MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2024, 20(12), em2553
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/15709

Mathematical processes for the development of algebraic reasoning in geometrical situations with in-service secondary school teachers

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Received 15 August 2024 ▪ Accepted 18 October 2024

Abstract

This paper starts from the hypothesis that algebraic reasoning can be used as an axis between different mathematical domains at school. This is relevant given the importance attributed to mathematical connections for curriculum development and the algebraic reasoning makes it possible to articulate it in a coherent manner. A definition of generalized algebraic reasoning is proposed, based on the notion of elementary algebraic reasoning of the onto-semiotic approach, and it is used to highlight the presence of typical algebraic processes in problem solving in geometrical contexts. To develop these ideas, a training course is designed and implemented with in-service secondary school teachers. Based on design-based research, the results obtained are contrasted with the expected answers. In this way, relevant information is obtained on how teachers mobilize different typically algebraic processes, that is, particularization-generalization, representation-signification, decomposition-reification and modelling. Actually, it is clear to affirm that teachers need specific training to improve their skills about how algebraic reasoning can help them to develop mathematical connections with their students.

Keywords: algebraic reasoning, particularization, generalization, algebraic modeling, geometry

INTRODUCTION

Already the NCTM (2000) established that "connections" is a standard for school mathematics from pre-kindergarten through grade 12. Thus, the instructional programs should enable all students:

– To recognize and use connections among mathematical ideas;
– To understand how mathematical ideas are interconnected and build on one another to produce a coherent whole;
– To apply mathematics in contexts outside of mathematics.

The importance of connections, whether intra-mathematical or extra-mathematical, remains central to mathematics education today (Ledezma et al., 2024). This is partly because problems that require making connections encourage the search for essential properties of mathematical concepts, which in turn helps to understand the fundamental characteristics and properties that define them (Hatisaru et al., 2024). In particular, Businskas (2008) already pointed out that the *particularization-generalization connection* is key, that is, when idea A is a generalization of idea B or, reciprocally, when B is a particularization of A.

Kaput (2008), for whom the action guided syntactically on symbols is essential in algebraic reasoning, justifies that these symbols represent classes

This article has been developed within the framework of the CAP PI1029 project, called "Generalized elementary algebraic reasoning for the development of mathematical competencies of the secondary education curriculum", funded by Pontificia Universidad Católica del Perú.

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Contribution to the literature

- This paper provides an extension of the common notion of Elementary Algebraic Reasoning like all kind of mathematical practice involving the processes of particularization-generalization, representation-signification, and decomposition-reification, as well as the complex process of modeling.
- Through design-based research, experimental data are obtained on how in-service teachers mobilize algebraic processes in geometric contexts.
- The statistical analysis links empirical variables which determine groups of in-service teachers with different levels of mastery of algebraic processes in geometrical contexts.

of objects and not concrete elements. So, according to Kaput (2008), generalization is a central aspect of algebraic reasoning. From the onto-semiotic approach (OSA) to mathematical knowledge and instruction, a mathematical practice is considered algebraic when certain algebraic objects and processes are present, among the latter, those of particularization-generalization stand out. As a result of a generalization process, a type of mathematical object called *intensive* is obtained, which corresponds to the rule that generates the class. As a result of the reciprocal process, *particularization*, *extensive* or *particular* objects are obtained (Godino et al., 2014b).

For OSA, the development of algebraic reasoning is encouraged through situations that require making mathematical connections. In particular, Burgos et al. (2022) analyze the importance of elementary algebraic reasoning (EAR) in the development of probability from an intuitive to a formal approach. Likewise, Pallauta et al. (2023) make a study of statistical data tables in textbooks relating levels of stochastic reasoning to those of EAR. Bueno et al. (2022) carry out a study on the definite integral as a measurement problem solved, first by approximate and numerical methods and, gradually, by more formal and general methods.

From other theoretical perspectives, research has also been conducted on the development of algebraic reasoning in a geometric domain (Barana, 2021; Boester & Lehrer, 2008; Silva, 2021). Despite all this, Strømskag and Chevallard (2022) observe that algebra is not a modeling tool in the current school curriculum and show how formulas used in geometry are currently reduced, in many cases, to arithmetic rules applied to numerical values. Therefore, it is necessary to determine a way to promote the development of algebraic reasoning in the school curriculum (Gaita et al., 2022).

It is the teacher's task to identify problems in different intra-mathematical or extra-mathematical contexts that require generalization processes to be carried out. This task is not trivial. Businskas (2008) reports that it is rare for secondary mathematics teachers to identify mathematical connections centered on a hierarchy of complexity. In order for the teacher to effectively perform this task, teachers must identify genuinely algebraic aspects in mathematical practice, as a prior step to progressively influence the algebraization of children's mathematical activity (Aké, 2021). In addition, it is necessary to create instruments for teacher training to show the role of algebra as a modeling tool. This can be achieved by studying situations in different contexts, particularly the geometric one, as well as studying teachers' mathematical performance in dealing with such situations.

The main objective of this article is to explore how a group of mathematics teachers recognize the presence of algebra and algebraic processes in problems in geometric contexts. To achieve this objective, firstly, the specific objects and processes of EAR are introduced, and a generalized perspective of EAR is proposed. This generalized perspective allows the mobilization of the EAR in different contexts, particularly geometric ones. Secondly, design-based research (DBR) is developed from which experimental data are obtained through a questionnaire designed ad hoc with tasks that require algebraic reasoning in a geometric context. The results obtained are discussed and related through implicative analysis, and then the findings are contrasted with expectations. Finally, a synthesis and implications for teaching, oriented to the development of algebraic reasoning, are presented.

THEORETICAL FRAMEWORK

Objects, Dualities, and Mathematical Processes

One of the objectives of the OSA is to analyze mathematical practices, as well as the types of objects and processes involved. Thus, it identifies a series of *primary objects* (problems, definitions, propositions, procedures, and arguments) and *elementary processes* associated with them (communication, problematization, definition, enunciation, algorithmization, and argumentation). Moreover, these objects and processes do not have an absolute nature but are relative to the context of use. Dualities are thus determined: *extensive (particular)-intensive (general)*, *expression-content*, *personal-institutional*, *ostensive-non-ostensive* and, finally, *unitary-systemic*. In addition, dualities are associated with processes (Godino et al., 2007). **Figure 1** represents these elements.

To exemplify the dualities, we considered the following: "3" may represent:

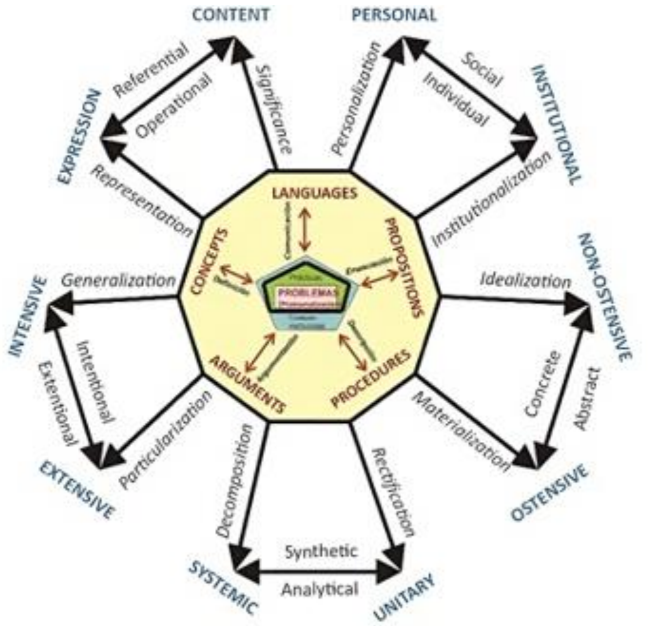


Figure 1. Onto-semiotic configuration of objects, processes, and dualities (Godino et al., 2016, p. 97)

- the number of bottles on a table (*extensive*),
- the way this quantity is written (*expression*),
- the activity performed by a child who counts how many bottles there are using the knowledge of "the last number indicates how many" (*personal*), and
- the verbal answer given by the child: "there are three" (*ostensive*).

If these dualities are not related to any other object, process, situation or context, they do not have a *unitary* character. However, the teacher has posed this counting as a phase prior to the situation "setting the table", where the objective is the cardinal and the correspondence between coordinable sets (*institutional*). Actually, after counting the bottles, the children establish a connection between the glasses and chairs necessary for 3 people to sit down to drink. This way, "3" is the common characteristic of all sets of objects (*content*) and can be used to complete a table with plates, glasses and cutlery (*intensive*). This general knowledge is not verbalized but observed "in the act" (*non-ostensive*). Finally, "cardinal" knowledge will be used in other contexts in relation to other mathematical notions (*systemic*).

With respect to the processes involved, writing "3" implies a *materialization* of this number, which could have been done differently: saying "three", writing "three" and showing "three" fingers. Now, this ostensive display of the number three in a given context can be abstracted or isolated to use it in other contexts. This abstraction implies an *idealization* of the number. In the same way, for the other dualities, processes are defined to characterize them and make their emergence possible. These processes are generalization-particularization, representation-signification, institutionalization-personalization, materialization-idealization, decomposition-reification. In turn, these processes are related.

For example, the process of idealization is sometimes mobilized through a process of *generalization* ("three as a class of sets with three elements"), through a process of *signification* that brings a different meaning ("three as a place in any list"). It could also consider a process of *decomposition* ("three as an example of cardinal or ordinal number as any natural number and, therefore, three is understood as part of the system of natural numbers"). Likewise, objects, processes, and dualities possess either a personal or institutional nature, depending on whether the mathematical practice is performed by an individual or whether it refers to the meaning attributed by an institution.

Finally, the processes referred to so far, associated with both objects and dualities, can be part of more complex processes, such as problem solving, modelling, or the study of relationships or structures (Godino et al., 2012). In the following section these concepts are adapted to define the *generalized algebraic reasoning* (GAR).

Generalized Algebraic Reasoning

In a "classical" school sense, algebra is the domain where formal manipulation of symbols begins. The expression "I was doing well until letters appeared" is a way of explaining the use of unknown quantities in solving equations or variables in defining functions, which appear in a generalized way in most curricula of different countries in the first years of secondary education (Castro, 2012).

This reductionist view of algebra, exclusively associated with the literal manipulation of symbols, leads to a pejorative interpretation of the term "curriculum algebraization", according to which problem solving is reduced to stereotyped exercises using given formulas. For this reason, some research shows concern about the role of algebra in schools. It is concluded that there is a need for "an imperative revitalization of the elementary algebra curriculum" (Strømskag & Chevallard, 2022, p. 1).

Based on the different theories that study algebraic reasoning, proposals have been made by the OSA to characterize objects and processes at stake in algebraic practices. Thus, a structure of *algebraization levels* is proposed to gradually develop *EAR*, from primary (Godino et al., 2012, 2014b) to compulsory secondary school (Godino et al., 2015). For a more detailed analysis, sub-levels have been characterized, considering different representations and transformations between them, degrees of generalization and functional reasoning, mathematical structures and structural reasoning, as well as analytical calculations (Burgos et al., 2024).

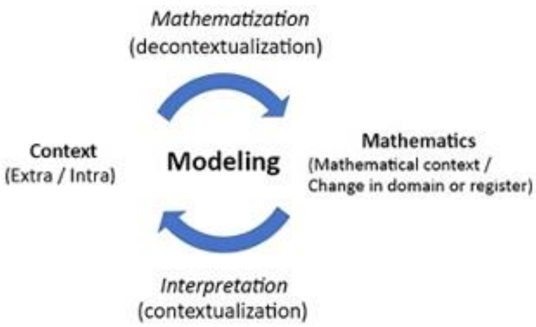


Figure 2. Modeling process (Source: Authors' own elaboration)

Dual processes are important in every mathematical activity, but in algebraic practice, the processes of particularization-generalization, representation-signification and decomposition-reification are particularly typical. It has long been recognized that generalization is a key feature of algebraic reasoning (Carraher et al., 2008; Cooper & Warren, 2008; Mason & Pimm, 1984), which has its dual side in particularization. Moreover, generalization is constituted as a criterion or rule that delimits elements of a set with a common characteristic. "The set then becomes something new, different from the elements that constitute it, as a unitary entity emerging from the system. Therefore, in addition to generalization, which leads to the set, there is a process of unitarization [or reification]" (Godino et al., 2014b, p. 205-206). Mathematical practice may require decomposing this new unitary object; this happens in the process of particularization. Finally, progress in algebraic practices is conditioned by giving meaning to a different way of representing mathematical objects. Thus, the dual process of representation-signification is key, allowing abstraction as a process that distinguishes expression and content.

EAR is a way of thinking and acting in mathematics, mainly characterized by the intervention and emergence of intensive objects at progressive levels of generality. However, generalization is not exclusively studied algebraically, nor do all algebraic activities involve generalization (Godino et al., 2012). In fact, this way of "reasoning algebraically" is applied transversally in the different domains of mathematics in secondary education (Burgos et al., 2024). For this reason, the process of modeling is considered particularly important. Modeling can be *intramathematical*, when a mathematical object is turned into another by changing the domain or register, or *extramathematical*, when a non-mathematical situation is "converted" to a mathematical one. In both cases, the change of context implies *mathematization*, that is, establishing a formulation of the situation in a different register or domain and identifying an operative and discursive practice for solving it. Once a solution to the mathematical problem has been determined, an *interpretation* of it is made in the original context. Modeling is thus a full schematic "mathematization-interpretation" cycle (**Figure 2**).

Thus, a "positive" vision of "curriculum algebraization" is adopted as a method that seeks to balance the dual processes of particularization-generalization, representation-signification, decomposition-reification, and to incorporate *modeling* (mathematization-interpretation) as an unavoidable complex articulating process. This leads us to propose a perspective of algebraic reasoning that exceeds the scope traditionally assigned to it in the curriculum, which we call *GAR*.

Elements of the Model of Mathematical Didactic Knowledge

In the OSA to mathematical knowledge and instruction (Font & Contreras, 2008; Font et al., 2008; Godino et al., 2006a, 2006b, 2007; Montiel et al., 2009) five components of analysis have been proposed:

1. *Onto-semiotic configuration*, which involves the elaboration of networks of mathematical objects and processes at three levels:
   a. referential network or framework of the instructional process,
   b. to be taught network and, finally,
   c. actually taught network.
2. *System of practices*, which implies the analysis of types of problems and their articulation in *operational practices* (what and how to do) and *discursive practices* (justification of relevance and discussion of their significance).
3. *Educational trajectories*, which involves the determination of potential trajectories, and the analysis of trajectories actually implemented, as well as the discussion of the didactical interactions observed.
4. *Contextual regulations*, which allows of the identification of the system of norms that, implicitly or explicitly, determines the relationships between teacher, students and the mathematical content under study.
5. *Didactic suitability*, which allows the assessment of the suitability of the planned and effectively implemented study processes in three global facets: *student-centered*, according to cognitive and affective dimensions; *teacher-centered*, according to interactional and mediational dimensions; and, finally, mathematics-centered, according to epistemic and ecological dimensions.

The teacher develops *spontaneous epistemology* (Brousseau, 1997) out of sheer professional necessity, which allows him to make decisions in the design, implementation and evaluation of instructional processes he/she is in charge of. However, while this spontaneous model is necessary, it needs to be improved in the pre-service and in-service teacher training.

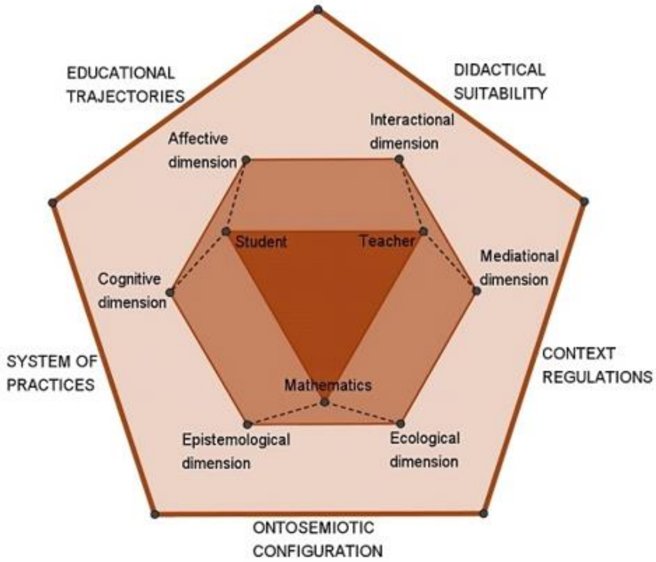


Figure 3. Dimensions and components for educational analysis and interventions (Source: Authors' own elaboration)

When the teacher plans a type of interaction and establishes the use of material means in a temporal sequence, he must consider both the mathematics to be taught and the students who will learn it. The *didactic-mathematical knowledge model* (Castro & Pino-Fan, 2021; Castro et al., 2018, Godino et al., 2017, Pino-Fan et al., 2015, 2018) is a tool to interpret and characterize the mathematical and didactic knowledge of the teacher 'in context'. This means that the possibilities and restrictions of the *dimensions involved* (epistemic-ecological, cognitive-affective and interactional-mediational) must be considered. Besides, the *educational moment* (design, implementation or evaluation of an instructional process) and the *component under study* (onto-semiotic configuration, system of practices, educational trajectories, contextual regulations or didactic suitability) must also be considered.

These components of analysis are mutually related and condition the design of instructional processes and their implementation in the classroom (**Figure 3**).

METHOD AND INSTRUMENT

In this sense, this paper seeks to contrast, by means of experimentation, the theoretical foundation proposed in relation to the GAR since it is located within the *DBR* (Godino et al., 2014a), which consists of 4 phases: preliminary study, design, implementation, and evaluation. The preceding sections constitute a brief preliminary study.

In the design phase, situations are constructed in geometric contexts. Besides, the presence of the typical processes of algebraic practices is justified when solving them. Based on this design, a questionnaire is developed to get information on didactic-mathematical knowledge from in-service secondary school teachers in relation to algebraic reasoning. The answers allows testing the hypothesis that secondary school teachers have a reductionist vision of "curriculum algebraization".

Questionnaire Design and Expected Answers

The questionnaire consists of four situations that in school are situated in the geometrical field because they involve "figures and regions", require "length or area calculations", or their interpretation requires *modeling* "three-dimensional objects". However, while solving these tasks, it becomes necessary to recognize algebraic objects and processes.

The four situations of the questionnaire have the following structure:

– *A mathematical statement, questions involving mathematical knowledge, and a mathematical solution to the problem*. Teachers solve the problem and use the given solution as a means of control. Thus, understanding the solution demands carrying out processes linked to GAR and is part of the professor's mathematical knowledge.

– *Issues to evaluate aspects of didactic-mathematical knowledge of GAR*. In this part, teachers explicitly identify algebraic objects and processes and relate them to answers from potential students of secondary education. This aspect is part of didactic knowledge in the epistemic facet.

The four problems of the questionnaire are presented below, as well as the answers expected by the teachers. This corresponds to the *a priori* analysis of dual processes that should be identified as necessary for task solving. The processes privileged by GAR are detailed: particularization-generalization, representation-signification, decomposition-reification, and modeling (mathematization-interpretation).

Problem 1. Regions on a circumference

The problem involves locating a certain number of points on a circumference, joining them two by two by means of segments, and determining the maximum number of regions into which the circle will be divided by these segments.

Figure 4 shows the statement and the question on mathematical aspects. A sequence of figures appears, corresponding to the first three elements of the sequence, accompanied by a sequence of numbers, corresponding to the number of regions determined by these points. The mathematical question demands three new elements of the sequence.

The expected answers in terms of the processes privileged by algebraic reasoning for problem 1 are:

- *Particularization-generalization*. Teachers are expected to recognize that a (*general*) conjecture cannot be established on the basis of the first 5 (*particular*) cases; this will be done by contrasting

Statement. In different circles are successively placed 1, 2, 3, ... points, which are joined by segments, forming the maximum number of regions. For example, the first three cases correspond to the following figure:

Number of points	1	2	3
Number of regions	1	2	4

Question. Determine the maximum number of regions with 4, 5, and 6 points.

Solution. The table shows the 6 terms:

Number of points	1	2	3	4	5	6
Number of regions	1	2	4	8	16	31

Issues

a) If you propose the same situation to your students, how do you think they would do the task and what answers would they give?

b) If you ask your students to determine the general term of a sequence whose first terms are 1, 2, 4, 8, 16,… What do you think they would answer? Why?

Figure 4. Statement, solution, and didactic-mathematical issues associated with problem 1 (Source: Authors' own elaboration)

with the actual construction of the sixth element of the sequence. On the other hand, teachers will indicate that a possible pattern identified by their students is simply to duplicate the previous term, based on the first 2, 3, or 4 terms.

- *Representation-signification*. While counting, a representation is made to maximize the number of areas, so the expectation is that such representation be taken as a "type" or representative of a class, preserving the meaning of a figure that meets the conditions of the problem. Thus, in the case n=6, a regular hexagon is not a representative of the situation, since the number of regions it generates is not maximal (30 regions instead of 31).
- *Decomposition-reification*. Based on experience, students are expected to obtain an expression linked to a known sequence type, whether arithmetic, geometric, or other types of progression similar to one previously studied.
- *Modeling* (mathematization-interpretation). It is intramathematical, and the graphical representation of the regions is mathematized by means of a table of values, which is interpreted in terms of the maximum number of regions that can be constructed.

Problem 2. Segment length

Figure 5 shows the statement and the mathematical question of the second problem. This is a task in which the location of points on the line is described, and relationships are established between the lengths of the segments defined by these points. The question is to determine the length of one of the segments.

Statement. Segment AD includes points B and C ($B \neq C$), with B closest to A and C closest to D. The following applies: $3\overline{CD} = 4\overline{AC}$ and $3\overline{BD} - 4\overline{AB} = 105$ units.

Question. Find $\overline{BC}$.

Solution. The length of each segment can be represented in the following way:

From the data, we have the following:

$$3n = 4(m+p), 3(p+n) - 4m = 105$$

Or, equivalently,

$$3n - 4m = 4p, 3n - 4m = 105 - 3p$$

Where:

$$4p = 105 - 3p \rightarrow 7p = 105 \rightarrow p = 15 = \overline{BC}$$

Issues

a) Do m and n take unique values? Justify your answer.

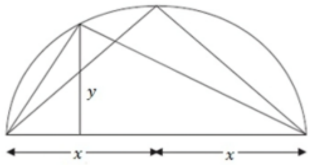
b) Show three pairs of integer values for m and n.

Figure 5. Statement, solution, and didactic-mathematical issues associated with problem 2 (Source: Authors' own elaboration)

The answers expected in terms of the processes privileged by algebraic reasoning for problem 2 are, as follows:

- *Particularization-generalization*: Particular solutions of integer doubles $(m; n)$ must be recognized, as well as the general and necessary relationship there is between the lengths of these segments, that is, $3n - 4m = 60$. As a result of interpreting this relationship, positive integer solutions are obtained, where "m is a multiple of 3 and n is a multiple of 4, being $n > 20$" applies.
- *Representation-signification*: In the solution, graphical (on the number line) and symbolic (by naming the segment lengths) representations are involved, between which relationships are expected to be established. In addition, the result of the symbolic manipulation must acquire meaning in graphical terms. Furthermore, the graphical representation in the solution is an ideogram, since, although distance $\overline{BC} = p = 15$ is fixed, measures $\overline{AB} = m$ and $\overline{CD} = n$ may take infinite values since the relative position of points A, B, C an D must be preserved on the number line.
- *Decomposition-reification*. Relationships of geometric lengths are interpreted by means of multiplicity properties of natural numbers and by imposing restrictions on the measures specific to the geometric context. This requires the establishment of mathematical connections between two domains and, therefore, understanding mathematical objects according to the unitary-systemic duality.
- *Modeling* (mathematization-interpretation). It is intramathematical; the collection of geometric solutions is mathematized by means of an indeterminate compatible system of equations,

Statement. Calculate the difference D between the areas of two triangles. The base of both triangles is the diameter of a circumference of radius x and vertex y opposite to the diameter is located on the circumference. One of the triangles is the one with the maximum area (that is, the one with the maximum height). Thus:



Question. Determine the expression of difference D using letters x and y.

Solution. Calculate the difference between the maximum area (A_{max}) and the area of the triangle with height "y" ($A_{alt(y)}$).

$$D = A_{max} - A_{alt(y)} = \frac{(2x)\cdot x}{2} - \frac{(2x)\cdot y}{2} = x^2 - xy$$

Issues.

a) For each circumference, what type of function is D?

Figure 6. Statement, solution, and didactic-mathematical issues associated with problem 3 (Source: Authors' own elaboration)

whose resolution allows determining the relations on the segments, which must be interpreted according to their relative position in the given ideogram.

Problem 3. Areas of triangles

Figure 6 shows the statement and the mathematical question of the third problem; a semi-circumference and two inscribed triangles are presented in such a way that one side of the triangle is the diameter of the semi-circumference. The task is to obtain the difference between the areas of the two triangles. The question requires recognizing that one of the triangles has a constant area and that the area of the second triangle depends only on its height.

The answers expected in terms of the processes privileged by algebraic reasoning for problem 3 are:

- *Particularization-generalization*. It is expected they recognize that the figure is a representative of all semi-circumferences; therefore, any conclusions drawn can be generalized to all of them. Measures "x" and "y" do not represent particular metric values, but play the role of parameter and variable, respectively.
- *Representation-signification*. The correct interpretation of formula D for the difference between areas requires the conventional use (in school) of "x" as a variable and as a parameter in the problem. In fact, this interpretation would not be a problem if instead of denoting the radius with "x", it would have been denoted with "r", and instead of "y", "x" would have been used as the "first" variable. On the other hand, the use of "r" (radius) and "a" (height) would have caused a dual problem to the one given, since both letters ("r" and "a") would have been interpreted as parameters based on which a solution would be obtained.

Statement. Imagine that the Earth is a perfect sphere and that a rope is placed tightly around its equator.

The rope is extended and 1m is added to its length (thick line at the end):

Question. Knowing that the length of the radius of the Earth is 6371km, if the rope is repositioned around the equator, will the rope still wrap around the Earth tightly; or if the two ropes are placed side by side, will you be able to tell which is which?

Solution. To answer the question, a strategy is to model the problem with two concentric circumferences:

Internal, whose perimeter is the perimeter of the rope surrounding the Earth at the equator (whose radius is 6371km).

External, whose perimeter is that of the previous rope plus 1m.

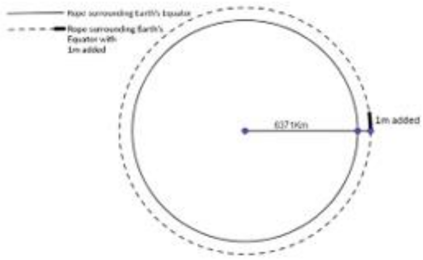


Figure 7. Statement, solution, and didactic-mathematical issues associated with problem 4 (Source: Authors' own elaboration)

- *Decomposition-reification*. It is expected they identify the relationship between the elements of the triangles and the semi-circumference. As a consequence, the notions of variable and parameter should be "explained" one in relation to the other, since both "vary". However, it is also expected they recognize that they play a different role; namely, "y" as an independent variable and "x" as a parameter that determines a class of objects.
- *Modeling* (mathematization-interpretation). It is intramathematical; it starts from a context of areas of triangles inscribed in a semi-circumference, and it is mathematized by means of a function, whose interpretation determines its domain of definition and the distinction between variable and parameter.

Problem 4. Radius of the earth

The fourth problem proposes to associate the shape of the earth to that of a sphere with a known radius and asks to compare the effect of encircling the earth at the equator with two ropes of different lengths, as shown in **Figure 7**.

The purpose of the issues is that, by repeating the procedure in other particular situations, it is recognized that the answer does not depend on the radius of the sphere considered.

The answers expected in terms of the processes privileged by algebraic reasoning for problem 4 are:

- *Particularization-generalization*. Once the problem has been solved in the plane, the expectation is to recognize that the procedure can be applied to any sphere, regardless of the size of its radius. To give meaning to the solution (constant function: $\frac{1}{2\pi}\ m \cong 15{,}9\ cm$), it should be noted that the ideogram of the solution–given by concentric circumferences–is valid for any sphere and that, therefore, the new "belt" is always separated by a constant quantity ($\frac{1}{2\pi}\ m \cong 15{,}9\ cm$), regardless of the size of the sphere, whether it is like a pea, a ping-pong ball, a basketball, or the Earth.
- *Representation-signification*. Understanding the solution requires understanding that the graphic representation, both of the earth and of the "extension" of its radius and the addition of 1 m, are ideograms because they have no metric sense (they are not represented to "scale"). Likewise, the representation of the concentric circumferences representing the equator, and the outer extended length is also an ideogram of a three-dimensional object in dimension 2.
- *Decomposition-reification*. Answering the question "what type of function can you associate with the solution of these problems?" involves relating the constant function, as a unitary object, to the "types of functions" system.
- *Modeling* (mathematization-interpretation). It is extra-mathematical; the earth is mathematized by means of a sphere (geometric object closer to the *geoid*). Moreover, mathematical work is not done with the sphere, but with its projection on a circumference. Thus, we have a different type of modelization, specifically an intramathematical one.

Experimentation and Results

After designing the instrument, which considers the processes favored by GAR applied to tasks in geometric contexts and their expected resolution, a study was carried out with secondary school mathematics teachers in continuing education.

Population and sample

The population was made up of active secondary school teachers, with several years of experience, from schools in medium-high socio-economic contexts, located in urban areas.

The sample was intentional and consisted of 32 mathematics teachers from private schools in Peru, comprised of 17 men and 15 women.

Application of the questionnaire

The instrument was applied in a continuing education class, whose objective was to show how to develop algebraic reasoning through different mathematical competencies proposed by the Peruvian national curriculum (MINEDU, 2016). There was particular consideration to the competency of solving problems of shape, space and location, linked to geometric knowledge.

The training course was carried out in person during two consecutive days of four hours each. Teachers answered the questionnaire in the second half of the first session, working individually for a 90-minute period.

Feedback and analysis

Participants' answers were subsequently evaluated. On the second day of training, feedback was provided focusing on the identification of algebraic objects and processes in geometric task solving.

RESULTS AND DISCUSSION: ANALYZING THE ANSWERS TO THE PROBLEMS POSED

This section presents and discusses the results obtained in the experiment, contrasting them with the expectations, that is, an *a posteriori analysis* or assessment is carried out, which is the fourth phase of DBR. This contrast between what was expected (*prediction*) and what was observed (*contingency*) seeks to control the *internal validity* of the observations. Moreover, DBR does not restrict data analysis techniques. Thus, in the first place, elementary descriptive statistics are used in the algebraic processes observed. Second, a statistical implicative analysis is performed, relating the mathematical activity identified in several problems. These two statistical techniques are *triangulated* so as to obtain consistent results and, therefore, a coherent analysis.

Descriptive Analysis of Answers to the Problems

Problem 1. Regions on a circumference

The analysis of the first problem reveals that 60% of participants (19 out of 32) based their reasoning on numerical manipulation, omitting graphical representation and figure-number correspondence. This suggests a limited perception of these elements as a system.

Teachers tend to look for patterns or general rules through numerical manipulation, although without interpreting these rules in the context of graphical constructions, that is, without considering that these numbers represent the maximum possible regions.

Dissociation of these two domains (numerical and figural) restricts the subjects' ability to give meaning to the formulas generated. Thus, the operative activity (manipulation of numbers and search for a rule) is not complemented by a discursive activity that gives meaning to the rule. An imbalance between the processes of representation and signification is thus observed. In addition, an independent manipulation of objects and their meanings became evident, without reuniting them in a unitary object linking both domains.

In terms of what responses, they would expect from their students after only presenting them with the number sequence, 53% (17 out of 32) of the teachers related the terms to their position in the sequence or to the previous term. This suggests that, in a school setting, few terms are considered "enough" to infer a rule.

Thirty-one percent of the teachers (10 out of 32) did not provide a correct rule for the first 5 terms of the sequence, suggesting a greater difficulty with geometric progressions compared to arithmetic ones. Finally, 28% (9 out of 32) of the teachers presented answers unrelated to the task, characterized by "general" explanations about the need to obtain rules linking particular cases to a formula, but without actually doing so in the given problem.

In all three cases, regardless of the correctness of the teachers' answers, it is clear that particularization-generalization processes are inherent to this type of task and, therefore, they are key to the development of algebraic reasoning.

Problem 2. Segment length

In the second problem, 78% of the teachers (25 out of 32) recognized the existence of infinite solutions for the lengths of segments m and n, but only 50% (16 out of 32) were able to present three particular cases. This shows a greater difficulty in recognizing the particular from the general.

Only 18% (6 out of 32) managed to translate "$3n - 4m = 60$" in terms of restrictions on n and m, that is, n and m must be multiples of 4 and 3, respectively, with m>0 and n>20. This indicates a disconnect between the use of literal language in the context of arithmetic and the particular meaning, for each of the lengths n and m, in the context of geometry. This finding reflects difficulties associated with the representation-signification processes.

Problem 3. Area of triangles

Eighty-four percent of the teachers (27 out of 32) associated the symbolic-literal writing "$D^2 = x^2 - xy$" with a quadratic function (24 teachers) or with a function with two variables (3 teachers). This reveals an interpretation of the formula without the geometrical context, which is linked to the conventional use in school, where letter "x" usually represents the independent variable in functions of real variables, and, in the university environment, "x" and "y" are commonly independent variables of functions of several variables. Therefore, an absence of contextual meaning is detected, associated with the geometric interpretation of function D for each particular circumference of radius "x". This indicates that representation-signification processes are not carried out appropriately.

Teachers did not recognize that the given geometric representation corresponded to only one (material, ostensive) representative of the set of all semi-circumferences with diameter $2x$, so "everyone" (intensive) met the same condition: the triangle of maximum area is the isosceles triangle whose base is the diameter and height is the radius (x). The prevailing idea is that this is a particular case, in which x is unknown, and not a general case, in which x is a parameter that allows describing a family of semi-circumferences. Moreover, these teachers did not distinguish between the function of "x" as a parameter (for each circumference) and that of "y" as an independent variable (height that makes the areas of all possible triangles vary). In other words, these teachers showed an inappropriate application of the decomposition-reification process that relates the parameter-variable as a system, where these objects are explained one in relation or in opposition to the other.

Problem 4. Radius of the earth

Sixty-eight percent (22 out of 32) stated that the function does not depend on r and that, therefore, the result is not exclusive to the terrestrial case, but it can be applied to any sphere, which evidences the occurrence of particularization-generalization processes. From the correct resolution, it can also be noted that the teachers understood the meaning of the proposed ideograms, which allowed them to transcend the material representation of the earth and the equator, demonstrating the realization of representation-signification processes.

However, 87% (28 out of 32) of the teachers did not identify the function as constant. This fact is contrary to the correct interpretation of the model, which allows stating that the extended ropes are equally separated from the sphere, regardless of its radius. The modeling process of the teachers is not evident. On the other hand, 34% (11 out of 32) associated the term "intuition" with "ease" in solving mathematical tasks and not with understanding how the difference in radii varies.

From the analysis of the answers, we conclude that, in addition to the processes of particularization-generalization and representation-signification, it is essential to consider the processes of decomposition-reification for the development of algebraic reasoning. The latter are crucial in order to obtain unitary objects that harmonize object systems. In fact, objects can be

Table 1. Variables identified in the answers

Question	Variable	Description
1	P1n	It reasons based on manipulating numbers and not on graphical representation.
	P1f	It reasons according to the figures and geometric patterns seen in them.
	P1e1	[Error] It gives a recursive formula that multiplies the previous one by 2.
	P1e2	[Error] It gives a formula to $a_n=2n$.
	P1e3	[Error] It gives a formula to $a_n=2^n$.
	P1ok	[Correct answer] It gives a formula to $a_n=2^{n-1}$.
	P1g	It provides an answer based on procedures, calculations, or generic knowledge, without tackling the given problem.
2	P2n	[Correct answer] It points out that *m* and *n* do NOT take unique values.
	P23	[Correct answer] It provides 3 specific examples.
	P2m	[Correct answer] It states that *m* and *n* must be multiples of 4 and 3, respectively.
	P2r	[Correct answer] Its sets restrictions on the value of $m > 0$ and n > 20.
3	P32	[Error] It points out that the given function is a 2-variable function.
	P3c	[Error] It points out that it is a quadratic function.
4	P4c	[Correct answer] It indicates that the function is constant.
	P4ok	[Correct answer] It gives the value $(1/2\pi)$m or 15.9 cm.
	P4n	[Correct answer] It provides a correct justification, indicating that the answer does not depend on the radius.
	P4l	[Error] It points out that the given function is a linear function.
	P4i	It points out that intuition is related to the resolution and not to the context.

represented with different languages and, thus, their meaning as an articulated whole might be compromised if a "new" object is not constructed to unify them.

Finally, it has been observed that adequate modeling, which implies a representation and manipulation of the mathematical model, does not in itself guarantee a correct interpretation in terms of the situation. This way, the importance of the subprocess of interpretation is highlighted, defined herein as the mathematical process that gives meaning to a mathematical model, beyond an operative and discursive practice restricted to a specific situation. Thus, interpretation in problem four requires stating that "if one meter is added to the length of a great circle (particularly, the equator) of any sphere, then the circumference concentric with the great circle formed by the new length has a radius that is approximately 15 cm larger". This statement not only generalizes the given problem, but also gives it a new meaning: not only is the radius constant, but the effect is the same for any sphere, whether it is a pea or the earth. The latter was verified, both by the absence of answers to the questionnaire reflecting this statement, and by the subsequent discussion with the whole group. Along these lines, it can also be noted that such a result goes against what intuition suggests.

Relational Analysis of the Answers to the Problems

The results observed and detailed "question by question" in the previous section reveal four fundamental processes in the development of algebraic reasoning: particularization-generalization, representation-signification, decomposition-reification, as well as the complex process of modeling. In this section, the answers are related in order to outline

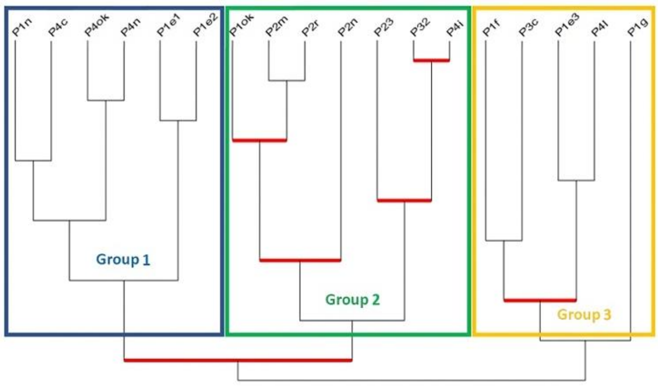


Figure 8. Similarity graph (Source: Authors' own elaboration)

distinctive "profiles" among the teachers to help understand the answers provided in general.

Since this is a DBR, variables are defined from the answers given in each problem. These isolated results are then linked by means of statistical implicative analysis (Gras et al., 2008). **Table 1** describes the variables considered; their names follow the same structure: they begin with "P1, P2, P3, or P4", which represent the problem they refer to, and then one or two characters are added to allow their precise identification.

The similarity graph (**Figure 8**) allows us to establish 3 groups of related variables, which determine different behaviors that we will call numerical, consolidated algebraic, and pedagogical or unconsolidated algebraic.

- *Group 1. Numerical*. This group only relates variables of problem 1 and problem 4. In regards to problem 1, a resolution based on numbers and not on the configurations themselves (P1n) gives two types of error by establishing a formation rule that is not contrasted with the particular cases available (P1e1 and P1e2). In regards to problem 4, the variables refer to the identification of the constant function obtained (P4c), to the explanation of a justification of this identification ("does not vary according to *r*", P4n) and, therefore, to a correct resolution (P4ok). However, there is no interpretation of the meaning of radius independence. All of this is indicative of a level of algebraic reasoning where the processes of generalization, signification, reification, and interpretation are not mobilized with flexibility in contexts that are not usual in the school environment.

- *Group 2. Consolidated algebraic*. This group is the only one that relates variables of the 4 problems, providing in general correct answers (P1ok, P2n, P23, P2m, and P2r) or answers related to the specific activity (P4i). The only related error is that they point out that the solution function of problem 3 is a two-variable function (P32), which shows a lack of differentiation between variable and parameter. In spite of this, the teachers contributing to this group show a balance between the four dual processes identified, and they do so in a stable manner throughout the questionnaire.
- *Group 3. Pedagogical or unconsolidated algebraic*. This group relates variables only from problems 1, 3, and 4. However, unlike group 2, their answers are incorrect for the three problems (P1e3, P3c, and P4l), and they also lack operational sense (P1f and P1g). Likewise, the answers given follow a principle of "expectation", according to which "it is presumed that the answer must be based on algebraic reasoning, and school indicators of the same are shown without meaning". This is why this group is called "pedagogical", since this type of "answer by expectation assumption" is not specific to mathematics, but to any training course regardless of the content.

The implicational graph (**Figure 9**) reveals that the correct answers establish a strong network between the variables associated with tasks 1, 2, and 4. This suggests that problem 3, where the use of letters is "not from school", has been particularly problematic and required specific measures to develop the dual processes of representation-signification.

Another result that emerges from the implication graph is the direction of the implication between the variables in problem 1 and problem 4. *A priori* we would expect a lower success rate in the last problem than in the first one, which would take the form of an implication P4ok→P1ok. However, the experimental data show a reverse implication, which means that problem 1 has had a lower success rate. This is explained by the fact that generalization from particular data is not evident and, therefore, what is determinant in this problem–more than the context or type of task–is the fact that the operative practice is far from the classical school activity, where the formula of the general term is usually determined by stereotyped procedures.

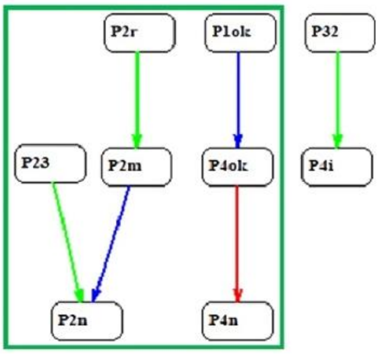


Figure 9. Implied graph at 95% (red), 90% (blue), and 85% (green) (Source: Authors' own elaboration)

Finally, there are no chains of statistical implication between variables in more than two problems. This is evidence that, beyond the context of mathematical activity (in this case, geometric), what is essential are the operative and discursive processes linked to algebraic practices centered exclusively on symbolic manipulation. Thus, the lack of chains of implications between variables is a sign of an inflexible handling of the mathematical processes common to the four problems.

The following section provides some recommendations for teaching aimed at the consistent mobilization of mathematical processes for the development of GAR.

SYNTHESIS, IMPLICATIONS FOR SECONDARY EDUCATION TEACHER TRAINING AND OPEN ISSUES

This paper expands the notion from EAR to GAR, which can be succinctly described as the mathematical practice involving the processes of particularization-generalization, representation-signification, decomposition-reification, as well as the complex process of *modeling* (mathematization-interpretation). Modeling integrates two dual sub-processes: mathematization-interpretation, the former consists of the abstraction of relevant information, either from an extra- or intra-mathematical context, to create and apply a model; and the latter refers to the reconversion of the mathematical result of the model to the original situation. Both sub-processes require a clear understanding of both contexts and the relationship between them.

In OSA, these processes have been defined and used to describe operational, discursive and regulating practices, with the exception of the dual process called "interpretation" which we associate here with modeling. These theoretical assumptions are contrasted through experimentation by analyzing the mathematical activity in problem solving in geometric contexts by practicing teachers. Under the perspective of *didactic engineering* (Godino et al., 2014a), the prediction made is contrasted with the experimental results, which allows us to draw implications for teacher training related to the mathematical processes of GAR.

- *Particularization-generalization*. Inductive reasoning (from particular cases to determine a general rule) or deductive reasoning (from a general law to determine the behavior in specific cases) are key in mathematics and require specific work where these processes are no longer routine and require specific means of control, as in problem 1.
- *Representation-signification*. In mathematics, ostensive objects are loaded with information, beyond the context in which they are used. For example, "x" represents an independent variable and "y" a dependent variable. However, this is no more than a convention based on a "linguistic economy" strategy. Problem 3 shows that in teacher training there is a need for an action that differentiates "convention" from "significance". Moreover, this would have an impact on the development of STEM projects, where the notation in physics, for example, usually differs from the one used in mathematics.
- *Decomposition-reification*. In classical teaching, algebra is associated with the resolution of equations, systems of equations, manipulation of polynomial functions (linear or quadratic at first). However, symbolic manipulation in other contexts, where relationships between variables are presented, should allow, on the one hand, improving the formal use of language and, on the other hand, obtaining unitary objects that structure the system of mathematical objects. Thus, in problem 2, teachers are required to interpret geometric length relationships by means of multiplicity properties of natural numbers and restrictions of measures, typical of the geometric context. This requires a flexible articulation of the processes of decomposition-reification.
- *Modeling* (mathematization-interpretation). Algebra has been studied as a modeling tool (Strømskag & Chevallard, 2022). In teacher training, this perspective should be introduced, emphasizing "back to context". Problem 4 is an example showing that a correct mobilization of the model does not necessarily imply a deep understanding of the situation. This is because, although teachers were able to point out the independence of the radius in their resolution, they did not conclude that the effect of adding one meter to the length of the perimeter would equally affect a pea and the earth.

Finally, we have exemplified how algebraic reasoning mediates geometric activity. Thus, GAR goes against a reductionist perspective of the curriculum, which seeks to identify the key mathematical processes of algebraic reasoning that allow the introduction and development of different domains of the curriculum. In other words, this GAR perspective allows algebra to be assumed as the backbone of the curriculum, connecting the different domains and, therefore, reinforcing the relational or systemic character of mathematics.

Author contributions: RCG, MRW, FJU & CSG: conceptualization, formal analysis, writing, review and editing; **RCG:** project management; **MRW:** methodology. All authors have contributed sufficiently to the study and agree with the results and conclusions. All authors agreed with the results and conclusions.

Funding: This study was supported by the Pontificia Universidad Católica del Perú Research Grant CAP PI1029 (2023-2025)

Ethical statement: The authors stated that the study was approved by the Research Ethics Committee for Social Sciences, Humanities, and Arts at Pontificia Universidad Católica del Perú on 31 August 2023 with approval number 087-2023-CEI-CCSSHHyAA/PUCP. Written informed consents were obtained from the participants.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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