



Inquiry-Based Mathematics Curriculum Design for Young Children-Teaching Experiment and Reflection

Su-Chiao Wu

National Chiayi University, TAIWAN

Fou-Lai Lin

National Taiwan Normal University, TAIWAN

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A group of teacher educators and practitioners in mathematics education and early childhood education generalized a set of inquiry-based mathematics models for Taiwanese young children of ages 3-6 and designed a series of inquiry-based mathematics curriculum tasks in cultivate the children's diverse mathematical concepts and mathematical power. In this paper, we mapped the blueprint of the whole curriculum with a brief section of "number" activities. We also compared this inquiry-based curriculum and instruction with the traditional mathematics teaching process in practical settings through a teaching experiment process, in which young children's mathematical learning performances were collected and analyzed as empirical evidence. Reflecting on the outcomes of this comparison, we presented exploratory findings of teaching experiments, and proposed relevant discussion for future works.

Keywords: curriculum design, early childhood mathematical learning, inquiry-based

INTRODUCTION

Scaffolding young children's mathematical learning using an inquiry-based approach

"Childhood is a special, magical time when the brain is metaphorically spongelike and when learning new skills can be both fun and effortless" (Diamond and Hopson, 1998, p. 4). This early learning can positively stimulate and enhance young children's potential for exploring the world in the future (Copley, 2010). For a long time, children in Taiwan have not been interested in or even have been afraid of learning mathematics, which has led us into putting more effort in resolving the obstacles of indifference and fear. Recently, Taiwan's educational reforms have highlighted the improvement of curriculum and instruction in mathematics, which primarily have been aimed at cultivating children's learning interests in mathematics and educating them to apply what they have learned in classrooms to real life situations (Ministry of

Correspondence: Su-Chiao (Angel) Wu,
Department of Early Childhood Education, National Chiayi University, Minsyong
Township, Chiayi County 621, Taiwan.
E-mail: angelwu421@yahoo.com.tw
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Education, 2008). Since mathematical learning in early childhood plays a significant role in children's future development of mathematical concepts, logical thinking, and learning attitude; how to enrich young children's mathematical learning experiences have stood at the core of the current educational reform. This preliminary action benefits to the whole process of mathematics learning.

In a series of studies of "Building Blocks" and learning trajectories, two authors (Clements and Sarama, 2007; Sarama and Clements, 2009) proposed a learning trajectory for young children from ages 1 to 5, and the thinking level of each stage; this trajectory provides directional indicators for early childhood mathematics education. They found that, without adequate learning experiences, some kids' mathematical learning performances are below the average; on the contrary, kids' performances are far better than the average with the guidance of high-quality education. In their studies, 4-year-old children who learned with well-planned constructive curriculum in the "Building Blocks" project reached or surpassed the mathematical capabilities of 5-year-olds. This encouraging result indicated a child-centered and child development-based mathematics curriculum could not only help teachers see how their kids learned mathematical concepts and understand their learning processes & trajectories while attaining these concepts, but also promote their high-level mathematical development. In addition, engaging children in the learning processes and providing inquiry-based learning opportunities were useful in enhancing their reasoning and problem-solving capabilities, which further furnished them with better understanding of mathematical concepts (Kennedy, 2009; National Council of Teachers of Mathematics [NCTM], 2000). Through these manipulative exercises, teachers could assess children's developmental status with respect to mathematical concepts. Some empirical evidence showed that teaching aids are valuable for promoting kids' learning outcomes (Fuson and Briars, 1990; Soydan, 2015). In fact, these constructive supports play a significant role in children's learning process, which echoes Vygotsky's (1978) concept of "Zone of Proximal Development (ZPD)". In this concept, scaffolding (i.e. assistance or support) is most effective when the designated support is matched to children's needs.

Additionally, an inquiry-based approach is seen as vital for children to obtain scientific content knowledge through the problem-solving processes (National Research Council [NRC], 2000), which includes identification of assumptions and use of critical and logical thinking, as well as consideration of alternative explanations (NRC, 1996). Based on a philosophy of mathematics education, Richards (1991) described inquiry as learning to speak and act mathematically by engaging in mathematical discussions, proposing conjectures, and solving new or unfamiliar

State of the literature

- The early learning experience could positively stimulate and enhance young children's potential for exploring the world in the future.
- The function of inquiry teaching is to organize activities based on the purpose of discovering problems or creating cognitive conflict scenarios.
- Meaningful learning of mathematics was to provide enriched mathematical learning experiences where children could be meaningfully guided to discover the structural relationships between concepts connected with numbers and their applications to problems arising from the real world. However, less study was conducted in mathematical learning, and even less in early childhood mathematics education.

Contribution of this paper to the literature

- We confirmed the feasibility of our inquiry-based mathematics curriculum and instruction and their advantages for young children's mathematical learning, as well as using it as the stepping stones to critically understand children's mathematical learning trajectories in the near future.
- Children are born to be mathematicians. Inquiry-based mathematics curriculum is beneficial for future studies of young children's learning trajectory and kindergarten teachers' professional development.
- We designed this inquiry-based mathematics curriculum with the goal of furnishing kids with enriched inquiry-based learning contexts in advancing their initial understanding of multiple categories of pre-K mathematical content and developing diverse mathematical power.

problems. Further, Yackel et al. (1990) claimed that children's mathematical knowledge could not be delivered simply from one teacher's instruction but must develop from participation in inquiry-based learning activities instead. Within this participative learning process, children were expected to comprehend what they saw and/or heard and bring up their own explanations. In short, inquiry-based mathematical instruction emphasizes a child-centered approach of "autonomy and manipulation", where children can discover and solve the problems firsthand. This instructional approach also offers teachers abundant opportunities to observe their kids' mathematical opinions and further bridge them toward a path of higher-level mathematical development. Up to the present, inquiry-based instruction has been employed broadly in the subject area of science (e.g. Park, M., Park & Lee, 2009). Only a few studies have been conducted with a focus on young children's mathematical learning (Erdogan & Baran, 2009; HodnikCadez & Skrbec, 2011; Soydan, 2015). Comparatively, in Taiwan, little study has been conducted in mathematical learning, and even less in early childhood mathematics education. Consequently, in this study, a group of practitioners and teacher educators in mathematics education and early childhood education gathered together to create/generalize a set of inquiry-based mathematics models for Taiwanese young children (i.e. ages of 3-6) from various theories and research evidence. Grounded on this theoretical framework, we designed a series of inquiry-based mathematics curriculum tasks for cultivating young children's diverse mathematical concepts; these tasks were subject to pilot instructional experiments in early childhood settings leading to revisions of the tasks. We hope that the designated curriculum can be applied in additional practical settings, in which more detailed learning processes and empirical data of young children's performances in learning mathematics can be collected for further improvements in the curriculum and additional research studies.

The study

In this study, a "two-stage" curriculum design process was employed to reach the main goal—designing, experimenting, and reflecting an inquiry-based mathematics curriculum for young children. At the first stage, through intensive meetings and dialogue, we dressed the curriculum up using core concepts of diverse mathematical theories, developmental theories of mathematical concepts, and existing research evidence for young children. . We examined every single lesson plan to confirm whether the instructional objectives were reached, as well as analyzing all lesson plans (as a whole, vertically for coherent concepts, and horizontally for every age level) to verify the model and the content of the whole curriculum with the established theoretical framework. At the second stage, we compared this inquiry-based curriculum and its implementation with the traditional mathematics teaching processes in practical settings through a teaching experiment process, in which young children's mathematical learning performances were collected and analyzed as empirical evidence. Reflecting on the induction of this empirical comparison, the designated model of inquiry-based mathematics curriculum and instruction was refined for future teaching and research purposes. Accordingly, in this paper, we have mapped the blueprint of the curriculum for a brief section of "number" activities, presented exploratory findings of the teaching experiments, and proposed relevant discussion for future studies.

THEORETICAL FRAMEWORK

Inquiry-based mathematics curriculum and instruction

Inquiry is essential for modern society/people (NRC, 1996). The cultivation of children's capability of inquiry is considered an instructional priority in contemporary mathematics and science education (Staer, Goodrum, and Hackling, 1998). The focus of inquiry teaching is to organize learning activities on the basis of discovering problems or creating cognitive conflict scenarios, which then provide young children with various opportunities for discovering scientific problems and developing the capability of using critical thinking while working on tasks and constructing problem-solving solutions (NRC, 2012). Thus, every child's role is that of an enthusiastic thinker, actively engaging in the process of questioning, observing, categorizing, explaining, applying, developing, and expressing their own opinions while accepting those of others, with the ultimate goal of being capable of solving problems and understanding their rationale (NRC, 1996; 2012). In this study, we employed the 5E Instructional Model (Bybee et al., 2006) in designing the inquiry-based learning activities of the targeted curriculum, including five phases (an inquiry-based learning cycle): engagement, exploration, explanation, elaboration, and evaluation. Based on constructivism, this model provides a basis for teachers to use diverse teaching strategies to advance their children's active learning (Bybee, 1997). By building up an inquiry learning environment, children can both explore and explain what they have learned (Orgill, & Thomas, 2007), as well as detect and solve life-related problems in their own ways (Chang & Wu, 2015).

As mentioned, coherent learning theories and related research evidence in early childhood and mathematics education have been generalized to design the designated curriculum. First, we endorsed the concept of Gestalt psychology that "the whole is different from the sum of its parts" (Wolfgang, 1992). In designing and executing the instructional activities, we emphasized giving children the whole picture of related concepts instead of using mechanical learning approaches that result in fragmented accumulation of knowledge or concepts. With this approach, children's learning processes were not "try and error" attempts but the systematic learning of problem-solving in a meaningful context. Moreover, according to the tenets of Constructivism, the learner constructs his/her own understanding through experience; "problematic experience can initiate the learning process and subsequent experiences lead to changes in understanding and action" (Osterman, 1998, p. 4). As a result of interactions between new and past experiences, authentic learning of a new concept is constructed. Therefore, learning is an active process requiring children's engagement. In the teaching of mathematics, teachers need to create an enriched learning environment full of life experiences and meaningful contexts; this precept was fully applied in our design.

Grounded on Piagetian theory, Dienes (1973) proposed a theory of mathematics learning, which was composed of four principles (constructivity principle, multiple embodiment principle, dynamic principle, and perceptual variability principle) and six developmental stages (free play, free experiments, comparison, representation, symbolization, and formalization). According to his theory, mathematics is the study of structure. Meaningful learning of mathematics provides enriched mathematical learning experiences where children can be meaningfully guided to discover the structural relationships between concepts connected with numbers and their applications to problems arising from the real world. In relation to the constructivity principle, it was suggested that teachers design hands-on activities with the use of realistic and concrete objects (manipulatives) whereby kids can be stimulated both physically and mentally to sense the structural relationships firsthand. The multiple embodiment principle leads to providing multiple learning contexts for children to make predictions from one structural situation to another. Mathematical abstractions occurred when they recognized structural similarities shared by several

related situations. The dynamic principle suggests that transformations within one model correspond to transformations in an isomorphic model even though the embodiments of these models might be dissimilar. The perceptual variability principle emphasizes that conceptual learning is maximized when children are exposed to a concept by means of diverse physical contexts or embodiments (e.g. concrete objects, graphics, written symbols, spoken language, or other representational systems). For example, teachers might “vary the perceptual details of a problem but include some common structural characteristics so that students have an opportunity to link structurally similar problems” (Sriraman and English, 2005, p. 258). Hirstein (2007) proposed instructional practices reflecting Dienes’s (1973) theory of learning mathematics: First, play and games are critical approaches in formulating young children’s initial awareness of new mathematical concepts. In designing curriculum, we should provide a proper instructional event in which children “can be introduced to very complicated ideas and can develop quite sophisticated approaches to problems if things are presented at the right level” (p. 169). Secondly, teachers need to provide sufficient opportunities for children to practice significant skills (i.e. abstraction and generalization) by engaging in diverse “embodiments” that bridge real world experiences to abstract mathematical concepts. Thirdly, since children, with traditional instruction, are asked to be familiar with the use of mathematical symbols before they can possibly understand the true meaning of those symbols, it was not surprising that their learning mainly involved memorization and is not permanent.

In summary, in this project, Dienes’s (1973) theory of learning mathematics was applied in designing the targeted curriculum for children, ages 3-6, in which an inquiry-based instructional approach was employed. The model was based on his six stages of learning mathematics. “Free experiments” and “comparison” were merged as a “problem solving” procedure. “Representation” and “symbolization” were integrated for stimulating children’s expression and communication. The result was that we developed a “four-stage developmental model” for structuring sequential learning procedures in every lesson: (1) free play: exploring problems; (2) exercising your brain: problem solving; (3) s/he, you, and I: expression and communication; and (4) just doing it: integrated inquiry.

Rationale and design of the curriculum

In ancient Chinese, the characteristics of a good teacher was recorded in “The Book of Rites”,

When a superior man knows the causes which make instruction successful, and those which make it of no effect, he can become a teacher of others. Thus in his teaching, he leads and does not drag; he strengthens and does not discourage; he opens the way but does not conduct to the end (without the learner's own efforts). Leading and not dragging produces harmony. Strengthening and not discouraging makes attainment easily. Opening the way and not conducting to the end makes (the learner) thoughtful. He who produces such harmony, easy attainment, and thoughtfulness may be pronounced a skillful teacher.

Grounded on this rationale, we defined the role of a good teacher for young children as “properly employing an inspirational and constructive approach to mathematics instruction”, as well as “guiding them (children) to think actively, inspiring them while exploring, and reinforcing them in a timely way”. A positive teacher-student relationship is established in assisting children’s adaptive development and stimulating their learning motives and outcomes. We differed from five big ideas highlighted by Clements, Sarama, and DiBiases (2002); our curriculum included four categories of mathematical content for young children: number,

measurement, space/shape, and logic. To scaffold the four mathematical content areas in the learning process, we emphasized the use of “mathematical power” in designing the lessons which included problem-solving, connection, reasoning, representation, and communication.

Additionally, based on the four-stage sequential learning procedures described above, we mapped out the mathematical teaching procedures corresponding to the 5E inquiry-based learning cycle (see Table 1). In each lesson plan, following this “exploratory path”, teachers are able to help children engage effectively in the inquiry-based learning cycle. In practice, the procedures for each mathematics concept are appropriately adjusted for each lesson.

Teaching procedures—An example

As mentioned previously only the “number” section of the curriculum is introduced in this paper. There are a total of 17 lesson plans in the “number” category, including four for ages of 3-4, six for ages of 4-5, and seven for ages of 5-6. For example, the lesson “Pulling out the carrot” for ages 3-4, dealt with the concept of natural numbers “1-10”, e.g. count, one-to-one correspondence, and order. We used the picture book (as named the lesson), animal headgear, and other decorations to construct the learning corner (the contexts for “free play”). After storytelling (using PowerPoint), kids would start select the role they would like to play in the story (in the meantime, teachers observed kids’ role play). To initiate the inquiry-based activity, teachers discussed the actions within the story with the kids and sorted all answers into several exploratory questions, e.g. “who and what kinds of animals are there”, “the quantity of every category”, and “orders in which these categories appear”. Then, kids would be divided into groups to search out possible answers, based on the procedures presented in Table 1. Another example was the lesson “Finding partners (pretty gloves)” for ages of 4-5, which focused on “odd and even numbers”. Real life objects were drawn into this lesson (e.g. chopsticks, gloves, and socks without right/left distinction); hands-on activities were provided as well. Within the exploratory path, kids could not only learn the concepts of “odd and even” but also work with the operational process of “pairing”, which was grounded on previous experiences provided for children ages of 3-4 as mentioned. In the lesson “Small storage helper” for ages of 5-6, diverse solid storage articles were provided to develop kids’ concepts of “solid, size relationship, inclusion”. More mathematical power (e.g., problem-solving, reasoning, communication, and scientific procedures) was included to help children to retrieve what they have learned for these new concepts. In sum, the design features in this study that emphasized basic principles of curriculum and instruction (i.e. scope and sequence) and inquiry have the potential to truly help our children learn mathematical concepts (e.g. number) and obtain mathematical power in a systematic and efficient manner.

Table 1. Exploratory path in parallel teaching and learning procedures

Mathematical teaching procedures	5E inquiry-based learning cycle
Constructing learning contexts and free play	Engagement
Choosing and initiating inquiry-based activity	Engagement, exploration
Posing and deciding exploratory question(s)	Exploration
Planning and executing problem-solving method(s)	Exploration
Observing cognitive behaviors	Explanation
Designing an activity which causes cognitive conflict(s)	Explanation
Concluding experiences of inquiry-based activity	Elaboration, evaluation
Providing resembling and/or extending activities	Evaluation

RESEARCH DESIGN

As mentioned above, the main purpose of this case study was to design an inquiry-based mathematics curriculum for young children, employing a two-stage process of implementation and reflection, as well as to conduct teaching experiments to evaluate the curriculum. We hope that the implementation and findings of the teaching experiments strengthen the feasibility of the designated curriculum in practical settings, in which young children's learning performance provided empirical evidence for further revisions.

First stage—Participants in the curriculum design process

Developing the designated inquiry-based mathematics curriculum was the main task of the first stage. Through intensive meetings and dialogue, we dressed up the curriculum for young children with core concepts of diverse mathematical theories, developmental theories of mathematics, and existing research evidence about mathematics instruction for young children. We reviewed every single lesson plan to verify whether the instructional objectives were achieved, and we analyzed all lesson plans to confirm the model and the content of the entire curriculum, to be consistent with the established theoretical framework. At this stage, nine members from both practical settings and academic institutions were recruited for the design team. One elementary principal (who has participated in mathematics textbook editing and who served as the consultant of "Counseling Committee of Grade 1-9 Curriculum — Mathematics") and one director of a kindergarten (who had extensive experiences designing and executing thematic integrated curriculum and who also served as a consultant in early childhood education) provided their practical suggestions in designing the curriculum. The rest were seven professors: three from department of mathematics education, three from departments of early childhood education, and one from a department of psychology; all of them specialized in mathematics education and were in charge of practical counseling tasks in pre K-12 settings. During the "one and half years" of the team's work, all members met every 1 to 2 weeks (four hours at least for each meeting). Records (videos and tape-recordings) of these intensive meetings were completed and analyzed for curriculum adjustment and team reflection. The model of the targeted inquiry-based curriculum and instruction for young children of ages 4-6 is reflected in Figure 1. All instructional activities were categorized into four dimensions: number, quantity, space/shape, and logistics. Because of the length limit for this article, we only provided one example for laying out all activities and main mathematical concepts of the four dimensions, that being for the age-6 class (see Table 2). In this paper, we present exploratory findings of the teaching experiments of the second stage of this project, as well as proposing relevant discussion for future work.

Second stage—Data collection and analysis of teaching experiments

The main tasks of the second stage were to conduct teaching experiments in practical settings for reflection and revision. A single-case holistic design is employed in this qualitative, explanatory, and descriptive case study at this stage. Participants were two kindergarten teachers (coded as "T1, T2") and their children (coded as "S1~S30") in one age-6 class of an affiliated kindergarten of a public experimental elementary school in southern Taiwan. In this stage, two tiers of instructional activities were executed in order to compare the designated inquiry-based curriculum and its implementation with a traditional mathematics teaching process; in this process children's mathematical learning performances were collected and analyzed as empirical evidence. For the first tier, we observed the instructional activities without any intervention, so that the two kindergarten

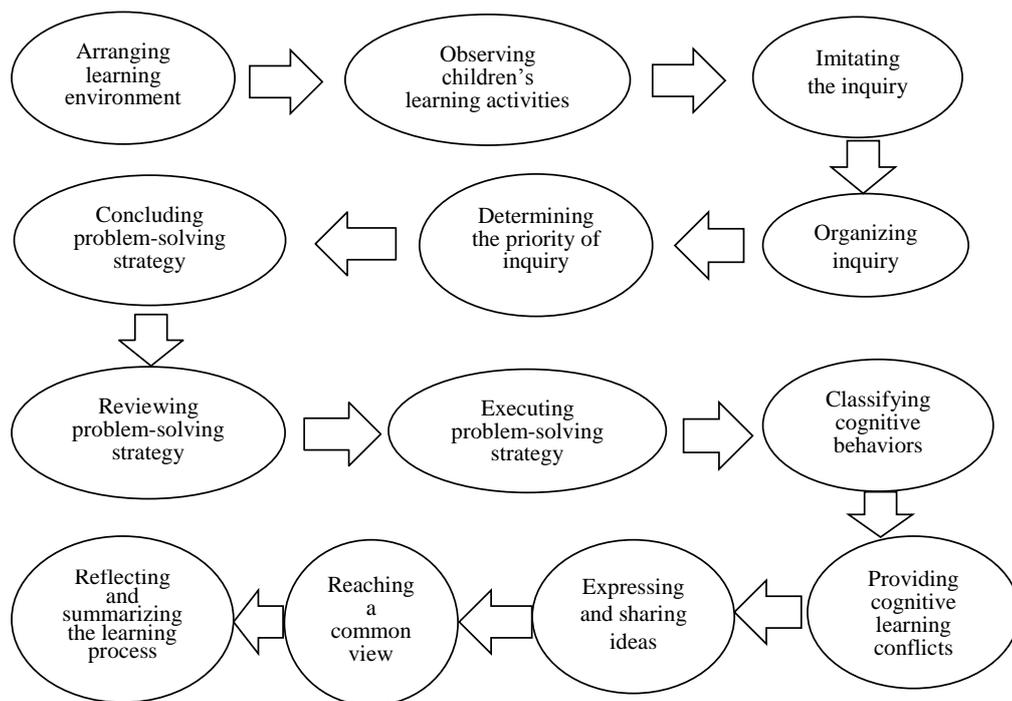


Figure 1. Model of the targeted inquiry-based curriculum and instruction

Table 2. Activities and mathematical concepts of four dimensions for Age-6 class

Dimension	Activity for Age-6 Class (X)	Main Mathematical Concept
Number	XN01-1: Super Shooter XN03-1: Same Amount in One Box Car XN05-1: Turn Two Cards Over XN07-1: Numerical Train	Adding Amount & Unit Addition & Subtraction with 10 Cardinal & Ordinal Number (1-10)
Quantity (measurement)	XQ01-1: Order Boxes XQ03-1: Insert Hercules XQ04-1: Cup Joint XQ05-1: Which Is Heavier	3D Size Relationship Measurement of Force Length Weight
Space / Shape	XS11-1: Junior Painter XS12-1: Where Did I Sit and Paint XS09-1: My Family Picture XS02-1: The Twins of Shape	Experiencing Visual Point Gauging Visual Point Describing Position Shape
Logistic	XL01-1: Supermarket XL07-1: String Necklaces XL02-1: Colored Sudoku XL05-1: A Life of Seed	Categorizing, Relationship Pattern Design Induction, Application Life Process

teachers used traditional approaches to teach mathematical concepts and their young children learned with the original approach for one month. During this traditional teaching process, teachers and children were intensively observed and interviewed to depict targeted children’s learning practices and processes while experiencing the traditional mathematics curriculum and instruction. For the second tier, interventions were provided in which the two teachers employed the inquiry-based mathematics curriculum and instruction in the classroom for three months. Professional development programs were supplied to help teachers advance their understanding of the designated curriculum and how to implement it in their classrooms; for example, intensive training programs were provided for basic understanding and professional dialogues was engaged for possible questions and advanced elaboration.

Within the process of the two-tier teaching experiments, data were gathered and triangulated through semi-structured observations (coded in transcripts as “OB” with

the date “month/day”), individual summative in-depth interviews and follow-up interviews (coded as “IN”), and various kinds of documents (such as researcher’s reflection notes, coded as “RN”) were first organized and pre-analyzed using the following five steps (Thomas, 2000): preparation of raw data files (making transcripts), closed reading of text, creation of categories, overlapping coding and uncoded text, and continuing revision and refinement of category system. Based on this pre-analysis, data were then analyzed qualitatively by template and editing analytic strategies (Crabtree & Miller, 1999). The template and editing analytic system, made use of organizing code topics, which were then used to make sure that the analyses focused on the development of inquiry-based mathematical concepts.

FINDINGS

Comparison of the two approaches

First, evidences related to the two teachers’ teaching tasks and their children’s mathematical learning events and processes during the two-tier teaching experiments were collected and compared. According to the data analyses, eight sub-themes were generalized as a result of the same phenomena emerging repeatedly (i.e. replication was reached), and, in turn, were the subthemes were synthesized into four themes. As shown in Table 3, four themes were extracted from the data corpus to present the differences between the traditional mathematics classroom and the inquiry-based mathematics classroom. Consequently, characteristics of the four themes were exhibited with empirical evidence as follows. A result of reflecting on the results of this empirical comparison was modification of the designated model of inquiry-based mathematics curriculum and instruction, for future teaching and research purposes.

Decorated learning environment and children’s learning motivation

[Theme 1] Decorated learning environment is beneficial to children’s motivation while exploring; teaching aids are interesting and selected so that children can learn from life-related experiences

During the first-tier instructional activities, the two teachers rarely changed the learning environment in the classroom because fewer mathematical activities were

Table 3. Themes extracted from the data corpus

Theme	Traditional Math Classroom	Inquiry-based Math Classroom
Learning Environment and Teaching Aids	<ol style="list-style-type: none"> 1. Focusing on students’ cognitive understandings of mathematical concepts and disregard of the environment. 2. Teaching aids consist mostly of workbooks and ready-made materials. 	<ol style="list-style-type: none"> 1. A decorated environment encourages young children to solve problems by asking life-related questions. 2. Teaching aids are concrete objects that children are familiar with so that they know how to play freely.
Teachers’ Role	<ol style="list-style-type: none"> 1. Emphasizing how teachers teach and what young children learn. 2. Teachers are experts of mathematical knowledge. 	<ol style="list-style-type: none"> 1. Encouraging young children to explore and make mistakes. 2. Teachers learn when to let go and create opportunities for cognitive conflicts.
Peer Relationships (among young children)	<ol style="list-style-type: none"> 1. Students learn independently. 2. Learning experiences are self-constructed, which come individually from the teacher and the operations of teaching aids. 	<ol style="list-style-type: none"> 1. Students learn from/with others, where they help scaffold each others learning, while learning themselves. 2. Students’ learning experiences are peer-constructed and socio-cultural.
Mathematical Teaching Process	<ol style="list-style-type: none"> 1. Learning activities are fixed, 40 minutes as one class period. 2. No direct connection between each activity. Activities are often followed by reviews and assessments. 	<ol style="list-style-type: none"> 1. The length of learning activities are flexible, and usually prolonged to the whole morning. 2. Learning happens in context, where children’s inquiry processes are visible and extended activities emerge.

included in the traditional teaching process. There were only a few demonstrations and operations of certain mathematical teaching aids.

T1: Let's play with these Montessori's number rods. Attention! There are two colors,

red and blue, which one is longer? [T1 displays the two number rods.]

S25: Red, blue, red, blue, red; this rod...red, blue, red, blue, red, blue....

T1: So, which one is longer?

S10: This one, this one [point out the longer one]...because it has one more blue. [Some children are just talking to others; two are crawling on the floor.]

T1: Attention, please! You can line them up, pay attentions to the colors, or I will teach you another strategy to count the number 1, 2, 3, 4.

Understand? Please follow my directions ! (OB-0923)

Within this traditional teaching event, these children just followed the teacher's order in which they used the teaching aids step-by-step. In fact, they were not interested in this operational process and felt bored by this fixed learning process.

Sometimes, the teachers would use picture books as a motivator at the beginning of the activity. For example (OB-0928): One day, T2 employed the picture book "Zoom" as a motivational activity while teaching the "part-whole" concept. T2 asked the children to sit in rows and look at the front board (i.e. a power point). The picture book was wordless, and T2 just pushed the play button and said nothing. After watching the power point silently, T2 ask some questions, such as: What did you see in the first page? Were you aware the picture in the second page was related to the first page? However, no one answered those questions. Most children seemed restless and did not really pay attention to the presentation and those questions (RN-0928). Based on these observations, we thought that even though using picture books was a good idea to motivate children's learning, the teacher failed to use it properly and wisely. Because they did not provide any clues or lively illustrations that corresponded to the picture of each page, these children gradually lost their attention to and interest in the picture book, which led to the no-response situation at the end of this activity. Additionally, this motivational activity was not directly connected with the subsequent activity (i.e. Jigsaw Puzzle Activity), which was the main learning activity for the concepts of space and shape.

Comparatively for the second-tier inquiry-based teaching experiments, two teachers started their teaching processes with the decoration of the designated learning environment. Within this environment, learning contexts were built based on what children encountered in their daily life situations or life-related problems, which were truly useful to motivate their interests in active inquiry. For instant, in the activity "Same Amount in One Box Car", there were small trains (box cars) and some animal dolls in a corner of the classroom. The first learning situation was to connect the learning process to these children's life experiences on taking cars, buses, or trains through use of the teaching aids and discussion. At the beginning, children engaged in exploring different ways of taking these vehicles. Some kids said, "only one animal in one box car", while others said, "some animals cannot catch this train; we need to put more animals in one box car" (OB-1103). Accordingly, one main question was raised during this operation and discussion process: How can we put the same number of animals in one box car so that every one of them can catch this train (RN-1103)? Children started to discuss how to solve this problem.

In short, young children were enthusiastically motivated by this life-related learning situation. They were interested in both the problem-based mathematical learning environment and life-related objects as teaching aids. In the designated inquiry-based mathematics curriculum and instructional design, all teaching aids are purposefully selected from life-related objects for children to manipulate practically; for instance: small sand bags for measurement, darts for addition, etc. In the activity

“My Family Picture”, children are asked to bring their family pictures and introduce all people in their families to classmates. Within this learning process, the concepts of “direction” and “reciprocal positions” are included, which may help these children to carry out an introduction correctly and successfully. At the end of this activity, one child said: “How fun it was! I learned how to distinguish “right” and “left”; if you stood on my left side that meant I was on your right side” (IN-S4-1028). In this case, it was evidently that these life-related teaching aids, which were merged in the inquiry-based learning process, helped young children to solve the confronting problem with high motivation and interest.

Teachers’ role and children’s learning

[Theme 2] Adjusting the teachers’ role to be student-centered allows young children to learn actively and naturally

For the first-tier traditional teaching process, teachers usually focused on designing the lesson plans in advance, which allowed them to teach step-by-step. Within the first month, we observed that teachers never modified their lesson plans (RN-1104). As T1 mentioned, “we set up the teaching objectives and designed the activities; then, we just follow this instructional design so that we could reach the objectives eventually. As a result, the final assessments are important” (IN-T1-1105). That is, these teachers emphasized how they taught and what their children learn in the classroom. In one interview that the following conversation occurred:

R: Which part of the mathematics curriculum and instruction do you think is the most important?

T2: Parents truly care about what their children learn in the school. So, we have to show them the final products or outcomes, e.g. learning sheets or teaching aids that they can take home and finish at home.

R: How are you sure that they are achieving the objectives?

T2: They will show you what they learn while they are counting, measuring, or writing learning sheets.

In brief, we noticed that the two teachers believed children’s abilities have to be visible (observable). Thus, the use of the assessments focuses more on the lower level of the cognitive domain, e.g. understanding or comprehension. Also, parents’ expectations influence teachers’ instructional strategies and roles. Since parents like to see the immediate effectiveness or direct behavioral performance, this conception often support a teacher’s role as the only authoritative leader of instruction and knowledge in the classroom. In fact, student-centered teaching and learning is the main educational philosophy of this modern era. Therefore, the curriculum (and instruction) usually is emergent in the learning process and incorporates children’s interests. However, at the first tier, the teachers used fixed curriculum and instruction that disregarded their children’s learning interests and life experiences.

Another example of failure to capitalize on real events was seen in the addition and subtraction activity that used animals in the zoo (pictures) as main objects. At the same time as the teacher was instructing, there were turtles in an aquarium at the back of the classroom. One turtle just climbed out the aquarium accidentally while the mathematical activity proceeded. Suddenly, children noticed this phenomenon and turned their attentions to the turtle instead of the ongoing mathematical activity (OB-0928). In this case, animal pictures simply were not as attractive as the live turtle. However, the teachers did not adjust the curriculum and instruction to incorporate the transference of children’s interests or attentions. On the contrary, they asked their children to sit straight and pay attentions to the teacher’s instruction. Our understandings of inquiry-based concepts lead us to think that this unexpected incident exactly matched the targeted learning concepts. If the teachers were able to follow children’s interests in this incident and were able to modify the curriculum and

instruction using the event, it would have helped the children learn enthusiastically and effectively: through observing the aquarium, they could count the total amount of turtles in the aquarium. How many turtles climbed out of the aquarium, how many were left in aquarium? How many legs does a turtle have, and how many are there for two turtles? By asking these questions, the targeted concepts (numbers 1-10, and even numbers — multiples of 2) and operations (e.g. addition and subtraction) could have been taught and learned in this learning context, which would have been more beneficial in advancing their learning motivation and outcomes.

Different findings were revealed at the second-tier inquiry-based mathematics activities. At the beginning of the second tier, the two teachers indicated that their children spent a great deal of time at the phase “free play”. During the professional dialogue meeting, they discussed with the researcher about how to help these children explore problems via free play. The researcher guided the two teachers to watch the teaching video and observe when the right timing was for providing children with cues (i.e. when do meaningful inquiry activities occur) and what kinds of strategies they might employ for the guidance. In this way, the inquiry activities were truly activated where children could explore multiple inquiry questions in different groups. As T1 mentioned,

We were not able to identify what they were doing related to the mathematical concepts during the “free play” phase, so we were worried about when and how to pull them back. But, the professor’s discussed with us key points and useful strategies for observation. Thus, we gradually found out what was going on in the learning process and how to manage this instruction. We definitely had to refine our roles in the classroom. (IN-T1-1011)

At the end of the semester, T2 also reflected her professional growth and role change in the classroom in an interview. She confirmed that children were active and happy learners in this inquiry-based curriculum, and it was a “win-win” situation. She said,

The major difference between this inquiry-based curriculum and instruction and the traditional teaching approach is the teacher’s role. In the past, I was the leader and the evaluator; but, I have multiple roles now: An observer, a person who provides guidance, an expert who generalizes inquiry problems, and a teacher who collects information about my children’s cognitive behaviors. I think the most difficult part is to create situations of cognitive conflict. This is what we never learned from the pre-service training programs. It is worthwhile to spend so much time to do it, and I am glad that I have this professional development that allows me to closely observe my children’s active learning and happiness. (IN-T2-0126)

In summary, these two teachers were actually disenthralled from the traditional and authoritative role while teaching. They learned how to be an activator of children’s inquiry activities, in which they knew how to “let go” and give ample time for children to make mistakes. Once teachers are learning and reflecting professionally, their children can experience the true advantages of inquiry-based mathematics curriculum and instruction.

Peer relationships and mathematical learning

[Theme 3] Peers are the best learning partners. Having good peer relationships are beneficial for expressing, sharing, and reaching the common view since mathematics is about communication

At the first-tier traditional teaching process, it was observed that children operated teaching aids after the direct teaching provided by the teacher. Children usually accomplished the operation of teaching aids, as prescribed, and filled out the learning sheets individually. Accordingly, children's learning experiences came directly from the teachers and the operational and self-constructed processes. Less activity was finished in groups. The sharing phase at the end of each activity became a routine where only questions and answers (but less answer was truly from children). Here was an instructional routine in one class period:

The teacher introduced all kinds of shapes in a jigsaw (10 minutes). Children operated their jigsaws individually (25 minutes). The teacher asked three children to answer her questions respectively, while others remained silent and clapped (5 minutes). (OB-0915)

During children's operations of teaching aids, the researcher randomly interviewed some kids. Their answers just reflected their true learning conditions.

R: Do you like mathematics classes?

S22: No, I don't like them.

R: Why?

S22: Because it is boring, just operations. I cannot talk to others.

R: Did you remember any interesting mathematics activity in this class?

S22: Yes, one time...we went outside of the classroom. We were divided into groups, and used different tools to measure the slide and trees. We discussed what and how to measure with my classmates in groups. (IN-S22-0930)

In brief, this interview actually reflects that children often do not like to learn alone; by contrast, they like to learn and interact with peers in a life-related situation.

In fact, we found that children possessed higher learning interests while they were working in groups with peers at the second-tier inquiry-based activities. Through expressing and sharing their ideas, they were able to elaborate the learned mathematical concepts. They reached a common consensus of their inquiry results after the democratic processes of negotiations and communications. For example, in the activity "Supermarket", kids were divided into groups, six kids in a group. Kids in group A discussed everyone's task after they reached the problem-solving strategy. They drew a working-process map: 1) cutting DMs of the supermarket; 2) categorizing all products; 3) sticking products (on advertisements) on different boxes (boxes as those products); 4) mapping the floor plan for all products; 5) slotting all products (boxes) (RN-1105, 1106). Even though there were some quarrels and arguments in the mapping and discussion processes, they still found a way to reach a common view (e.g. they played the finger-guessing game to decide what they wanted). This inquiry process and problem-solving approach is one of the best ways of presenting the notion "mathematics is about communications", in which peer interactions provide scaffolding to all children's learning.

Flexible time, teaching in context, and emerged inquiry process

[Theme 4] Inquiry-based instructional time is flexible, teaching activities are contextual, and children's inquiry process is emergent and extendable

At the first-tier traditional teaching process, every class period is 40 minutes. In such a short time for each period, time is always a concern for teachers. Children are usually asked to follow teachers' orders, where less active inquiry learning occurs. Two activities in two periods are not connected or related. All instructional activities are designed to reach pre-determined objectives, which follows the three steps:

raising the motivation, developmental activity, and synthetic activity with the assessment. As T1 mentioned,

Regarding the limitation of children's attention, one class period is 40 minutes. Also, we teach one small mathematical concept in every class period. We are afraid that they do not understand or absorb it if we teach too many concepts or use too much time. (IN-T1-0909)

Comparable to the developed curriculum, there were eight mathematical activities in the first tier, which were composed of five activities of number and quantity, two activities of shape, and one activity of measurement. These eight activities are relevant to the children, and no extended activity is found for any of them. The teachers used "review" and "assessment" as the synthetic activity in the last class period. However, "it is a critical concern that children's learning is fragmented because these individual activities are actually independent." The two teachers' may have a misconception of "mathematics is algorithm" (RN-0929). Fortunately, this concern gradually disappeared during the second tier teaching experiments.

At the second-tier inquiry-based teaching experiments, the inquiry-based teaching path of the designated curriculum and instruction was generalized from theories of early childhood development, curriculum and instructional design, and mathematics education. The result was that the teaching content consists of four dimensions: number, quantity (measurement), space/shape, and logistics. It emphasizes the use of the "5E inquiry learning cycle" for designing the inquiry-based teaching processes, which lead to young children's learning as an in-depth inquiry process. Additionally, learning materials and teaching aids are developed or selected based on a problem-based and life-related approach, in order to connect these children's life experiences to their learning in the classroom. All instructional activities are integrated and connected, and associated with an extendable activity design for future learning.

For example, in the activity "A Life of Seed" project, through reading a picture book, children actively discovered the maturity process of the seed, which inspired them to actively operate the picture cards. However, these kids were not satisfied with the picture cards. In one group, one kid proposed that they used "real-object" photos as substitutes. Therefore, they decided to conduct an activity of "planting green beans". Kids of another group decided to draw sketches of the seed's maturity process. In addition, they discussed how to record the maturity process, e.g. "how many days are better for observing an obvious but acceptable change, two, three, or more days as an interval?" (RN-1218). Finally, this activity lasted three weeks (i.e. there were other ongoing activities since it employed a thematic curriculum approach). The discussions happened in the regular class periods, at lunch time, or even at home (discussed with parents). Through these inquiry-based teaching experiments, the children's learning is ubiquitous. What they learned in this activity "A Life of Seed" included not only how to use observations and logistics to ascertain the sequence of those picture cards of the maturity process, they developed and thought how to make their own cards (e.g. photos or sketches) to present the seed's maturity process through a real planting experience. In summary, we actually could see the advantages of the designated inquiry-based mathematics curriculum and instruction, which included a flexible time frame and contextual teaching activities, emergent and extendable inquiry processes. These advantages are authentically beneficial both to young children's capabilities of inquiry learning, and their development of basic mathematical concepts, all of which are influential for children's future learning.

CONCLUSION AND DISCUSSION

Children are born to be mathematicians; Learning is more important than teaching in designing curriculum and instruction

Recently, some theorists expressed concerns about Piaget's theory of cognitive development by proposing new theories and models that purport to show evidence that violates Piagetian propositions. Actually, the argument about the cognitive-developmental theory of Piagetian and Post-Piagetian theory is an ongoing controversial issue. Based on Piaget's viewpoint, young children at preschool and kindergarten (pre-K) levels do not yet comprehend concrete logic and are not able to manipulate information mentally (Piaget, 1965). In this "pre-operational" stage, thinking is still egocentric, which means that it is difficult for a child to understand others' viewpoints. In learning mathematical concepts, young children at this stage are short of conservation and show centration. However, this conception of being incapable in learning mathematics is often doubtful and challengeable. Conversely, Antell & Keating (1983) argue that preschoolers and infants may have the ability of conservation, and this conservation concept is also teachable or trainable. With regard to Post-Piagetian viewpoint, Vygotsky's (1978) conception emphasizes that "learning leads development". He believes that social interaction plays an important role in the process of cognitive development, where social learning precedes development. Besides, the "zone of proximal development" is the difference between "the actual development level" (a child can solve problems independently) and "the potential development level" (a child can do with teachers' or peers' help) (Vygotsky, 1978). From this viewpoint, learning happens in this zone. Related to this argument, newer philosophies of modern mathematics education suggest that every child is born to be a mathematician since s/he has the most powerful gift "curiosity". Children can learn mathematics in a natural way when they can explore patterns or rules and are able to speculate according to their observations (NCTM, 1995; Marinas & Furner, 2010).

Also, some scholars claim that young children's ability of comprehending mathematical concepts is inherent (Koechlin, Dehaene, & Mehler, 1997; Rips, Bloomfield, & Asmuth, 2008). In fact, instead of being unaware of some mathematical concepts, young children at pre-K levels possess the ability of informal arithmetic, which is established through life-related understandings and applications (Baroody, 1987; Ginsburg, 1989; Lee, 2010). In addition, children may have a sense of number, space, and shape from birth (Clements and Sarama, 2007; Sarama and Clements, 2009). If so, these initial mathematical concepts and children's mathematical thinking definitely influences their subsequent thinking and development of more complicated mathematical concepts. Recapitulating the design and findings of this study, we designed the designated inquiry-based curriculum in terms of this inherent viewpoint about children's mathematical ability. Additionally, we chose a great number of life-related issues in connecting mathematics to their daily experiences, as well as engaging them in manipulating real objects or concrete teaching aids, which were proven to be effective in previous studies (Erdogan & Baran, 2009; HodnikCadez & Skrbec, 2011; Soydan, 2015). For instance, in the activity "Mating Gloves" (i.e. this is an activity designed for the second semester of Age-6, but not included in this teaching experiment process), the concepts of odd and even numbers are taught by operating real objects, such as: chopsticks, gloves, shoes, and socks. In this way, young children's learning is meaningful and interesting, and they are able to learn the targeted mathematical concepts effectively. In summary, we designed the targeted inquiry-based mathematics curriculum to furnish our kids with enriched inquiry-based learning contexts in order to advance their initial understandings of multiple categories of pre-K mathematical contents and to develop diverse mathematical power. This curriculum was also considered to provide stepping stones so that young children were able to progress on their way to further learning in the elementary level.

Inquiry-based mathematics curriculum is beneficial for future studies of young children's learning trajectory and kindergarten teachers' professional development

In this study, we observed both a traditional mathematics teaching process and the process of the designated inquiry-based mathematics teaching curriculum, and we conducted teaching experiments. Through the comparison of these two teaching processes, we confirmed the feasibility of our inquiry-based mathematics curriculum and instruction and its advantages for young children's mathematical learning. Although this study (in this paper) aimed to execute the teaching experiments for examining young children's learning progresses at age-6 and verifying the proper theoretical framework for mathematics teaching and learning, we also found some evidence of these young children's mathematical learning trajectories. Consequently, the researchers plan to use this inquiry-based mathematics curriculum and instruction as stepping stones to critically understand children's mathematical learning trajectory blueprints in the near future.

The authors support the notion that children's learning and their intellectual development were based on young children's intrinsic developmental progressions. Especially for young children, their learning of concepts and skills in mathematics have distinctive features which cannot be compared with adult leaning trajectories (Clements, 2007; Lee, 2010). Therefore, we as educators have to understand deeply the nature of these children's mathematical learning processes at all stages, and design a series of inquiry-based learning activities that are matched their developmental needs (Kennedy, 2009). Through the implementation of these activities in practical settings, we have been able to clarify children's mathematical learning paths and developmental progress, which are critical elements of understanding their mathematical learning trajectories, deciding the targeted curriculum design and instructional plans, and promoting teachers' professional development (Clements, 2007). In the United States, many country-level studies have emphasized young children's developmental processes of acquiring mathematical concepts in order to provide research-based evidence for establishing the main goals of the designated national curriculum (Clements & Conference Working Group, 2004; National Mathematics Advisory Panel, 2008). The kind(s) of developmental environments and cultural contexts we provide for our kids' mathematical learning deserves to be deeply researched. Classical and unique mathematical learning trajectories will be broadly explored, in future studies, to assist us (educators) in both advancing teacher professional development and creating high-quality curriculum.

In the curriculum design and teaching experiment project, we found that kindergarten teachers were not familiar with the rationale and the theory of inquiry-based mathematics. Teacher knowledge and its application need to be refined through more intensive professional development programs. Based on the findings of this study, the researchers also intend to explore further how to promote kindergarten teachers' mathematical concepts, theoretical understanding of inquiry teaching, and instructional design and implementation capabilities. In summary, in order to be well-prepared in confronting this new generation's mathematics education, kindergarten and preschool teachers must devote themselves to advanced professional development, which will equip them with multiple teaching strategies to create an adaptive and enriched learning environment. Additionally, they can learn how to respect young children's natural learning instincts and help them develop higher level of thinking abilities for their future lives.

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