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Networking theories to study teachers' semiotic potential awareness of mathematics artifacts: A pilot lesson study

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Abstract

Although artifacts bridge mathematical abstract concepts and concrete learning, teachers often lack tools to intentionally leverage their semiotic potential. We addressed this gap by developing and testing teachers' semiotic potential awareness (Te-SPA) analysis—a preliminary framework for assessing teachers' awareness of the meaning-making potential of mathematical artifacts. Through action-research across three Italian lower secondary schools (N = 7 teachers), we networked semiotic mediation theory, cultural transposition theory, mathematics teacher's specialized knowledge model, and lesson study to identify contextual enablers and barriers to semiotic awareness development and examine the feasibility of the Te-SPA instrument. Our preliminary analysis revealed three divergent trajectories of awareness development, teacher-program alignment emerging as a pivotal factor. While acknowledging the pilot phase's constraints—limited sample, bounded timeframe, and preliminary validation—we show that the Te-SPA analysis offers a viable approach for future research and design of effective teacher preparation programs centered on intentional artifact use.

Keywords: lesson study, artifact in mathematics education, semiotic potential awareness, community of practice

INTRODUCTION

Teacher training is crucial for improving teaching practices and building professional communities that support motivation and ongoing skill development. However, literature shows that traditional training programs often fails to produce lasting changes in daily practices or sustain teacher motivation over time (Blankenship, 2013; Iori, 2007).

A key step is recognizing teachers as members of a community of practice sharing goals, values, and professional vision. This sense of belonging can enhance teacher commitment and promote more conscious, lasting educational practices (Vescio et al., 2008; Wenger, 1998).

Communities of practice provide a collaborative environment that supports shared reflection and collective knowledge-building, essential for continuous improvement and sustainable educational innovation.

In this regard, lesson study (LS) (Dudley, 2014) emerges as a training methodology that, through collaborative planning, mutual observation, and shared lesson discussions, facilitates professional communities among teachers (Bianco & Di Paola, 2025; Ramploud et al., 2022). LS promotes belonging and collective reflection on teaching, countering professional isolation—a widespread issue in education (Arbia et al., 2023; Qi et al., 2025). Introducing this originally Japanese methodology into Italian schools calls for careful cultural transposition (CT) (Bartolini et al., 2017; Capone et al., 2024; Mellone et al., 2019; Spagnolo & Di Paola, 2010). However if group reflection relies primarily on intuition or shared experience without explicit theoretical tools, it risks remaining at a merely descriptive level. This limitation suggests the need for networking (Artigue, 1990; Bikner-Ahsbahs & Prediger, 2008) LS with robust theoretical frameworks that can structure and deepen teachers' analytical capabilities.

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Contribution to the literature

- This study contributes by networking theory of semiotic mediation (TSM), CT, and mathematics teacher's specialized knowledge (MTSK) theories into a LS process to deepen teachers' understanding.
- It deepens the awareness of the semiotic potential of artifacts (Te-SPA).
- It also suggests that this combination can foster reflective, collaborative professional communities that can sustain teacher engagement.

For this reason, we propose networking LS with the TSM, a framework that provides structured, theoretically-grounded tools for interpreting teaching and learning processes mediated by various artifact (Bartolini Bussi & Mariotti, 2008), with the CT theory as framework useful for defining and interpreting teaching and learning awareness in relation to different cultural contexts (Bianco & Di Paola, 2025; Mellone et al., 2019) and with the MTSK model (Carrillo-Yañez et al., 2018). Rooted in Vygotskian perspectives on mediated learning, TSM posits that mathematical meanings are constructed through teachers' deliberate use of artifacts as semiotic mediators. By combining LS's collaborative, practice-based methodology with CT and TSM's explicit focus on artifact design and semiotic potential, teachers can develop a "semiotic awareness"—the capacity to recognize and intentionally leverage how artifacts support meaning-making processes in students' learning. To document the development of this semiotic awareness, we developed the teachers' semiotic potential awareness (Te-SPA) analysis—a systematic analytical framework that translates TSM constructs into observable indicators of how teachers recognize and exploit, artifacts' semiotic potential.

In this study the networking (Artigue, 1990; Bikner-Ahsbahs & Prediger, 2008) application of TSM, CT, MTSK and LS was used in a pilot training program with math teachers from three Sicilian schools, aimed at developing their semiotic potential awareness of educational artifacts (Eyal, 2025). Teachers worked on three key topics: the meaning of division, enhancing mental calculation and numbers of ratios and relationships. The semiotic analysis of artifacts and collaborative lesson design encouraged a reflective and shared approach. Although less pronounced in one of the schools, in the other two this strongly stimulated sharing and a sense of belonging to the educational community, highlighting how differences in beliefs and expectations can influence teacher involvement and the success of such initiatives in different cultural teaching contexts. In this report we present a comparative overview of the results from the three schools, while a detailed manuscript on the experience at one of them (the "F-BPG" school) is forthcoming (Amore & Di Paola, 2026).

BUILDING COMMUNITIES OF PRACTICE THROUGH NETWORKING OF THEORIES

Putting up communities of practice requires thoughtful choices based on analyzing specific contexts and comparing effective practices (Bismuth & Merzel, 2025). In this research we hypothesize that this process can be strengthened by adopting a networking of theories approach (Artigue, 1990; Bikner-Ahsbahs & Prediger, 2008) that networks four frameworks: the TSM (Bartolini Bussi & Mariotti, 2008; Maffia & Mariotti, 2020), the CT theory (Bartolini et al., 2017; Capone et al., 2024; Di Paola & Buttitta, 2022; Ferretti et al., 2018; Mellone et al., 2019; Noviyanti et al., 2025; Silvaggio, 2024; Spagnolo & Di Paola, 2010), the LS (Bianco & Di Paola, 2025; Dudley, 2014; Fujii, 2014), and the MTSK model (Carrillo-Yañez et al., 2018).

TSM, grounded in a Vygotskian perspective, views mathematics learning as a social process mediated by artifacts and signs, with teachers acting as mediators in constructing shared mathematical meanings (Bartolini Bussi & Mariotti, 2008; Eyal, 2025), by promoting conversions between different representation registers (Duval, 2017a, 2017b).

LS methodology (co-planning, mutual observation, shared reflection) that transforms theory into professional practice, emphasizing communities of practice and student learning (Bismuth & Merzel, 2025; Dudley, 2014; Fujii, 2014). It requires careful management of roles and dynamics for effective growth (Cardoso et al., 2025; Richit et al., 2021) .

CT highlights adapting educational models to local cultural contexts, promoting innovative and open lesson studies and consequent teaching practices (Bianco & Di Paola, 2022, 2025; Capone et al., 2024; Mellone et al., 2019; Spagnolo & Di Paola, 2010).

Lastly, MTSK conceptualizes teacher knowledge as specialized mathematical and pedagogical knowledge integrated with personal beliefs, all influencing teaching decisions and practices (Carrillo-Yañez et al., 2018; Padilla-Escorcia et al., 2025).

We propose that these frameworks operate synergistically: TSM and MTSK would provide conceptual tools for deep practice analysis; LS would activate them in collaborative cycles; CT would ensure cultural relevance in different teaching application contexts. This networked approach (Artigue, 1990; Bikner-Ahsbahs & Prediger, 2008) was expected to

overcome limitations of isolated theories by supporting structured reflection, strengthening professional belonging, guiding artifact design, and adapting innovation to the Sicilian context.

This theoretical structure guided our pilot LS design, with empirical results analyzed in subsequent chapters.

MATERIALS AND METHODS

Methodological Stance and Epistemological Status of the Accompanying Researcher

In this research, which employed observational methods to monitor the development of teachers' competencies (in analyzing, designing, and using conceptual tools from the TSM), the accompanying researcher acted both as project planner (in teams with the other two researchers, authors of this paper) and as a teacher training in the TSM. Explicitly stating the researcher's own learning stance helped foster an equal and dialogic interaction, promoting shared reflection on theoretical concepts and teaching practices. This approach aligns with participatory qualitative research, which values researcher's reflexivity and transparency in the knowledge production process. Epistemologically, it fits within an interpretative and constructivist paradigm, where knowledge is understood as a dynamic, situated, and co-constructed process (Benvenuto, 2015; Pagani, 2020). In this context, explicitly acknowledging the researcher's role is essential to ensure the credibility and validity of the study. Consistent with this stance, the researcher embraced emergent insights throughout the process; the creation of the final questionnaire (described later) is one way this evolution was translated into investigative tools. To mitigate bias from the dual role (trainer/analyst), reflexive notes on personal influences and triangulated multiple data sources (observations, questionnaires, transcripts) were discussed between all researchers, authors of this paper, during and after the LS implementation. This strengthened the study's interpretive credibility.

Operational Contexts, Challenges, and Opportunities

The experimental program involved three Sicilian schools located in distinct geographical and socio-economic areas. Each school comprises at least two campuses, typically divided between primary (1st-5th grade) and middle school (6th-8th grade), with some variability in grade distribution. In each school, the 'Mathematics and Science' department comprises teachers of grades 6-8 who meet regularly, sometimes including grade 1-5 teachers in joint sessions; grade 1-5 teachers also convene weekly separately. The teachers involved in this professional development program (indicated simply as "program" in the following text) participated as volunteers without remuneration and interacted with the accompanying researcher for the duration of the whole program. Activities were carried out separately within each teaching team.

The "F-BPG" Institute in Barcellona Pozzo di Gotto (around 40,000 inhabitants in the Province of Messina), includes twelve campuses, with two serving grades 6th-8th, in a socially diverse area. The school actively fights school-dropout. The two participating teachers have 27 and 18 years of experience, regularly attend mathematics education events, organize school initiatives, and contributed to the learning environment didactics, managing scheduling, procurement, and space customization, often beyond school hours.

The "SP-PA" Institute has seven campuses in a populous, complex neighborhood (the "Sperone" district, in the south-east outskirts of Palermo) marked by poverty, school dropout, crime, and poor services. Recent urban renewal brought social centers and cultural projects. The school reduced dropout from 27% to 1% and runs social inclusion activities. The three participating teachers have 2, 6, and 10 years of experience, with varying school roles.

The "RG-PA" Institute is located in the Libertà district (a residential area of Palermo), it has two campuses, including a historic villa for the lower secondary school. It focuses on STEM, arts, and science labs, digitalization, and continuous staff training. The school supports a diverse student body with inclusive activities and strong family collaboration. The two teachers involved in the program have over 10 years of experience, coordinate classes and organize math competitions.

Initial Survey Questionnaire: Attitudes, Beliefs, and Theoretical Knowledge

Before starting the program, teachers were asked to complete an anonymous questionnaire addressing in-service training (perceived formats and usefulness of attended programs, topics of interest for further training); theoretical knowledge (aspects of the TSM); and beliefs about teaching practices (classroom work methods, use and selection of artifacts, observations and/or hypotheses). Each question provided several statement options, with respondents asked to rate their level of agreement on a scale from 1 to 10 (1 = strongly disagree, 10 = strongly agree). Where appropriate, respondents could select "other" and specify further.

Artifacts' Analysis, Design and Lesson Planning Meetings

These meetings (hereinafter referred as "design meetings") followed an action research approach grounded in participatory observations (Benvenuto, 2015). Their purpose was to analyze existing artifacts, design new ones, and plan lessons using a "lesson plan" within the LS framework, adapted according to the

principles of CT. The lesson plan was in three sections: the first, describing the class (composition, skills, disciplinary focus, objectives), the second describing the artifact (structure, purpose, functioning, semiotic potential), and the third to detail the classroom intervention in terms of timing and working methods.

Preliminary Analysis of the Design Meetings

All discussions were recorded through notes and videos. An observation grid was used for a preliminary analysis, which was designed to capture behaviors during the artifact analysis phase, the artifact design/choice phase (new artifacts were designed by the "F-BPG" and "SP-PA" teams while an existing one was chosen by the "RG-PA" team) as well as during lesson planning. The behaviors monitored are indicated in **Figure 1**. Frequencies of occurrence, recorded at 10-minute intervals and marked with "+" or "-" to indicate alignment or non-alignment with the indicator, are presented in six sets of bar charts (in blue for "+" and in orange for "-"). The first two sets relate to the analysis

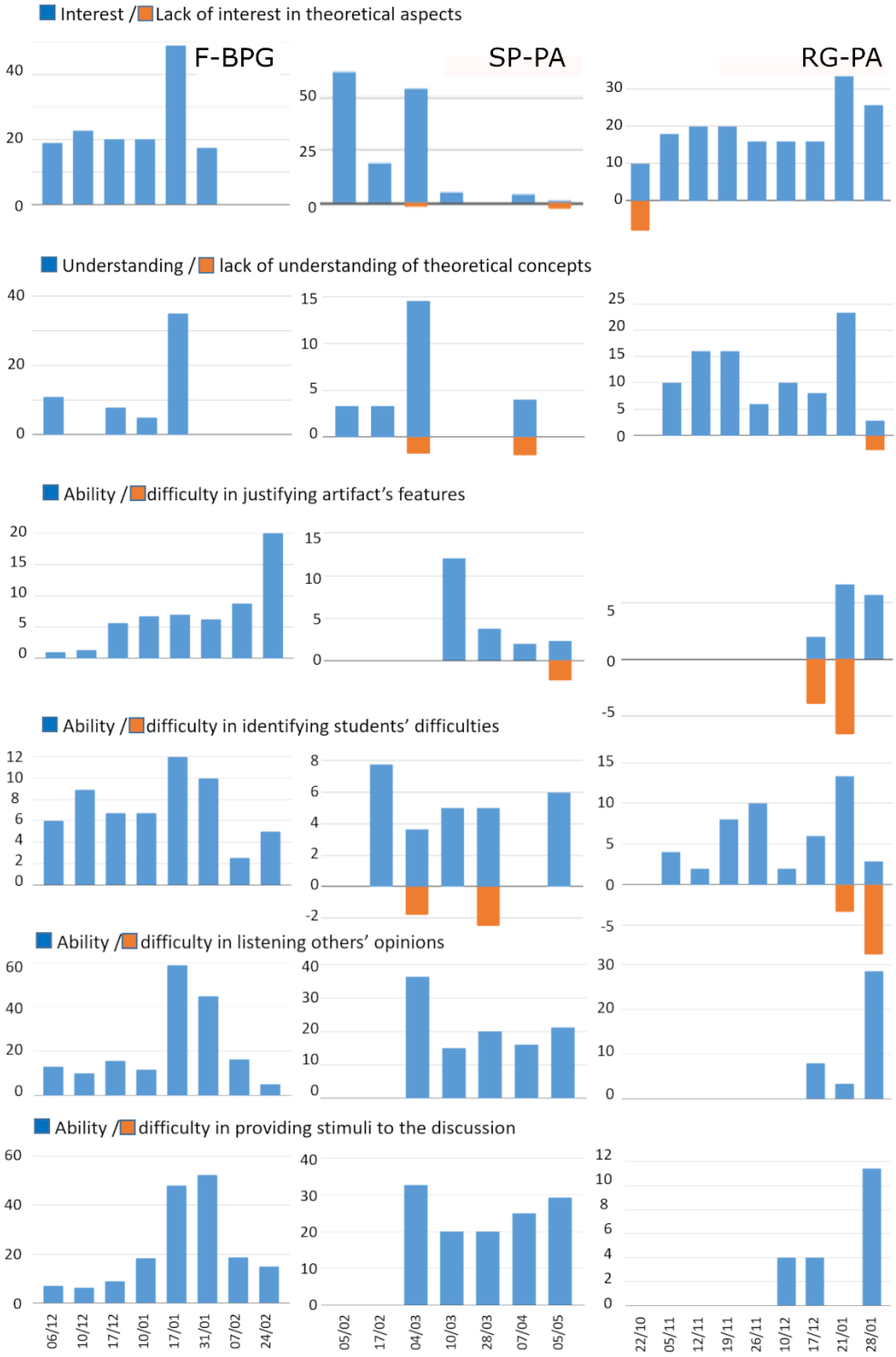

Figure 1. Analysis of the design meetings (Source: Authors' own elaboration)

Table 1. Chronology of data collection activities

School	Design meetings	Observation of classroom activities	Post-lesson discussions
F-BPG	6 December 2024/24 February 2025	12 March 2025	12 March 2025
		13 March 2025	13 March 2025
		31 March 2025	12 April 2025
		2 April 2025	
		7 May 2025	7 May 2025
		21 May 2025	21 May 2025
SP-PA	5 February 2025/5 May 2025	9 May 2025	22 May 2025
		12 May 2025	
RG-PA	22 October 2024/28 January 2025	25 March 2025	7 April 2025
			14 April 2025
		29 May 2025	9 June 2025

phase, while the remaining four refer to the artifact design/choice and lesson planning phases. Behavioral frequencies were normalized to 100 minutes to enable comparison between meetings of varying lengths. These data were employed to qualitatively explore the evolution of discursive and planning dynamics over time, without assigning strict quantitative significance to the numerical values.

Classroom Lesson Observations and Post Lessons Meetings

In each school, classroom lessons were observed by other teachers from the school team and, when possible, by the accompanying researcher (at "F-BPG" and "SP-PA" schools). Observers took notes and completed an observation grid, provided by the accompanying researcher. This was designed to capture specific teachers' choices with respect to classroom management and artifact utilization during activities. Its primary function in this study was twofold:

(a) to support researchers in observing lessons with a structured focus and

(b) to serve as a facilitation tool for structured discussions with the observing teachers in post-lesson meetings.

Therefore, data from this grid was not coded systematically. Post lessons discussions focused on the analysis of artifacts' efficacy (especially in enabling the production of local signs and in their transformation). These were later analyzed using the Te-SPA framework (described below).

Dates of design and post-lessons meetings are indicated in **Table 1**.

Interpretative Data Review

The qualitative data collected during the study were analyzed using an interpretative and reflective approach, prioritizing meaning-making over rigid classification. Various sources were examined, including questionnaire responses, meeting notes, and informal conversations between the accompanying researcher and teachers. The analysis was iterative, with evolving meanings emerging progressively, without a predefined coding system (Braun & and Clarke, 2006). Particular attention was paid to distinguishing individual perceptions from collective dynamics, in line with research emphasizing shared reflection and the building of professional teacher communities (Wenger, 1998). This approach also informed the design of the final questionnaire, administered at the end of the study.

Final Survey Questionnaire

The questionnaire aimed to assess the overall impact of the action research on teaching practice. It gathered reflections on the effectiveness of theoretical knowledge, tools used, perceived benefits and challenges in artifact use. The importance of collaborative work for developing planning and analytical skills emerged during the study. This was reflected in questions exploring collaboration, shared reflection, and collegial attitudes. Responses were given on a numeric scale (1-10), with space for comments, complementing previously gathered data.

Teachers' Semiotic Potential Awareness (Te-SPA) Analysis

To assess the evolution of teachers' awareness of the semiotic potential of artifacts (Te-SPA), an analytical grid was developed through a critical adaptation of the TSM (Bartolini Bussi & Mariotti, 2008; Duval, 2017b; Rabardel, 1995). To this end, concepts originally formulated to describe semiotic mediation processes with students were utilized to derive indicators of teachers' awareness of artifacts' semiotic potential. The grid encompassed five dimensions: **Rap** (representational), **Int** (interpretative), **Com** (communicative), **Str** (instrumental and design-related), together with a category (**Gen**) accounting for manifestations of generic awareness without explicit theoretical anchoring. For each dimension, the following several indicators are detailed in **Table 2**. In this exploratory pilot phase, coding was conducted by researchers. Transcripts and reports from design and post-lesson meetings were first processed through Copilot (an AI-assisted tool) to generate preliminary

Table 2. Te-SPA analytical grid

Dimension	Theoretical anchoring	Indicator	Focus of teachers' reflection
Representational (**Rap**)	Conversions between representational registries (Duval, 2017a)	Rep1: Thematization of representational value	The representational role of the artifact
		Rep2: Generalization and theorization	How artifacts support abstraction
Interpretative (**Int**)	Sign production and transformations (Bartolini Bussi & Mariotti, 2008)	Int1: Production of situated signs	How artifacts allow creation of situated signs
		Int2: Sign transformations (treatments/conversions)	How artifact allow treatments and conversions
Communicative (**Com**)	Artifacts as communication mediators (Bartolini Bussi & Mariotti, 2008)	Com1: Role as mediators in collective discussion	How artifacts mediate communication
		Com2: Sign transformation through discussion	How artifacts mediate sign transformations during communication
Structural/ instrumental (**Str**)	Artifact structural analysis (Rabardel, 1995)	Str1: Concrete analysis of artifact structure	Structural details of the artifact
		Str2: Task facilitation recognition	How structural details facilitate tasks
Generic (**Gen**)	Awareness with no theoretical anchoring	Gen1: Educational intentions	General educational intents
		Gen2: Teacher as mediator	The role of teachers as mediators in classroom discussions
		Gen3: Critical review of artifact	Critical aspects of the artifact that need revision
		Gen4: Generic reflections on difficulties	Pupils' difficulties in math

sentence-level attributions to the Te-SPA indicators. This automated pre-screening step was deliberately introduced to mitigate potential selection bias in the identification of relevant excerpts, reducing the risk that the researchers might—consciously or not—privilege certain utterances during the initial screening. Each attribution was then individually reviewed and validated by researchers, who retained only those deemed theoretically aligned with the TSM constructs. Attributions were normalized by meeting duration to enable cross-context comparisons, yielding the heat maps presented in the Results. No inter-coder reliability checks were applied at this stage, reflecting the developing status of Te-SPA framework; future studies will incorporate multiple coders for validation.

Data Collection and Ethical Considerations

All data collection complied with consent and data protection regulations. Participants were fully informed and provided with written informed consent. The study received approval from the Bioethics Committee of the University of Palermo (approval 253/204; 11-7-2024)

RESULTS

Comparative Analysis of Initial Survey Responses

The initial survey responses (**Table 3**) revealed distinct baseline profiles across the three schools that shaped their subsequent developmental trajectories. All teachers recognized the value of professional development, though format preferences differed: F-BPG teachers strongly favored self-study and in-person courses; SP-PA showed varied ratings across reading, workshops, and webinars; RG-PA teachers preferred dedicated courses but with differing enthusiasm levels.

TSM familiarity varied clearly between schools. F-BPG teachers reported moderate knowledge, with one showing critical insight; SP-PA teachers had mixed understanding without strong theoretical grasp; RG-PA presented contrasting levels—one with moderate familiarity, the other showing complete unfamiliarity.

These differences extended to teaching practices. F-BPG consistently preferred hands-on laboratory approaches; SP-PA combined frontal teaching preferences with mixed methodologies; RG-PA teachers split between strong laboratory orientation and mixed methods. Artifact use patterns followed similar lines: most systematic at F-BPG (regular concrete models); more limited at SP-PA due to self-reported skill constraints; divergent at RG-PA (one frequent user vs. one less engaged).

Key finding

F-BPG began with theoretical familiarity and innovation orientation; SP-PA demonstrated practical diversity with underlying curiosity; RG-PA showed contrasting teacher profiles. These baseline differences predicted the developmental trajectories observed throughout the program phases.

Design Meetings

The meetings with the three teams of teachers revealed different trajectories in how they approached theoretical reflection, analysis of students' difficulties,

Table 3. Responses to the initial survey questionnaire

Category	F-BPG	SP-PA	RG-PA
Training & professional development	Very high interest in books, seminars, and further courses (scores 9-10)	Mixed interest: Strong in problem solving and special needs, moderate in ICT and history	Medium-high interest, especially in probability/ statistics and use of artifacts
Preferred classroom mode	Strong preference for hands-on/lab work (9-10), very low for lecture (1-2)	More balanced: Some prefer lecture (up to 8), others blended (6-9)	Clear preference for hands-on (7-10), blended also appreciated (6-10)
Current use of artifacts	Regular use (score 8), mainly models/objects; software and diagrams less frequent	Discontinuous use (0-6), varies across teachers	Medium-high use (6-8), models/objects most common, software and diagrams less
Desired future use of artifacts	Very high desire to increase use of models (10), software and diagrams (7-9)	High interest in models (8-10), moderate in software (6-7) and diagrams (6-8)	Medium interest in models (8), lower in software (2-8) and diagrams (4-8)
Functions attributed to artifacts	Mainly for explaining concepts and facilitating problem-solving (scores 9-10)	For explaining concepts and problem solving (8-10)	For explaining concepts and problem solving (7-8)
Perceptions of effectiveness	Artifacts perceived as very effective (10), even with minimal teacher intervention	More cautious: Some see them useful only with strong teacher guidance (6-9)	Variable: From very effective (10) to requiring constant teacher support
Risks & criticisms	Concerns about distraction and over-reliance on concrete objects (scores 6-8)	Moderate risks perceived (3-5), mainly distraction and classroom management	Low risks perceived (1-3), fewer concerns overall
Obstacles (Sperone T2 only)	–	Main obstacle: Lack of confidence/ preparation (8). Other reasons: cost, disruption, extra workload	–
Future likelihood (Sperone T2 only)	–	High willingness: Models (8), software (8), diagrams (9)	–

and artifact design (data from the analysis of these meetings are shown in **Figure 1**).

F-BPG school

The two teachers were always available to meet with the accompanying researcher outside school hours for extended sessions. From the outset, they showed interest in identifying conceptual challenges and student difficulties (the concepts of "relationship" and the difficulties with the division algorithm). They agreed on the necessity of cooperative activities to overcome the perception of division solely as "distribution." During the meetings, strong engagement with theoretical aspects emerged, especially once the artifact took its definite shape. Attention to students' obstacles remained constant, though it waned in the final phases focused on concretizing the artifact. The request to fill in the lesson plan stimulated renewed analysis of the artifact's semiotic potential, leading to increased contributions on epistemological aspects.

SP-PA school

The three teachers showed consistent commitment by attending meetings outside school hours. Initially, they engaged deeply with theoretical concepts of semiotic mediation and their application through didactic artifacts. Over time, their focus shifted to practical concerns, such as analyzing student difficulties and designing activities. In final meetings, their ability to justify design choices improved by relating them to didactic and epistemological principles. Active participation and shared strategies resulted in two artifacts focused on mental calculation and number sense.

RG-PA school

Meetings were held during short breaks between lessons. Interest in theoretical concepts appeared early but varied. Classroom interventions used pre-existing artifacts, with one teacher struggling to justify her choice. Analysis of student difficulties was present but not comprehensive; participation focused mainly on practical matters.

Key findings

F-BPG sustained theoretical depth; SP-PA evolved from theory to practical-theoretical synthesis; RG-PA remained practically focused. These patterns in **Figure 1** reflect how team cohesion influences reflection quality

Classroom Implementation Activities and Post-Lesson Discussions

In-class activities and post-lesson discussions conducted in the three schools showed different dynamics, both in terms of artifact design and methods of observation, discussion, and accompaniment. These

differences, due to organizational factors and the availability of participants and school administrators, influenced each group's path, resulting in non-overlapping outcomes.

F-BPG school

The activities were implemented in a structured manner, with two cycles of lessons cross-observed by the teachers and the accompanying researcher, followed by timely discussions and a phase of redesign and further lessons and discussions. The artifacts used were newly designed by the teachers. The artifact used in the first cycle, consisted of "image cards", "number cards", and a worksheet with guiding questions (details are given in Amore & Di Paola, 2026). The aim of the artifact was to guide students in creating problems that needed to be solvable with a division, to broaden pupils' ideas beyond "division as distribution". Reflection on the misconceptions (D'Amore, 2022): "in a division the dividend should always be greater than the divisor", that emerged during classroom activities, led to the creation of a second artifact (used in the second cycle). This consisted of cardboard pieces depicting rectangular pizzas to be divided among a number of people, accompanied by a worksheet with numerical data. This tool fostered abundant production of sign-artifacts by students, a highly participative collective discussion, and an effective conversion between semiotic registers. This allowed to overcome misconceptions, leading to a mathematical discourse on the meaning of the division result, with connections to fractions and the decimal system. The process concluded with a final lesson in which remaining doubts were addressed and resolved, linking it more explicitly to fractions and deepening the concepts related to the representation of numbers and decimal digits in particular.

The teachers' willingness to attend long and in-depth meetings, immediately after lessons and outside school hours, eased shared reflection and experimentation. Post-lesson discussions were systematic, participatory, and always based on direct observation. The teachers showed progressive refinement of analytical skills, which evolved towards analyzing students' misconceptions, the semiotic function of the artifacts, and the role of didactic mediation. A growing ability emerged to connect observed practices to the theoretical concepts of the TSM, with conscious use of specialized language as a tool to analyze learning dynamics.

SP-PA school

The program showed a collaborative and well-articulated process, where two teachers designed educational artifacts focusing on students' difficulties on mental calculation. Two main artifacts were created: cards representing monetary units (euros) and rectangular tiles of various sizes featuring a grid of unit squares. The common goal was to stimulate students' awareness of number composition, to enhance mental calculation (Gaidoschik, 2024). The two lessons with these artifacts (one in each teacher's class) were held on closely spaced dates and observed by the whole team. One teacher chose to separate the manipulation phases from the symbolic formalization phases, favoring a gradual consolidation of operational skills before transitioning to mathematical language, supported by metacognitive moments of reflection. The other teacher preferred a more immediate integration between manipulation and formalization, encouraging early use of mathematical language. Both classes were followed by one in-depth post-lesson discussion, where the group showed growing openness to the concepts of the TSM, progressively acquiring interpretative tools to understand the functions of the artifacts in didactic mediation. Collaboration among teachers allowed refinement of analytical and design skills. The importance of adjusting the timing and structure of activities according to class characteristics and artifact complexity was also emphasized.

RG-PA school

The whole program took place in a more fragmented context compared to the other groups. The teachers used a pre-existing artifact composed of "flat beads" and worksheets designed to address students' difficulties with number decomposition and numerical relationships. The first teacher used the artifact focusing mainly on practical aspects; she did not analyze the worksheets returned by the students, limiting opportunities to reflect on and address the strategies and difficulties that emerged. In contrast, the other teacher initiated a process of instrumentalization by modifying and adapting the original worksheets to her class's specific needs, demonstrating a more conscious appropriation of the artifact, as described in Rabardel's (1995) models. Due to administrative decisions, the accompanying researcher was unable to directly observe the lessons. Therefore, post-lessons discussions were based on teachers' notes and observations. Furthermore, these took place after a delay, in short sessions during breaks between classes. In these meetings, teachers focused mainly on practical aspects and the perceived effectiveness of the artifact, with a limited attention to its role as a semiotic bridge between students' concrete actions and formal mathematical language. It also became clear during these discussions that no collective classroom conversation explicitly connected 'sign-artifacts' to 'mathematical signs,' revealing a limited teacher awareness of the artifact's semiotic potential.

Key findings

F-BPG achieved complete LS cycles with deep TSM integration; SP-PA demonstrated collaborative variation in mediation approaches; RG-PA remained practically

Table 4. Comparison among the key aspects of each of the three programs

Aspect	F-BPG	SP-PA	RG-PA
Artifacts' source	Teacher-designed: Two new artifacts	Teacher-designed: Two new artifacts	Pre-existing, with some adaptation
Classroom observation	Accompanying researcher + teachers	Accompanying researcher + teachers	Teachers only
Post-lesson discussions	Immediate, engaged and thorough	Immediate, engaged	Remote, based on teachers' notes, less integrated
Cycles and re-design	Two, with re-design	One, no re-design	One cycle, long gap between two lessons; no re-design
Didactic objectives	Division; decimal representation	Number composability; mental calculation	Numerical ratios and proportions
Availability for meetings	High, beyond school hours	High, beyond school hours	Low, in breaks between classes
Support from management	Very high	High	Limited

Table 5. Responses to the final questionnaire

Category	F-BPG	SP-PA	RG-PA
Confidence & use of artifacts	High design: 8-9 Adapt: 7-8 Frequent use: 8-9	Medium-high design/adapt: 6-8 Frequent use: 7-9	Medium confidence design/ adapt: 6-9 Use: 5-9
Concepts & tools	Concepts became working tools: 9-10	Became working tools: 7-10	Less consolidated: 4-6
Collaboration & professional community	Group analysis of semiotic potential: 9-10 Collaborative design: 9 Strong community growth: 10	Group analysis: 8 Collaborative design: 8-9 Community growth: 8-9	Group analysis: 5-6 Collaborative design: 6-7 Community growth: 3-5
Expected benefits	Motivation, understanding, Problem-solving, collaboration, language all very high: 9-10	Motivation and problem-solving: 10 Collaboration and language: 8-10	Motivation and problem-solving moderate: 7-9 Language weaker: 3-6
Difficulties	Main issue: Time (7)	Issues: Time (4-6), design (5-7)	Stronger issues: Time (2-4), design (5-6)
Researcher & program	Researcher impact: 10 Aims aligned: 9 Doubts resolved: 9-10	Researcher impact: 7-10 Aims alignment: 6-8 Doubts resolved: 6-8	Researcher impact: 5-6 Aims alignment: 4-7 Doubts resolved: 7
Impact on practice	Artifacts/semiotic reflection: 9-10 Lesson plan design: 8 Lesson delivery with observers: 9	Reflection: 6-10 Lesson plan design: 6-8 Lesson delivery with observers: 6-9	Reflection: 5-8 Lesson plan design: 6-8 Lesson delivery with observers: 4-5
Sharing & leadership	Sharing practices: 8-9 Principal support: 9-10	Sharing practices: 7-10 Principal support: 6-7, 10	Sharing practices: 2-8 Principal support: 2-4

focused. Organizational support differentiated reflection depth across contexts. A comparison among the three programs is presented in **Table 4**.

Comparative Analysis of Final Questionnaire Results

At the end of the three interventions, a questionnaire was administered to the teachers. Its initial purpose was to assess the development of knowledge and awareness regarding the semiotic potential of the artifacts. As the project progressed, informal conversations with teachers revealed additional themes that were not fully captured by the original items. This led to a refinement of the questionnaire, integrating questions aimed at exploring these emerging aspects. **Table 5** summarizes the main results, while the full dataset is provided in the **Supplementary Material**.

The first theme concerned teachers' perception of professional change. As some teachers described a clear distinction between "before" and "after," emphasizing that the most significant developments stemmed from collective work and shared dialogue rather than from individual effort alone. This prompted the inclusion of items designed to investigate whether design and reflective competences were experienced as group properties, and to what extent collaborative processes supported individual growth.

The second theme related to the construction of an educational community. Several teachers expressed the desire to share the experience with colleagues in the mathematics department and with primary school teachers. This highlighted the importance of collegiality and cooperative work for professional development, in line with the notion of communities of practice (Wenger,

1998). The questionnaire therefore explored both the individual impact of the training and the perceived contribution of peer collaboration, shared reflection, and the sense of belonging to a professional community.

Responses across the three schools revealed different trajectories in the transformation of teaching practices and in the development of collaborative cultures. Variations in scores and comments reflected not only the degree of engagement with the project but also the organizational and relational conditions characterizing each context.

Professional Transformation: "Before and After"–The Effect of the Work Group

At F-BPG school, the teachers perceived a clear evolution in didactic awareness. The ability to design artifacts was rated very highly, as was the capacity to adapt them. The theoretical concepts introduced during the course were perceived as operational tools, with concrete effects on lesson planning and classroom practice observation. One of the teachers emphasized having learned to "value material, spatial, and communicative choices," recognizing their impact on educational processes. The experience also strengthened personal motivation and the sense of belonging to the group. The teachers regarded the path as an important professional growth opportunity, made possible mainly by systematic and structured dialogue within the work team. This dialogue did not rely solely on personal experience and sensitivity but benefited from sharing tools—such as lesson plans, periodic meetings, and planned and observed lessons—that helped focus discussions on classroom events. This together with the use of theoretical references from TSM provided fundamental interpretative keys to critically and collaboratively analyze didactic practice and classroom dynamics, fostering group development that supported and deepened individual growth.

At SP-PA school, the teachers' responses indicated a positive evaluation of the training path at both the individual and collective levels. The teachers self-reported competence in designing and adapting artifacts, with a high anticipated frequency of future use. This individual growth was strongly attributed to the work group, emphasizing that artifact and lesson design skills were fundamentally rooted in collaboration. The group was also perceived to possess an increasing shared capacity for reflection and innovation and the collective dimension emerged as predominant in the assessment of perceived benefits. While individual awareness of the strengths and weaknesses of didactic practices scored highly, "reflection with fellow participants in the experimentation" received even higher scores, highlighting the importance of peer interaction and exchange for professional development. The concepts introduced throughout the process were regarded as concrete operational tools. Regarding didactic benefits, all teachers assigned high ratings across various dimensions, including increased student motivation, facilitation of comprehension and problem-solving, enhancement of collaboration, and acquisition of mathematical language. A positive impact on inclusion was also noted. Anticipating future practice, the reflective use of artifacts was recognized as important, with a lesser emphasis on the adoption of lesson plans and co-designed lessons with peer observation. The program further promoted sustainable practices and qualitative improvements in teaching, with particular attention to meeting student needs. Anticipated challenges pertained mainly to designing adequate artifacts and managing the time required for their production, while the integration of artifacts into the curriculum was perceived as more manageable. Interaction with the accompanying researcher was viewed as an opportunity for growth, although teachers expressed a desire for more hands-on classroom activities and better alignment between content covered and specific curricular demands.

At the RG-PA school, teachers reported generally limited individual professional growth as a result of the action research pathway, with noticeable differences between the two participants. Both teachers valued in similar way their artifact design skills, slightly above the passing mark. They diverged in terms of adaptation and future intentions: one teacher actively appropriated and adapted the artifact, while the other displayed more superficial and less intentional use. This divergence impacted on their attitudinal change, one teacher being moderately more open to new approaches, while the other remaining largely unchanged. The influence of theoretical content was modest overall. Both teachers showed limited perceived professional growth through group work in supporting individual skill development.

Building the Educational Cultural Community

At the F-BPG school, the collegial dimension was experienced as central. Group work was rated very highly and the growth of the team as a professional community received the maximum score. The teachers expressed the desire to share the experience with the department and with the primary school, emphasizing the value of building shared practices. Interaction with the accompanying researcher was perceived as a stimulus for analytical reflection and conceptual systematization.

At the SP-PA school, a strong propensity to share the experience emerged: although at different degrees, all three teachers expressed the desire to share within the department and to involve the primary school. These data are particularly interesting when juxtaposed with the teachers' years of experience and involvement in school life. Despite differences in experience,

institutional engagement, and perception of administrative support, all three participants completed a program which, concentrated within a few months, led to the conclusion of a first cycle of experimentation (design, classroom testing, and subsequent discussion).

Intentions to share experiences with the wider school community at RG-PA were mixed and varied significantly between teachers. One teacher reported a moderate inclination for sharing within the mathematics department and with the primary school, while the other showed much less interest. Both perceived poor levels of support from the school administration, with limited opportunities for organizing classroom observations and comprehensive collaborative practices. Ineffective interaction with the accompanying researcher, prior familiarity with the ideas discussed, lack of alignment with school programming, and concerns about researcher competence were not seen as major barriers. Shared reflection was valued by both teachers but did not constitute a decisive factor for group growth. Overall, while the process led to some recognition of the relevance of joint reflection, collaborative dynamics and the desire for dissemination remained limited and varied by individual. The unsupportive role of the administration reinforced these trends.

Key findings

Overall the three schools experienced the program in distinct ways, influenced by organizational conditions and professional attitudes. The comparison highlights varying degrees of willingness to turn experimentation into shared practice: where time and space for dialogue were ensured (F-BPG and SP-PA), team discussions fostered openness and the intent to disseminate, strengthening attention to the semiotic potential of the tools and promoting continued development. In contexts with fragmented time and limited dialogue, effects were confined to individual initiatives. Interpretations of "artifact" also varied, shaping the depth of reflection and the connection between action and symbolic representation. The accompanying researcher's role was crucial: when recognized, it supported a more conscious transition between theory and practice; where marginal, the dialogue weakened. Ultimately, participants' openness to change and the quality of shared professional time emerged as key factors for growth and for raising awareness about the semiotic dimension of the artifacts.

The TeSPA Analysis of the Design and Post-Lesson Meetings

The analyses presented thus far provide a picture of how teachers engaged with the intervention and perceived its impact. These analyses document observable behaviors, interactions, and self-reported experiences. To move from description to interpretation, in this section we apply the Te-SPA analytical framework to understand what these patterns reveal about teachers' developing semiotic awareness of artifacts' role in mathematical meaning-making. To this aim, transcripts from design and post-lesson meetings from the three teams were analyzed (**Figure 2**). In what follows "horizontal analysis" traced the progression of indicators over time, while "vertical analysis" captured the richness of each discussion through the co-occurrence of multiple indicators.

As for the F-BPG school, operational indicators (Str1) were present from the outset, while epistemological and interpretative dimensions (Str2, Int2) increased markedly in the second part of the process, peaking between March and April. Representational (Rap1, Rap2) and communicative indicators (Com1, Com2) showed a discontinuous trend, with significant peaks in the central and final phases. Vertically, discussions before March 12 (when the first post-lesson meeting was held) already displayed multiple indicators, though with lower density; over time, reflections became progressively richer, reaching maximum complexity in the post-lesson sessions, where several dimensions co-occurred. The presence of Gen 1-4 throughout indicates that not all reflections were fully anchored to the theoretical framework, a sign of an ongoing learning process. Overall, F-BPG's trajectory reflects a shift from design based on hypotheses to integrated reflection stimulated by classroom observation. These dynamics are consistent with the initial questionnaire (operational familiarity with artifacts, preliminary knowledge of semiotic mediation) and with observations of design meetings' behaviors, where practical concerns intertwined with theoretical openings. The final questionnaire confirmed this trajectory, with high scores for collective design and the perception of semiotic mediation concepts as working tools.

Reflection remained focused on a restricted set of dimensions in the Sperone team's meetings. Rap1 grew progressively to reach high values in the final sessions, while Str1 showed some oscillations. Com1 appeared only from March onwards and consolidated in the final phases, while Int2 emerged in two meetings, one of which was the last. Other indicators remained absent, and the constant presence of Gen3-4 indicates a reflection not always fully anchored to the theoretical framework. Unlike F-BPG, Sperone held only one post-lesson discussion (the delayed start of the program limited the number of in-class activities). In that meeting no new indicators appeared compared to the design meetings. Reflection therefore became gradually richer but remained focused on a few dimensions, showing an evolution more stable than dynamic. These trends parallel the initial questionnaire (limited theoretical knowledge, artifacts not shared) and the design meetings (collaborative participation but focused on practical and organizational aspects). The prevalence of

F-BPG	Dec 6 2024	Dec 10 2024	Dec 17 2024	Jan 10 2025	Jan 17 2025	Jan 31 2025	Feb 7 2025	Feb 24 2025	Mar 12 2025	Mar 13 2025	Apr 12 2025	May 7 2025	May 21 2025
Rap1	0,68	0,76	0,76	0	0	0	0,75	0	2	0	1,3	1,33	0
Rap2	0,68	0	0,76	0	1,82	0,66	0,75	1,5	2	0	1,3	1,33	1,71
Int1	0	0,76	0	0	0	0	0	0	2	2	1,3	1,33	1,71
Int2	0,68	0,76	0	0	0,61	0	0	0	2	2	1,3	4	3,42
Com1	0	0,76	0	1,07	0	0	0,75	1,5	2	2	0	0	0
Com2	0	0,76	1,52	1,07	0	0	0,75	1,5	0	2	0	0	0
Str1	0,68	0	0,76	0	0,61	1,32	1,5	3	2	2	2,6	1,33	1,71
Str2	0,68	0,76	0	4,29	1,82	1,32	1,5	3	2	6	1,3	1,33	1,71
gen1	0,68	0,76	0,76	1,07	0,61	0,66	1,5	1,5	0	2	1,3	1,33	1,71
gen2	0	0	0	0	0,61	0	0	1,5	2	2	0	0	1,71
gen3	0	0,76	0	1,07	0,61	0,66	0	1,5	2	2	1,3	2,67	1,71
gen4	1,36	0,76	0	1,07	0,61	0,66	0	1,5	2	2	2,6	1,33	1,71

SP-PA	Feb 5 2025	Feb 17 2025	Mar 4 2025	Mar 10 2025	Mar 28 2025	Apr 7 2025	May 5 2025	May 22 2025
Rap1	0,67	2	2,18	3	2,25	4	0,55	5,14
Rap2	0,67	0	0	0	0	0	0	0
Int1	0	0	0	0	0	0	0	0
Int2	0	0	0	0	0	2	0	3,42
Com1	0	0	0	1	1,5	1	0,55	3,42
Com2	0	0	0	0	0	0	0	0
Str1	0	2	2,18	3	1,5	2	1,65	3,42
Str2	0	0	0	0	0	0	0	0
gen1	1,33	0	0	0	0	0	0	0
gen2	0	0	0	0	0	0	0	0
gen3	0	0	0	1	0,75	0	0,55	3,42
gen4	1,33	2	3,27	2	1,5	2	0,55	3,42

RG-PA	Oct 22 2024	Nov 5 2024	Nov 12 2024	Nov 19 2024	Nov 26 2024	Dec 10 2024	Dec 17 2024	Jan 21 2025	Jan 28 2025	Apr 7 2025	Apr 14 2025	Jun 9 2025
Rap1	0	0	0	0	0	0	0	0	3,64	0	1,28	0
Rap2	0	0	0	0	0	1,43	0	2,14	0	0	1,28	0
Int1	0	0	0	0	0	0	0	0	0	0	0	2
Int2	0	0	1	2	2	0	3	2,14	0	0	1,28	0
Com1	0	0	0	0	0	0	0	0	0	0	0	0
Com2	0	0	0	0	0	0	0	0	0	0	0	0
Str1	0	2	0	4	2	0	3	2,14	3,64	0	1,28	0
Str2	0	0	0	0	0	0	0	0	0	0	0	0
gen1	0	0	0	0	0	0	0	0	0	2,86	0	0
gen2	0	0	0	0	0	0	0	0	0	0	0	0
gen3	0	0	0	0	0	0	0	0	0	1,43	1,28	2
gen4	1,5	4	2	4	4	1,43	4,5	2,14	3,64	4,29	2,56	6

Figure 2. Te-SPA analysis of the design and post-lesson discussion meetings (Source: Authors' own elaboration)

Rap1 highlights attention to representations, while Str1 underscores structural focus. The final questionnaire confirmed this orientation, with positive but more modest scores compared to F-BPG.

The trajectory of RG-PA school's meetings was characterized by discontinuity. In the initial phase (October–November 2024) only episodic Str1 and Int2 were present. In the intermediate period (December 2024-January 2025) reflection broadened irregularly with values of Rap2, Str1 and Int2 fluctuating. Rap1 remained absent until late January, with an isolated peak (January 28) that was not maintained afterward. In the subsequent months (April-June 2025) the trajectory returned fragmented: Rap1, Rap2, Int2 and Str1 appeared just once with modest values. Vertically, Gen4 was always present with medium-high values. Str1 and Int2 were the most consistent, although discontinuous, indicators; Rap1 and Rap2 appeared late and without stability. Com1, Com2, Int1, and Str2 were absent. These results are consistent with the divergence between the two teachers revealed by the previous analyses. One showing interest in artifacts and moderate familiarity with semiotic mediation (consistent with the episodic appearance of Str1 and Int2); the other reporting limited knowledge (consistent with the prevalence of Gen4). Observations confirmed discontinuous reflection, alternating between theoretical openings and operational focalizations without progressive growth. The final questionnaire reinforced this interpretation: differences in artifact adaptation and future use mirrored the discontinuity of indicators. Medium-low scores for group growth and theoretical concepts confirmed the prevalence of generic reflections, while

high scores for perceived student benefits reflected pre-existing pedagogical beliefs rather than new practices.

Key findings

the triangulation of indicators, questionnaires, and observations reveals three distinct trajectories: progressive enrichment (F-BPG), stable but limited reflection (Sperone), and fragmented reflection (Rapisardi).

DISCUSSION

In this section we present and analyze some key points that emerged in the course of the program, to provide an interpretative key to the observed dynamics. These points reflect the interaction between theoretical models, motivational factors, adopted methodologies, and specific contexts, offering insights both for understanding current practices and for improving future training programs.

The MTSK as an Interpretative Key to Teachers' Choices

The MTSK model (Carrillo-Yañez et al., 2018) confirms itself as a powerful theoretical lens for understanding teachers' choices regarding didactic design in mathematics, structured on two fundamental dimensions.

First, the selection of the teaching topic contemplates not only specific knowledge of the subject but also awareness of its relations with other concepts in the mathematics curriculum. A very representative case is that of F-BPG school. Here, attention to division as a relation reveals a deep understanding of the topic (knowledge of topics) and its organic connections with the curriculum (knowledge of the structure of mathematics), consistent with expected competences at the lower secondary school. Similarly, in SP-PA school, the choice of geometric tiles builds a bridge between arithmetic and geometry.

Second, the pedagogical reflection on students' needs and difficulties (pedagogical content knowledge) led to choices such as

(1) using cards with euro representation, motivated by how the concrete and familiar dimension of money makes mental calculation operations more accessible than abstract numbers;

(2) proposing geometric tiles provides concrete support to the development of number sense and quantity;

(3) using rectangular "pizzas" to allow students to perform fair division (avoiding the more difficult round ones).

These choices reflect attentive consideration of the most effective strategies for learning, applied to a concrete context.

Teachers' Motivation and the Success of the Proposed Cultural Transpositions: Dynamics and Contexts

Teachers' motivation is a key factor for successful training CTs. In this research it did not depend on factors such as years of teaching experience, involvement in school life, or support from leadership. Indeed, teachers with very different experiences and work contexts completed long-term experiments (e.g., RG-PA school teachers, involved from October to May) or those requiring considerable extra-school time investment (teachers from F-BPG and SP-PA schools). However, success—meaning the availability to reconsider practices and to analyze professional skill development—seems linked more to alignment between individual expectations and project objectives than simply perseverance. This emerges from both final questionnaire responses and informal teacher comments, which illuminate some observations emerged during the program implementation. A revealing example comes from F-BPG and SP-PA schools, where initial doubts about theoretical and methodological premises were overcome by progressive recognition of personal and group professional growth —another factor supporting motivation and success, consistent with findings by other authors (Guskey, 2002). Additionally, "resonance" with personal beliefs (Carrillo-Yañez et al., 2018; Kohanová et al., 2025), such as viewing school as a mean for social emancipation (derived from comments during informal conversations with the teachers) and the consequent need to improve educational practices, acted as an additional motivational lever. According to the aim of CT theory, the relationship with the accompanying researcher, seen more as reflective dialogue and support than simple knowledge transfer, played a stimulating role in sustaining commitment and reflection, confirming the centrality of interpersonal and collaboration dimensions in teacher training. In RG-PA school, lack of alignment manifested as impatience regarding prolonged design phases (the "Too much time dedicated to theory and too little to practice" item received the highest scores from these teachers) and initial participation driven by curiosity about already known artifacts (unreported comments) rather than deep interest in didactic transformation. This misalignment appeared explicitly in final questionnaire responses.

Networking Lesson Study and the Theory of Semiotic Mediation

Consistent with Bikner-Ahsbahs and Prediger (2008), networking TSM interpretative tools with LS methodological framework (Dudley, 2014), provided teachers with tools facilitating more conscious analysis of teaching practice and more effective artifact use. A significant example is F-BPG school, where critical reflection on the first artifact's features (situation cards) revealed that the absence of opportunities for students to

produce "sign-artifacts" limited its effectiveness. The second artifact (rectangular "pizzas") enabled rich sign production, facilitating acceptance of division where dividend is less than divisor (overcoming an initial misconception), connection with fractions, and understanding of decimal digits—concrete proof of semiotic mediation's potential to facilitate learning.

Final questionnaire responses confirm that when this integration was effective, introduced concepts were perceived not as mere abstract theory but as practical and operational tools guiding reflection on methodological choices and conscious artifact use. However, this meaning-making activity requires time, space, and favorable conditions that were not always present.

The Contribution of the Te-SPA Analysis

The Te-SPA analysis represents the methodological innovation of this study, offering a systematic way to document teachers' semiotic awareness during collaborative meetings. It provides a lens to interpret the different trajectories observed across the three schools and to relate the results from other analyses such as questionnaire and meeting observations, teachers' prior knowledge, collaborative practices, and institutional contexts. Therefore, the Te-SPA analysis can complement other approaches and contribute to making visible the processes through which teachers reflect on artifacts and their semiotic potential.

By coding transcripts and normalizing indicators, Te-SPA analysis allows both temporal (horizontal) and structural (vertical) perspectives on reflection. This dual view enriches the analysis of professional development, showing how awareness evolves and how discussions gain density over time.

More broadly, Te-SPA analysis can provide a framework that can be integrated with other models of teacher knowledge, supporting comparative studies across contexts. Its contribution may lie in operationalizing semiotic awareness in a way that is transparent, replicable, and adaptable to different institutional settings.

The Potential of Training Experiences for Building a Learning Community

One of the program's most significant outcomes is the strong desire for sharing, observed in F-BPG and SP-PA schools, both in the final questionnaire and informal discussions. Participants recognized the value of confrontation and collaboration as essential engines of professional growth and expressed interest in extending the experience to other colleagues, fostering an educational community. This phenomenon can be interpreted as a strategic lever to support lasting innovation, as it values collective professional learning and promotes a climate of mutual responsibility and support.

Limits and Strengths of This Study

A significant methodological limit concerns the lack of formal validation of the observation instruments used, the examining status of Te-SPA (the framework is exploratory and at a preliminary stage of development) and the qualitative and local nature of the investigation, which reduce generalizability. However, the triangulation between different observational methods can (at least in part) compensate for the first aspect. Furthermore, the presence of three very different school contexts may be seen as a notable strength, in that it enables identification of critical conditions and success or failure factors across different training processes. These include alignment between teachers' expectations and programs' scopes (which in this study was probably the most prominent aspect), availability of time and space for shared reflection, active voluntary teacher participation, leadership support, and qualified accompaniment facilitating theory-practice connection. Where these conditions exist, training experiences translates into opportunities for authentic and sustainable transformation. Conversely, organizational constraints, lack of dialogue, and poor collaboration risk fragmented paths with limited practical impact.

CONCLUSIONS

This study's results fit within broader research showing how collaborative, theory-grounded training fosters didactic improvement, teachers' cultural awareness, and professional community-building that sustains innovation over time (Arbia et al., 2023; Bartolini Bussi & Ramploud, 2018; Bianco & Di Paola, 2025; Kohanová et al., 2025; Ramploud et al., 2022; Vescio et al., 2008). Specifically, active teacher involvement in lesson design and analysis, combined with structured reflection on didactic choices and artifact use, emerges as a key lever for teacher professionalization and the diffusion of shared, sustainable practices (Dudley, 2014; Ramploud et al., 2022; Wenger, 1998). These results confirm that effective teacher training must integrate theoretical, practical, and collaborative dimensions, promoting a reflective, contextualized approach that considers teachers' beliefs and knowledge (Carrillo-Yañez et al., 2018; Kohanová et al., 2025).

While the scope of this research was limited to three schools, and the training process reached full maturity only in one context—characterized by experienced and highly motivated teachers—the findings nonetheless offer valuable insights. In particular, the data suggest that when training initiatives do not resonate with teachers, this may be due to a misalignment between their expectations and the program's goals, a factor that deserves further attention in future design. Conversely,

when conditions are favorable, training can promote a shared awareness of the group's collective strength in analysis and planning and encourage a shift toward collaboration and mutual support. These dynamics, although observed in a limited setting, point to the potential of structured, context-sensitive training to initiate processes of professional identity transformation and community building.

Such outcomes align with existing literature on reflective practice and community-based development (Ferretti et al., 2018; Noviyanti et al., 2025; Silvaggio, 2024) and resonate with view that teacher development involves not only technical growth, but also engagement with epistemological beliefs (Kohanová et al., 2025) and implicit professional philosophies. Their work emphasizes the importance of cultivating both mathematical and didactic culture and suggests that meaningful change arises when teachers are supported in critically rethinking their assumptions and interpreting classroom dynamics through shared theoretical lenses. The observed evolution in teachers' discourse and planning practices in this study suggests that such training not only enhances technical competence but also fosters a deeper professional identity grounded in didactic awareness.

Finally, the introduction of the Te-SPA analysis represents the methodological contribution of this study. By operationalizing teachers' semiotic awareness through normalized indicators, it provides a tool that can complement existing approaches and potentially support future comparative research across diverse educational contexts.

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AI statement: The authors stated that AI-assisted tools (Microsoft Copilot, Perplexity, SciSpace) were used exclusively to support consistency checks across data sources and to enhance clarity in the drafting process, with all analytical and editorial decisions rigorously reviewed and validated by human researchers. In the Te-SPA analysis, Microsoft Copilot was used solely to generate preliminary, non-binding attributions of transcript excerpts to the indicators, with all classifications subsequently reviewed, corrected, and validated by human researchers; this AI-assisted step served only as a technical aid to reduce initial selection bias and did not replace human analytical judgement

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