of new problems when studying geometrical problems. *Mathematics in school*, 6, 62-65.
- Kanin, E. S., & Nagibin, F. F. (1982). *The final stage of solving mathematical problems*. Moscow: Prosvesheniye.
- Karpushina, N. M. (2006). Dynamic problems in teaching geometry. *Mathematics in school*, 3, 48-54.
- Kolyagin, Y. M. (1977). *Problems in teaching mathematics. Part I*. Moscow: Prosveshcheniye.
- Kryukova, N. I., Zakharova, A. N., Dulina, G. S., Yusupova, Z. F., Belonovskaya, I. D., & Bogdanova, J. N. (2017). Didactic features of pedagogical interaction as the basis of university education. *Man in India*, 97(3), 29-41.
- Kuznetsova, L. I. (1992). Developing heuristic methods of mental activity in students when solving geometrical problems. *Development of students when teaching and learning Mathematics: Inter-university collection of research papers*. N. Novgorod: NSPU named after M. Gorky, 33-45.
- Kvon, G. M., Vaks, V. B., Masalimova, A. R., Kryukova, N. I., Rod, Y. S., Shagieva, R. V., & Khudzhatov, M. B. (2018). Risk in implementing new electronic management systems at universities. *Eurasia Journal of Mathematics, Science and Technology Education*, 14(3), 891-902.
- Lester, F. K., & Cai, J. (2016) *Can Mathematical Problem Solving Be Taught? Preliminary Answers from 30 Years of Research. Posing and Solving Mathematical Problems. Research in Mathematics Education*. Cham: Springer
- Likhota, E. A. (1983). Variability of problem statements at extra-curricular lessons. *Mathematics in school*, 6, 81-82.
- Melnik, N. S. (1986). On interconnected geometrical problems. *Mathematics in school*, 6, 48-50.
- Mostov, A. I., & Nakonechny, M. N. (1976). Solving geometrical problems in different ways. *Mathematics in school*, 5, 44-48.
- Mutavchi, E. P., Prokopyev, A. I., Kostyleva, G. V., Blinov, L. V., Fedorov, V. V., & Polichka, A. E. (2018). Scientific-methodical resource of student training and vocational motivation development in university. *Espacios*, 39(20), 15.
- Nedogarok, G. P. (1989). *Composing geometrical problems by students as a means of generating and developing general problem solving skills* (PhD Thesis). Moscow: Moscow State Pedagogical Institute.
- Olbinsky, I. B. (1998). Development of the problem. *Mathematics in school*, 2, 15-16.
- Olbinsky, I. B. (2002). *Methodology of teaching high school students to reflexive studying mathematical problems* (PhD Thesis). Moscow: Moscow Pedagogical State University.
- Pleskach, Y. A. (2014). Academic studies on Geometry as a factor of broadening learning experience in students. *Achievements and prospects of Psychology and Pedagogics Collection of papers of the international research and practice conference*. Ufa, Russia: Aeterna Publishing.
- Ponarín, Y. P. (1992). One problem – different solutions. *Mathematics in school*, 1, 15-16.

- Potapova, S. V., Danilova, I. Y., Prasolov, V. I., Makarova, E. V., & Kryukova, N. I. (2018). Mythological foundations of ontology. *XLinguae*, 11(2), 672-685. <https://doi.org/10.18355/XL.2018.11.02.53>
- Poya, D. (1991). *How to solve the problem*. Lvov: Quantor.
- Prasolov, V. V. (1988). Several proofs of the theorem on the altitudes of the triangular. *Mathematics in school*, 1, 72-73.
- Roshchina, N. L. (1996). Solving problems in different ways is the first step to aesthetic perception of Geometry. *Mathematics in school*, 3, 17-19.
- Roshchina, N. L. (1998). *Developing aesthetic sense of students when solving planimetric problems*: PhD Thesis. Moscow: Moscow Pedagogical State University.
- Rukshin, S. E. (1981). Series problems in extra-curricular activity. *Mathematics in school*, 6, 62-63.
- Russiyskaya Gazeta. (2013). *Development Concept of Mathematical Education in the Russian Federation*. Retrieved from <http://www.rg.ru/2013/12/27/matematika-site-dok.html>
- Sarantsev, G. I. (1995). *Exercises in teaching mathematics*. Moscow: Prosveshcheniye.
- Sarantsev, G. I. (2002). *Methods of teaching mathematics in secondary schools*. Moscow: Prosveshcheniye.
- Sarantsev, G. I. (2003). *Aesthetic motivation in teaching Mathematics*. Saransk: Mordovia State Pedagogical Institute named after M. E. Evsevev.
- Sarantsev, G. I., & Kalinkina, T. M. (1994). Methods of obtaining knowledge as a means of arranging geometrical problems. *Mathematics in school*, 6, 2-4.
- Semenov, E. E. (1995). Reflections on heuristics. *Mathematics in school*, 5, 39-43.
- Skrabich, S. N. (2005). Developing the skill of composing planimetric problems based on the stated task. *Educational Technologies*, 4, 74-78.
- Sullivan, P., & Clarke, D. (1992). Problem solving with conventional mathematics content: Responses of pupils to open mathematical tasks *Mathematics Education Research Journal*, 4(1), 42-60. <https://doi.org/10.1007/BF03217231>
- Tokmazov, G. V. (1994). Dynamic problems. *Mathematics in school*, 5, 9-12.
- Tsukar, A. Y. (1982). Extra work on the problem. *Mathematics in school*, 1, 42-43.
- Yasinovy, E. A. (1974). *Composing mathematical problems by students as a means of stimulating their cognitive activity* (PhD Thesis). Yaroslavl: Yaroslavl State Pedagogical Institute.
- Yudina, N. A. (2011). *Teaching Analogy to students at the final phase of solving planimetric problems*: PhD Thesis. Volgograd: Volgograd State Socio-pedagogical University.
- Zelenina, H. A. (2003). Functions of the final stage of working with the problem. *Problems of Mathematical Education and Culture: Collection of abstracts of International scientific conference*. Tolyatti, Russia: TSU, 92-93.
- Zelenina, N. A. (1998). Creative activity of students in modern educational concepts. *Humanization and humanitarization of mathematical education in the school and university: papers of all-Russia scientific conference*. Saransk, Russia: Mordovia State Pedagogical Institute Press, 50-52.
- Zelenina, N. A. (2004). *The final stage of solving geometric problems in the main school*: PhD Thesis. Kirov: Vyatka State University of Humanities.
- Zelenina, N. A. (2005). The structure of the final stage of solving a mathematical problem. *Mathematical bulletin of pedagogical institutes and universities of Volga-Vyatka region: periodic inter-university collection of scientific and methodological papers*, 7, 205-214.