MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2025, 21(3), em2604
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/16069

In-service teachers' mathematical work on quadrilaterals and their technological knowledge

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Received 26 July 2024 ▪ Accepted 03 January 2025

Abstract
This article aims to analyze the mathematical work of teachers, associated with their technological knowledge (TK) of quadrilaterals teaching. The research subjects are secondary mathematics teachers. The study uses as a basis the technological pedagogical content knowledge model and the theory of mathematical working spaces. It follows a qualitative research methodology and a case study method. Mathematics teachers interact with digital artifacts to increase their TK in the development of their mathematical work by solving a task on quadrilaterals as their students would. This study presents the analysis of a task involving the construction of a quadrilateral by using GeoGebra. The study revealed that the mathematical work of teachers was favored by developing their TK and their confidence in the use of GeoGebra since using this digital technology implies recognizing the importance of exploring and visualizing representations of mathematical objects.

Keywords: geometry, technological knowledge, in-service mathematics teachers

INTRODUCTION

This article is part of an ongoing research project and uses aspects of Théry (2023) research to analyze the technological knowledge (TK) in the mathematical work of secondary teachers when teaching quadrilaterals. To do so, this study points out research associated with geometry teaching, the use of digital technology, and teachers' TK.

Regarding the teaching of elementary geometry in school, D'Amore and Duval (2023) state that the purpose of teaching is not only to present students with figures to see, recognize, and construct, but to use specific terms to recognize them, understand their representations, and describe them in situations. This confronts the cognitive relationship between "seeing" and "saying", where common language has limitations. Therefore, cognitive activity is more demanding in geometry than in other areas of mathematics, since figurative and discursive transformations must be carried out simultaneously.

In this sense, the literature highlights the importance of teaching geometry, as it involves processes of perception, representation, construction, and naming of geometric objects (Bonelo & Maca, 2021; Buitrago, 2023; Henríquez Rivas & Kuzniak, 2021; Marmolejo, 2021; Prior & Torregrosa, 2020). Furthermore, research carried out by Houdement and Kuzniak (1999, 2006) focused on characterizing three paradigms for teaching and learning geometry, which they called geometry I (natural geometry), geometry II (axiomatic natural geometry), and geometry III (axiomatic formal geometry).

Thanks to this characterization, Kuzniak and Rauscher (2011) point out that it is possible to analyze and understand some difficulties present in geometry teaching and learning. In addition, they also emphasize the need to articulate geometry with other areas of mathematics.

Also, Kuzniak and Nechache (2021) characterize the forms of geometric work that teachers perform when

This article is the result of an international research project and uses aspects of the second author's thesis (2023).

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Contribution to the literature

- The research provides an initial link between the theory of mathematical working spaces (MWS), to analyze the mathematical work of teachers, and the technological pedagogical content knowledge (TPACK) model, that favors the design and analysis of professional training courses for mathematics teachers when interacting with digital technology.
- The research shows the relationship between TPACK and the mathematical work of teachers in teaching quadrilaterals.
- The results highlight that the development of teachers' TK might promote the use of technology which could in turn help their students have a meaningful mathematical learning process.

solving a geometric task to estimate the area of a piece of land. The analysis of the mathematical work of teachers described the geometric paradigms, mathematical work and results, and forms of geometric work. These five forms were identified as dissector, surveyor, explorer, constructor, and calculator. In addition, in another research on teacher training, Henríquez Rivas and Kuzniak (2021) analyze the mathematical work of teachers in the domain of geometry using, in the same task, pencil and paper and GeoGebra, in which iconic and non-iconic visualization is evident. The authors highlight the use of prototypical designs and an incipient knowledge of GeoGebra. In this sense, there is a need to incorporate technology into teachers' training and education to strengthen their fundamental digital skills.

On another hand, in this study, digital technology plays a key role, where "digital artifacts" become important for teaching and learning mathematics. According to Salazar et al. (2022), digital artifacts are a set of propositions characterized by being executable by an electronic machine that contain historical intelligence and relative epistemological validity.

The use of technology turns students and teachers into active participants since their mathematical working space is generated; in addition, it reinforces the semiotic, instrumental or discursive work that is carried out when developing a task (Arancibia et al., 2021).

In addition, Bueno et al. (2021) conducted research with in-service mathematics teachers in a postgraduate education course. Using the TPACK model for analysis, in-service teachers designed a sequence of activities to develop applications and logic games with GeoGebra.

Likewise, Padilla and Conde-Carmona (2020) used the TPACK model to explore whether mathematics teachers are prepared to incorporate into their classes technologies that contribute to promote skills and respond to the new characteristics and needs of students. Results indicate that there is a disconnection between teachers' discourse and their pedagogical practice. Teachers value and refer to the benefits of technological tools; however, in practice they continue to follow traditional methods.

The creation of frameworks including descriptions of the profile a teacher should achieve in terms of teaching capabilities demonstrates the importance of teachers' TK. In this sense, regarding the skills that a teacher should have, the United Nations Educational, Scientific and Cultural Organization (2023) has created a digital competence framework that can be used to assess different levels of teacher development.

Since we are interested in analyzing the mathematical work of secondary school teachers and their TK in teaching quadrilaterals, we have considered "connecting" the theory of MWS and the model of TPACK. Both theoretical constructs have elements that allow us to use them harmoniously according to the needs of this study. MWS allows analyzing the mathematical work of teachers, and the TPACK model offers the necessary support to design and analyze professional training courses for teachers–in our case mathematics teachers–when interacting with digital technology.

THEORETICAL FRAMEWORK

This study is based on the theory of MWS and the model of TPACK, which are explained below.

Regarding the MWS theory, Kuzniak et al. (2016) point out that mathematical work consists of solving mathematical problems, identifying problems, and organizing content within a specific domain. They also explain that a task is considered to be any type of mathematical problem, with questions established in an explicit and clear manner, requiring a predictable time for its resolution.

In MWS, the epistemological and cognitive planes are articulated through three geneses: the semiotic genesis, which is the process associated with signs and representamen (or signifiers). It is related to mathematical objects to their signifying elements. The visualization process interprets and relates these objects through cognitive activities associated with the semiotic register (identification, processing and conversions); the Instrumental genesis, which allows artifacts to be made operational through construction processes to achieve mathematical work; and the discursive genesis, which is activated on the basis of a referential by definitions, properties or theorems, and the discursive reasoning based on a proof that can be pragmatic or intellectual.

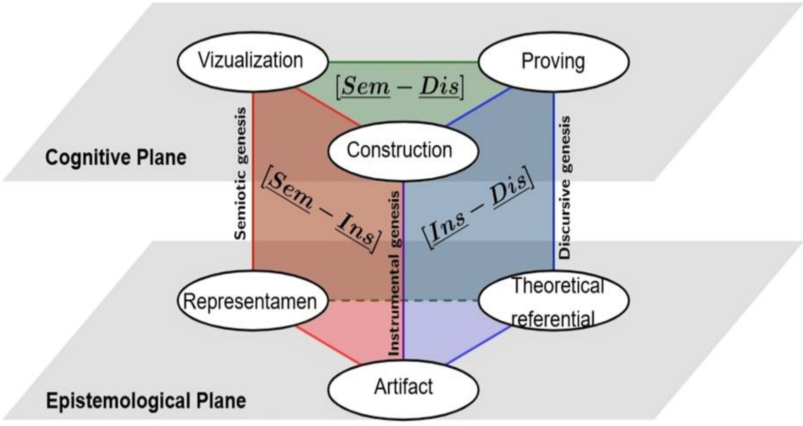


Figure 1. MWS diagram (Adapted from Kuzniak et al., 2016, p. 248)

Furthermore, as shown in **Figure 1**, the authors identify three vertical planes. Each plane is defined by the interaction of two geneses: semiotic and instrumental [Sem-Ins], instrumental and discursive [Ins-Dis], and semiotic and discursive [Sem-Dis].

The [Sem-Ins] plane, associated with the semiotic genesis and the instrumental genesis, is oriented towards the construction of results (figures and graphs) and the interpretation of the information provided by artifacts. The [Ins-Dis] plane, associated with the discursive genesis of proof and the instrumental genesis, comprises the appropriation and selection of artifacts through deduction and induction. The [Sem-Dis] plane, associated with the semiotic and discursive geneses, is distinguished by argumentative reasonings.

Furthermore, the circulation process is considered to be an internal activation of the MWS in a mathematical task, in other words, the articulation of its components and processes.

On the other hand, the TPACK model, in the case of mathematical content, states that teachers' knowledge determine how teaching will be. Teachers make decisions based on their knowledge. Therefore, it is desirable that teachers have TK to incorporate digital technologies into teaching (Koehler et al., 2015). Mishra (2019) establishes that this model has maintained its elements unchanged for the past 15 years; therefore, the use of references from earlier studies to describe the model is still relevant. Nevertheless, he adds a contribution to fix an inconsistency about the context. Initially, context was not considered to be knowledge, but he incorporates the element of context knowledge to improve the model. As shown in **Figure 2**, teachers' knowledge can be included in the following dimensions: pedagogical knowledge (PK), mathematical content knowledge (in the case of TPACK for mathematics) (CK), TK, contextual knowledge or context (XK), and their intersections (PCK, TPK, technological mathematical content knowledge [TCK], and TPACK).

Since teachers must know and choose the best option for the particular mathematics teaching situation, they need knowledge that allows them to develop their TPACK (Drijvers et al., 2016). This implies the use of digital technology for communication, documentation and presentation, which are essential to exchange mathematical ideas, such as spreadsheets, computer algebra system, and dynamic representation environments.

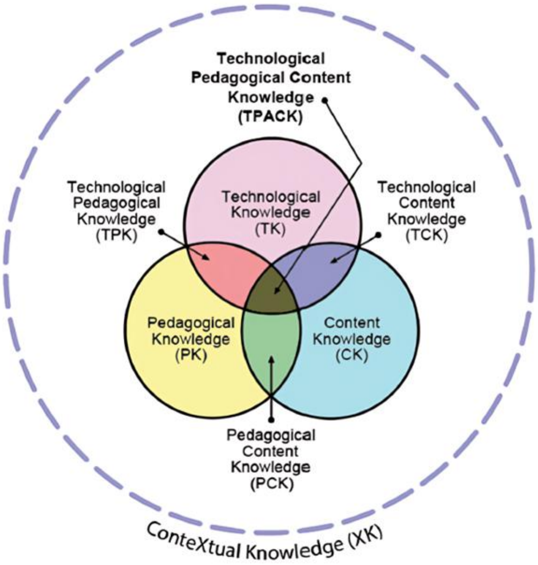


Figure 2. Revised version of TPACK and its knowledge components (Mishra, 2019, p. 77)

Engin et al. (2022) conducted a study in which they describe and exemplify the six levels of technological integration in teachers' TPACK. Data were collected from semi-structured interview forms, lesson plans, micro teaching experiences, and interviews. They observed that, as teachers make progress in the levels by integrating technology, there is also a change in their teaching methodology as technology promotes student-centered instruction. By increasing teachers' TPK, they focused on their students' understanding of mathematics, minimized the impact of difficulties, and used several teaching strategies. The authors concluded that "Technological developments have deeply affected education and therefore changed teaching competencies.

Table 1. Criteria to analyze circulation in the MWS

Criterion	Component	Description
Semiotic genesis	Representamen	Relates mathematical objects and their significant elements.
	Visualization	Interprets and relates mathematical objects according to cognitive activities linked to the register of semiotic representation (identification, treatment and conversions).
		The visualization process considers two levels of the objects' visual identification (iconic visualization, non-iconic visualization).
Instrumental genesis	Artifact	Uses the material, symbolic or digital artifact.
	Construction	It is based on actions triggered by the used artifacts and the associated usage techniques.
Discursive genesis	Referential	Uses definitions, properties or theorems.
	Proof	Discursive reasoning is supported by proof (pragmatic, intellectual).

Note. Taken from Henríquez Rivas and Kuzniak (2021, p. 129)

A good teacher is expected to integrate the CK, PK, and TK into education" (p. 4791).

In the present study, coordinating TPACK and MWS is useful to understand how mathematics teachers' knowledge relates to their MWS.

METHOD

This research considers a case study method, based on a unique integrated design (Yin, 2018), in which the units of analysis are considered to be the mathematical work in the domain of geometry that two mathematics teachers show when solving a task on quadrilaterals with GeoGebra.

For this study, a blended training course with online and in-person sessions for in-service mathematics teachers (secondary level) has been designed. This course is part of a semester-long subject with 3 weekly sessions of 2 hours each, in which knowledge about quadrilaterals and technology is explored using digital technology. The course includes the development of four tasks on the study of quadrilaterals, their properties and classification.

Six in-service teachers who teach secondary school pupils (12-14 years old) participated in the training course. Two teachers were selected for this article, who will be referred to as Andrés and Sofía to protect their identities. The selection criteria used were comparable: pre-service training as mathematics teachers, similar in-service experience, and both teachers performed the task in a similar way to what we expected.

It is important to mention that all ethical aspects of scientific research were considered (Creswell & Creswell, 2018). For this reason, the participants signed an informed consent protocol, which explained the purpose of the research, the strict use of the data collected exclusively for research purposes, and the confidentiality of the identity of all the teachers and their connection with the results obtained.

It should be noted that the present article only considers the fourth task called "construction with properties". This task required building representations of quadrilaterals from their definitions and properties. For the development of this task, participants used GeoGebra as a digital artifact. The analysis of the MWS was performed using the criteria for circulation analysis in the MWS based on Henríquez Rivas and Kuzniak (2021), which are shown in **Table 1**.

In addition, the Kuzniak and Nechache (2021) method is considered for the analysis of the mathematical work of teachers. This method consists of two stages: top-down analysis, in which the actions of the subjects are described when solving the task. This analysis allows identifying and describing work episodes and the circulation in the MWS. Then, a bottom-up analysis is carried out. In this stage, the episodes are systematized based on the MWS scheme (see **Figure 1**).

To identify the TK of teachers according to the T-PACK model, a final questionnaire (Google Forms) is developed and implemented. It comprises questions related to technological, pedagogical and mathematical aspects and their relationships. The task and its respective analyses are presented below.

Quadrilaterals Task

For this article, as explained above, the participating teachers completed the fourth task of a teacher training course. The task requires the configuration of a quadrilateral by constructing parallel lines and others in different directions, dragging and rotating the representations. The task requests the following: "Build representations of quadrilaterals from their definitions and properties." **Table 2** shows what is expected in the development of this task. Regarding the development of this task (experimental part), virtual meetings were held on the Zoom platform. The instruments used for data collection were PDF files with instructions and the tasks that the teachers had to complete, GeoGebra files, videos of the Zoom meetings, and a final questionnaire (Google Form).

RESULTS AND DISCUSSION

As stated above, we considered elements of the Kuzniak and Nechache's (2021) method and the

Table 2. Task development with GeoGebra

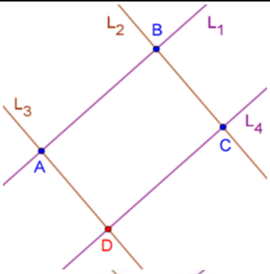
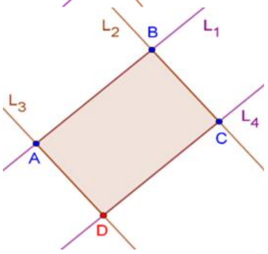
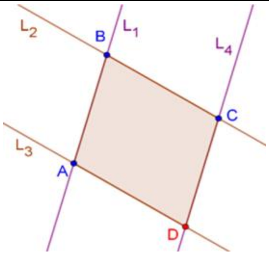
Task attributes	Description of possible actions
 	The "point" tool is used to represent points A, B, and C in any position on the plane. Then, the "line" tool is used to represent straight lines L_1 and L_2 so that each of them passes through two of the previous points. In this case, L_1 passes through A and B, and L_2 passes through B and C. To draw two straight lines parallel to L_1 and L_2, we use the "parallel" tool. We select crossing point A and L_2 as direction, defining straight line L_3 // L_2; in an analogous way, we obtain line L_4 // L_1. Then, with the "intersection" tool, point D is obtained, which represents the intersection between L_3 and L_4. Finally, the intersection points of the four lines are identified as the vertices of parallelogram ABCD. Then, the "polygon" tool is used to represent the quadrilateral
	The vertices of parallelogram ABCD are moved and dragged to identify and explain the properties of parallelograms

Table 3. Task episodes

Episodes	Sequence of mathematical actions
E1: Construction of the parallelogram	Uses GeoGebra tools such as "point", "line", "parallel", and "polygon" to generate the representation of the parallelogram, and then hides them.
E2: Justification	Uses properties to justify the construction of the requested parallelogram.

circulation criteria of MWS to analyze the production of the teachers. Then, we proceeded to analyze the mathematical work of teachers Andrés and Sofía. In the top-down analysis, it is worth highlighting that both teacher Sofía and teacher Andrés performed similarly to what was expected (**Table 2**), which means that the same episodes were identified: E1, called construction of the parallelogram; and E2, called justification (see **Table 3**).

Teacher Sofía's construction in E1 was carried out in a similar way to what was expected for this task (see **Table 2**), except that she used GeoGebra functions to hide all the auxiliary construction elements in the resulting polygon. In E2, to justify the construction, teacher Sofía added, as shown in **Figure 3**, a list of properties that include the congruence between the measurements of opposite angles, the bisection of diagonals, and the linear pair formed by two consecutive angles. She verified these properties using GeoGebra tools.

In the circulation analysis, we observe that in E1 teacher Sofía mobilized her TK of GeoGebra tools to build the parallelogram, that is, she activated her semiotic and instrumental geneses. This means the [Sem-Ins] plane was activated.

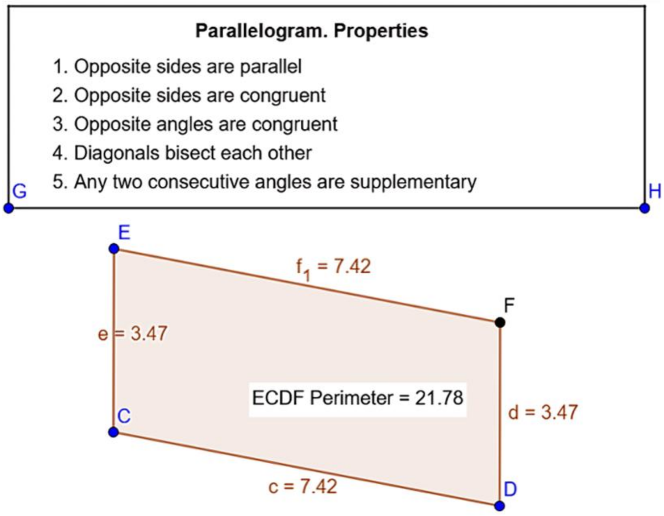


Figure 3. Construction of a parallelogram with GeoGebra–teacher Sofía (translated from Spanish) (Source: Authors' own elaboration)

Likewise, in E2 the teacher activated the theoretical referential by using properties such as parallelism of sides, and congruence of angles and sides. She does this by using GeoGebra. That is, the theoretical referential activated the instrumental genesis when the teacher

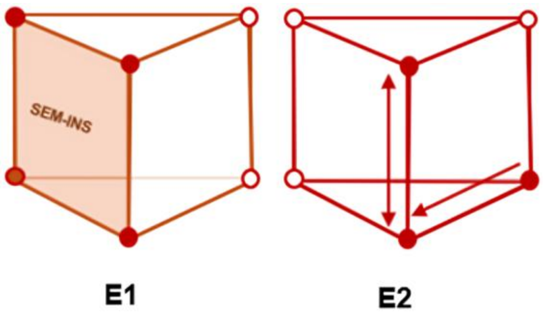


Figure 4. Global description of the MWS–Sofía (Source: Authors' own elaboration)

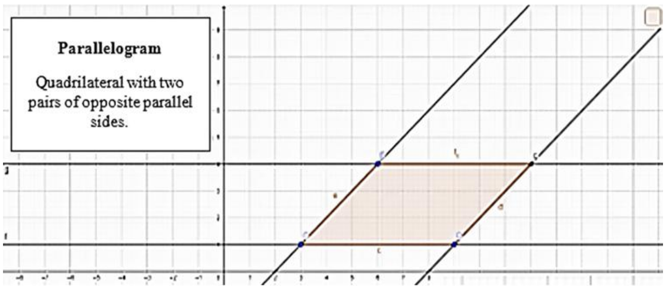


Figure 5. Construction of a parallelogram with GeoGebra–teacher Andrés (translated from Spanish) (Source: Authors' own elaboration)

used the digital artifact GeoGebra to represent the parallelogram.

In the bottom-up analysis, the MWS diagram (**Figure 4**) is used to make the global description of the work.

We observe the preponderance of the [Sem-Ins] plane activated by the theoretical referential of the epistemological plane.

Regarding the work carried out by teacher Andrés (top-down analysis), as explained above, the same E1 and E2 episodes are identified (see **Table 3**). In E1 teacher Andrés's construction is similar to the expected construction of the parallelogram (see **Table 2**). Using the relations of parallel lines in parallelograms and intersection of points, he obtained parallelogram ABCD. He did not hide the construction lines (see **Figure 5**).

In E2, to justify the construction, teacher Andrés added the definition of a parallelogram, which he verified using GeoGebra tools. He worked in the algebraic view of GeoGebra (in which the coordinate axes are shown) and fixed the vertices of the parallelogram.

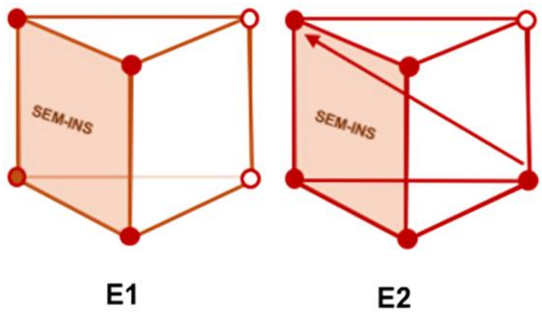


Figure 6. Global description of the MWS–Andrés (Source: Authors' own elaboration)

In the circulation analysis, in E1 we observed that the teacher activated his TK about the GeoGebra tools, that is, the semiotic and instrumental geneses and, consequently, the [Sem-Ins] plane. In E2, for the justification, he activated the theoretical referential and the semiotic and instrumental geneses because he used the GeoGebra artifact and the algebraic window appropriately.

In the bottom-up analysis, the global description of the work is made through the MWS diagram (**Figure 6**). Using GeoGebra, the teachers built a parallelogram based on its properties. The teachers made conjectures about the steps they had to follow, justifying their proposal with the properties of the object represented.

At the end of the task and the teacher training course, the teachers answered a questionnaire (Google Forms) that aimed to identify their TPACK. The implementation of this instrument provided information to identify the TK of teachers and its links with mathematical and didactic content. **Table 4** shows some questions from the questionnaire that are directly related to the task analyzed in this article.

About item 1, in general, the teachers agreed that they would use the digital artifact (GeoGebra) because they consider it is motivating for their students. In addition, they perceive GeoGebra as an efficient resource to show the variations that the represented mathematical objects can undergo (treatments in the figures). This shows the link between TK and CK, mathematical in this case. Thus, this references the intersection with teachers' TCK.

Regarding item 2 and item 3, about their interest in using the digital artifact, the teachers answered on a scale from 1 to 5 (where 1 means they completely disagree and 5 they completely agree) that they are both

Table 4. Questions from the questionnaire related to the task (adapted from Théry, 2023, p. 89)

No	Final questionnaire
1	Would you use any digital artifacts to teach quadrilaterals? (yes or no) What advantages or disadvantages do you see with respect to teaching without digital technology?
2	Are you interested in learning more about these or other digital artifacts to teach quadrilaterals?
3	Will you use the digital artifacts used in this course in teaching classes about quadrilaterals?
4	Are you interested in learning more about integrating technology, teaching strategies, and knowledge about quadrilaterals for teaching?
5	Do you think that your knowledge about digital technology affects your students' learning about quadrilaterals?
6	Final comments or reflections.

interested in learning more and using digital technology in their classes. For example, in item 3, teacher Andrés's response shows the relationship he establishes between pedagogical content knowledge (PK and PCK) and his TK.

> Teacher Andrés: The use of technology is very interesting, but the way it is used is decisive because it can benefit the development of the class at all stages of students' work, but one must have some knowledge about its use and about the problems that can be solved.

Both teachers' answers in item 4 and item 5 of the questionnaire show the relationship between TK and CK, mathematical in this case. So, this means that teachers' TCK is evidenced in their interest in incorporating technology–GeoGebra in particular–in teaching quadrilaterals in their classes for understanding concepts and experimenting on the properties of quadrilaterals. For example, teacher Sofía answered:

> Teacher Sofía: It is necessary to progressively incorporate technology into mathematics classes so that teachers can master it to support us with visuals and exploration of the properties of quadrilaterals and with other mathematical contents as well.

Finally, in the comments or reflections (item 6), the teachers expressed a positive attitude and disposition about the incorporation of digital technology in their mathematics classes, specifically to teach geometry (quadrilaterals).

CONCLUSIONS

The findings in this research highlight that teachers showed increased TK, especially in using the GeoGebra digital artifact. This improvement translates into greater confidence and competence integrating technology into their pedagogical practices by including it in key learning moments of their planning, which is crucial for effectively teaching complex geometric concepts such as quadrilaterals (Bueno et al., 2012; Drijvers et al., 2016; Henríquez Rivas & Kuzniak, 2021).

For example, teacher Sofía showed an interaction between her TK and her mathematical work when using the GeoGebra digital artifact to construct the parallelogram. This is evidenced when she showed her acquired knowledge about the digital artifact.

On another hand, the use of GeoGebra can facilitate the exploration and visualization of the properties of quadrilaterals, which could allow teachers and students to interact more dynamically with mathematical representations by dragging points and verifying the permanence of the properties of quadrilaterals (Salazar et al., 2022). This interaction is essential to develop a deeper and more meaningful understanding of geometric properties, as it allows students to observe how transformations affect figures. In addition, the task allowed teachers to demonstrate their development of TK and gave them confidence to use it in their class sessions, that is, it showed that they are developing their TPACK compared with what they initially demonstrated.

The research also suggests that greater integration of digital technologies in teacher training should be encouraged (Henríquez Rivas & Kuzniak, 2021; Koehler et al., 2015). This could include developing specific programs that address both TK and innovative teaching methodologies, thus ensuring that teachers are better prepared to face current educational demands.

The observation made in this study that teachers consider using technology with digital artifacts such as GeoGebra implies that they recognize the advantages of exploration and visualization. In addition, this type of task seeks to configure how quadrilaterals can be taught using digital artifacts that allow their representations to be visualized and manipulated. This means they promote activation of the semiotic and instrumental geneses and, consequently, the vertical [Sem-Ins] plane.

In addition, some relationships were identified between the TPACK model and the MWS theory because both provide relevant aspects and elements for understanding the characteristics of teachers' knowledge to efficiently incorporate digital technology into teaching. It is also observed that a solid knowledge of mathematical content allows teachers to include tasks that not only involve the use of technology, but also promote complete mathematical work, thus facilitating more effective learning in students (Engin et al., 2022; Mishra, 2019).

Based on this research, new research could be generated, such as the influence of teachers' knowledge of other technological artifacts and whether there is variation by teaching level. Teachers' TK is interrelated with their decisions in the mathematics teaching process, which opens up several research and in-depth opportunities that could be developed.

In addition, this research highlights the importance of teachers' TK in mathematics teaching, specifically in geometry, and suggests that better training for in-service teachers could support them in the process of teaching mathematics.

Finally, a potential effect of this research is that this type of analysis can be carried out now from the perspective of network theories, which could have theoretical implications and contributions in the future.

Author contributions: All authors have been involved in designing, analysing and writing process. All authors have agreed with the results and conclusions.

Funding: This study was supported by the Dirección de Fomento de la Investigación de la Pontificia Universidad Católica del Perú, through grant DFI-2022-C-0025 [PI0890].

Acknowledgments: The authors would like to thank the Instituto de Investigación sobre la Enseñanza de las Matemáticas of the Pontificia Universidad Católica del Perú and the Red Iberoamericana de Investigación en Trabajo Matemático for their collaboration.

Ethical statement: The authors stated that the study was approved by the Ethics Committee at the Pontificia Universidad Católica del on 16 August 2022 approval code 080-2022-CEI-CCSSHHyAA/PUCP. Written informed consents were obtained from the participants.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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