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Parameterization-based instruction for symmetry transformations: A quasi-experimental study of students' intra-mathematical modeling skills

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Abstract

This article examines the potential of the proposed didactic framework for developing students' intra-mathematical modeling skills. The aim of the study is to develop students' model-building abilities, generalization competencies, and general logical thinking by teaching topics related to symmetry through a parametric approach. A mixed-methods study methodology was employed, including comparative analysis and pre- and post-test, combining quantitative data from student evaluations with qualitative data gathered from interviews. The experiment was conducted with 11th-grade students of the Academic Lyceum of Fergana State University. The results indicate that parameterization is a potent method for improving generalization and modelling skills in mathematics education. It showed that the percentage of high-level modelling students in the experimental group increased from 0% to 10.7%, exhibiting statistically significant differences ($\chi^2 = 45.755$, $p < .001$), while the indicators at the low level decreased. Statistical analyses confirmed the effectiveness of the didactic framework based on the parameterization method. As a result of the study, an original instructional development, a system of tasks designed for teaching symmetry transformations on the basis of parameterization—was created. It underscores the necessity for educators to implement varied strategies that promote improved mathematical modeling.

Keywords: GeoGebra, didactic framework, mathematical modeling, learning activity, parameterization method, symmetry transformations

INTRODUCTION

In contemporary mathematics education, it is regarded as a priority pedagogical goal that students should not be limited to merely applying ready-made algorithms, but should also be able to analyze mathematical situations, identify relationships between variables, construct a general model, and draw conclusions on the basis of that model (Carlson & Bloom, 2005; Herold-Blasius et al., 2026). From this perspective, the concept of a parameter and the parameterization method have particular didactic and methodological significance in mathematics teaching. This scenario is especially significant in areas like symmetry transformations, where points, lines, and functions may alter based on a rule, yet certain structural links stay unchanged. In academic mathematics, symmetry transformations are predominantly presented using pre-established formulae and conventional examples (Leikin et al., 1997; Savaş & Köse, 2023). Such guidance may enhance procedural competence, although it fails to explicitly illustrate how learners might go from particular transformation instances to overarching algebraic and geometric frameworks (Wadoń-Kasprzak, 2009).

In particular, it may be interpreted as a fundamental didactic unit that reveals the general nature of mathematical relationships and serves as a means of shifting students' thinking from static solutions toward the perception of dynamic relationships (Šedivý, 1976).

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Contribution to the literature

- This study enhances mathematics education literature by presenting a parameterization-based method for teaching symmetry transformations, providing an alternative to conventional procedural instruction. This strategy enhances both conceptual comprehension and students' abilities in mathematical modelling, generalization, and reasoning skills.
- The study used a mixed-methods methodology, yielding quantitative proof of enhanced student performance and qualitative insights into students' cognitive processes, so facilitating a more thorough comprehension of learning outcomes.
- The study incorporates participants' interview-based comments to elucidate how students navigated the shift from formula application to technology-enhanced generalized parametric reasoning.

Alternative methodological approaches (Kurudirek, 2026; Kurudirek et al., 2026) aimed at fostering the acquisition of the parameter concept in the school mathematics curriculum, particularly in the process of solving equations and inequalities, are of considerable didactic importance for developing students' variational thinking, comparative analysis, generalization, and systematization skills (Gonda & Tirpáková, 2019). In studies devoted to the methodology of teaching parameter-based problems, such problems are viewed not merely as a class of difficult tasks, but also as a means of developing higher-order mathematical thinking (Bilgili & Çiltaş, 2025) in students. The study of parameter-based problems should begin not with routine algebraic manipulations, but with didactic situations that reveal the meaning of the parameter itself (Ledovskikh et al., 2020). Their capacity to encourage students to move beyond standard procedures, engage in inquiry, and independently analyze problematic situations defines their motivational and didactic value (Shivrinskaya, 2002). Thus, parameter-based tasks and the parameterization method are significant not only as cognitive tools, but also as didactic factors that activate learning activity. Detailed mathematical examples that examine symmetry transformations using the parameterization method within a didactic framework and that develop students' intra-mathematical modeling skills are provided in **Appendix A**.

This research frames this shift as a kind of intra-mathematical modeling, highlighting students' capacity to create, analyze, and verify generalized mathematical representations within the discipline of mathematics. Generalization involves identifying shared patterns among particular instances; logical reasoning pertains to elucidating and validating mathematical connections; and procedural competency signifies the accurate utilization of established formulae or regulations. Mathematical modeling necessitates the integration of various procedures to develop a comprehensive model, represent it symbolically or geometrically, evaluate it with particular values, and elucidate its mathematical significance (Çakıroğlu & Muştuoğlu, 2025; Ellis et al., 2022; Hidayat et al., 2022; Karali & Durmus, 2015; Rivera, 2018).

Current studies reveal a notable deficiency concerning the influence of parameters on mathematical education and parameter-driven problem-solving. The information regarding the utilization of a progressive parameterization sequence for instructing symmetry transformations and its potential impact on students' mathematical modeling capabilities is scant. It is particularly ambiguous how students react when directed from specific instances of symmetry to variations in parameters, the formulation of general rules, model creation, and the ensuing phases of interpretation and validation (Hernández-Zavaleta et al., 2026). This research tackles this deficiency by exploring a parameterization-oriented pedagogical framework for instructing symmetry transformations in the realm of grade 11 algebra. Consequently, the findings of the research pertain not to overarching mathematical aptitude, but specifically to within-mathematics modeling proficiency, evaluated via a scoring rubric that delineates this proficiency based on students' capacities to represent symmetry relations parametrically, develop generalized transformation models, interpret these models both algebraically and geometrically, and validate them through particular instances.

From this perspective, the main research questions (RQs), null hypothesis and alternative hypothesis of the study may be formulated as follows:

RQ1. Once baseline performance is accounted for, is there a statistically significant disparity in post-test intra-mathematical modeling performance between students instructed through parameterization and those taught via traditional methods regarding symmetry transformations?

RQ2. In what manner do students in the experimental group articulate the role of parameterization in transitioning from specific instances to general principles and model development?

RQ3. What learning challenges and instructional supports do students report during parameterization-based instruction on symmetry transformations?

H_0: Upon adjusting for pre-test outcomes, no statistically significant disparity exists in post-test intra-mathematical modeling ability between the experimental and control cohorts.

H_1: Upon adjusting for pre-test outcomes, a statistically significant disparity in post-test intra-mathematical modelling ability is seen between the experimental and control cohorts.

LITERATURE REVIEW

Intra-Mathematical Modeling as a Learning Construct

In this research, within-mathematics modeling denotes the creation, interpretation, and verification of abstract mathematical representations independent of any external real-world environment. This concept is associated with, although separate from, generalization, logical deduction, and procedural expertise.

Mathematical modeling has emerged as a significant emphasis in mathematics education research (Blum & Leiss, 2007; Blum & Borromeo Ferri, 2009; Kaiser, 2020). Mathematical modeling is typically examined in the context of real-world scenarios; yet modeling endeavors can also take place inside the realm of mathematics. From this viewpoint, parameterization serves as an instructional instrument that assists students in transitioning from specific instances to generalized mathematical frameworks. In the realm of symmetry transformations, this indicates that students do not simply compute the representation of a certain point, line, or function. Instead, they discern the framework of the change, articulate the relationship in algebraic or geometric terms, establish a universal principle, and verify the consistency of that principle across many instances. Through the comparison of various symmetry scenarios, the alteration of parameter values, and the observation of resultant changes in algebraic and geometric representations, students may discern common patterns and establish general transformation principles. Generalization entails identifying a consistent pattern across instances; logical reasoning requires substantiating the validity of a relationship; and procedural competency consists of accurately implementing an established norm. The method of within-mathematics modeling necessitates the integration of these procedures in the development of a mathematical model. This process is intricately linked to comparison-based learning and the enhancement of mathematical proficiency (Hagena et al., 2017; Niss & Højgaard, 2019). This is consistent with research on issue presenting and task creation, which highlights the pedagogical significance of creating families of mathematical problems through systematic variation (Schoenfeld, 2016; Silver & Li, 2025; Xu et al., 2021). Accordingly, in this research, within-mathematics modeling is characterized as students' performance evaluated by a rubric in parametrically depicting symmetry relationships, formulating generalized transformation models, interpreting these models both algebraically and geometrically, and substantiating them through particular instances.

Parameterization as a Tool for Moving from Particular Cases to General Models

This is especially crucial for symmetry transformations, as features such the axis of reflection, center of symmetry, slope, point of intersection, or coordinates of a point may be regarded as changeable components within a universal transformation framework.

Analyzing the experience of training prospective mathematics teachers to solve parameter-based problems, Prus and Lenchuk (2024) show that working with such tasks requires a broad mathematical outlook, multi-case analysis, generalization, and structural thinking on the part of the teacher. Parameterization offers an educational framework for transitioning from discrete instances to overarching mathematical constructs. When constant values in a problem are substituted with parameters, learners can explore a collection of interconnected scenarios instead of only one numerical instance.

This is directly related to our study, since teaching symmetry transformations through parameterization likewise requires teachers to perceive mathematical objects through changing parameters and to represent them effectively in didactic terms. The problem of analyzing algebraic problems through parameterization therefore constitutes a direct theoretical basis of our study. Symmetry, by its nature, is a geometric phenomenon associated with motion, form, and changes in position, and graphical and visual approaches (Xonkulov et al., 2026b) significantly deepen students' understanding. The effectiveness of graphical and visual approaches in parameter-based problems has been analyzed in a number of studies (Koryanov & Prokofiev, 2011). Teaching parameter-based problems through graphical methods has been shown to yield strong results; in particular, the advantages of this approach become especially apparent in helping school students form visual representations (Xonkulov et al., 2026a) of abstract relationships (Küçük, 2023; Zelenina et al., 2019). From this viewpoint, parameterization is employed in this research not just as a mathematical method. Instead, it functions as a conduit linking particular instances to mathematical representation. Learners initially examine specific instances, subsequently contrast consistent and fluctuating components, adjust the parameters, establish a universal principle, and ultimately understand and corroborate that principle. Parameterization facilitates the shift from the application of procedural rules to the development of models.

Learning Symmetry Transformations through Algebraic, Geometric, and Dynamic Representations

In parameterization-based instruction, learners may see the impact of altering a parameter on the related point, line, or function, but the fundamental structure of the transformation is unchanged. Dynamic geometry platforms like GeoGebra facilitate this approach by rendering parameter changes graphically apparent.

Symmetry transformations need students to synchronize various representations, such as coordinate rules, algebraic equations, geometric figures, and visual motions. This representational coordination is crucial for learning transformations, since students must comprehend both the nature of the changed item and the rationale behind the transformation rule. Dynamic geometry environments, particularly GeoGebra, provide an especially convenient platform for modeling symmetry transformations through parameters (Benning et al., 2023). However, in this research, GeoGebra is not regarded as a standalone learning element. Instead, it is situated as a supportive representation inside the wider parameterization framework. Its function is to assist learners in linking algebraic equations with geometric significance and to facilitate the understanding and verification of the models they create.

This is because students are able to observe changes in parameter values simultaneously at the levels of algebraic expression, coordinate transformation, and geometric representation. These conclusions are especially important in the study of symmetry transformations; in this regard, the study by Adamchuk and Samsikova (2020) is of particular scientific significance. The authors substantiate the parameterization method as a tool for analyzing the conditions of geometric problems and show that expressing different elements of a geometric situation through parameters makes it possible to reveal relationships among particular cases of a problem and to generalize solution strategies (Adamchuk & Samsikova, 2020). This approach is especially effective in the study of symmetry transformations, since components such as the axis of reflection, the center of symmetry, the angle of rotation, the scale factor, or the law governing coordinate changes can all be represented within a parametric model. As a result, what emerges is not merely a collection of isolated examples, but a unified didactic system. By understanding the close relationship among algebraic expression, coordinate transformation, and geometric image, students begin to discover the parameter as a mediating concept between algebra and geometry (Abramovich & Norton, 2006). Rather than directing students toward finding a single result, the parameterization method guides them to identify regularities among changes, thereby creating a methodological and cognitive foundation for the development of intra-mathematical modeling skills (Hernández-Zavala et al., 2025; Zeytun et al., 2016). Expressing relationships between mathematical objects through parameters, as well as the parameterization and model-construction approach, is not limited solely to the didactic process of teaching school mathematics.

Theoretical Justification for the Instructional Sequence and Assessment Dimensions

For this reason, investigations in this area hold considerable importance both theoretically and practically from a methodological perspective. The instruction utilizing the parameterization technique starts with straightforward and lucid examples that elucidate the mathematical idea in focus and is structured to progress from specific instances to a broader regularity. In this procedure, learners derive overarching conclusions through the examination of instances, juxtaposing them, and discerning the connections between them. In the subsequent phase, specific constant values inside the problem are substituted by parameters, resulting in the formulation of a generic mathematical model. This enables the elucidation of various specific instances under diverse parameter values inside a cohesive model. At the same time, in developing students' algorithmic and analytical thinking, the analysis of incorrect cases and the identification of exceptional situations are of considerable methodological importance (Tupouniua, 2022). In studying symmetry transformations through parameterization, students likewise analyze under which parameter values properties are preserved and under which they change. In this context, the system of parameter-based algebraic tasks developed as a result of our study may serve as an effective didactic tool for organizing assessment, verification, and reflective activities. In practice, prospective teachers sometimes demonstrate persistent difficulties in their knowledge and understanding of analytic geometry (Zulu & Brijlall, 2024); methodological shortcomings on the part of teachers and recurring errors on the part of students are also frequently observed (Benecke & Kaiser, 2023). This situation may also manifest itself in the study of the concept of symmetry within the algebra curriculum, since working with coordinates, equations of lines, distance, angle, and spatial representations requires a high level of conceptual precision.

In mathematics lessons, teachers should evaluate students' errors not merely as deficiencies to be eliminated, but as didactic resources that can promote a deeper understanding of mathematical concepts. Such an approach makes it possible, in particular, to analyze students' thinking processes, identify the causes of errors, and transform them into a means of reflective learning. This study is informed by a conceptual framework derived from the reviewed literature, which connects parameterization, generalization, and intra-mathematical modelling in the instruction of symmetry transformations. Collectively, these investigations indicate that parameterization may enhance mathematical modeling when systematically arranged from particular instances to parameter modification, general principle development, model creation, and verification. The theoretical framework outlined below elucidates the manner in which this conceptual approach is implemented in the current research.

Conceptual Framework of the Study

This study's conceptual approach connects parameterization, generalization, and intra-mathematical modeling as a sequential learning process. Instruction starts with specific symmetry examples, enabling students to discern invariant and variable elements. By systematically varying parameters, students analyze analogous scenarios and discern linkages that stay consistent across examples. These connections are further articulated as general principles and converted into algebraic or geometric models. Ultimately, students analyze and corroborate the model by replacing specific parameter values, verifying the outcome geometrically, and elucidating the significance of the change. Within this framework, parameterization serves as a conduit between specific instances and abstract mathematical constructs; intra-mathematical modeling is defined as the creation, interpretation, and validation of models within mathematics, rather than modelling external real-world scenarios.

METHOD

Research Design

This research employed a matched-groups quasi-experimental pre-/post-test control group design, supplemented with an explanatory qualitative follow-up (Flores & Salimaco, 2025). The research did not constitute a completely randomized trial. Participants were chosen among male students in grade 11 attending the Academic Lyceum of Fergana State University. Among the qualified students who consented to take part and fulfilled the necessary evaluation protocols, 57 individuals were incorporated into the ultimate analysis.

A total of 28 students constituted the experimental group, whereas 29 students comprised the control group. The groups were established by a matched-group methodology. Before the intervention, an analysis of students' prior mathematics training and first pre-test scores was conducted to see if the two groups were generally equivalent. The study was deemed quasi-experimental due to group allocation being determined by pre-existing institutional circumstances and group comparability instead of total individual randomization.

Didactic Framework and Instructional Principles

According to this approach, instruction began with the presentation of simple, explicit examples that made the underlying mathematical concepts transparent. These examples focused on tasks such as determining the reflection of a point across a line, formulating the equation of a line after reflection about a point, and identifying general rules for reflecting functions with respect to a point or a line in a way that enabled students to rediscover these rules independently.

In the initial stage, students worked with carefully selected particular cases that captured the essence of symmetry transformations, with the level of complexity gradually increasing. In this context, increasing complexity refers specifically to the introduction of parameters: students first engaged with problems without parameters aimed at identifying general rules, followed by structurally similar problems that incorporated parameters.

Students initially observed, analyzed, and identified relationships among symmetric cases through teacher-guided tasks. This sequence supported the comparison of cases, the identification of common properties, and the transition from specific instances to general conclusions. Throughout the process, students' activity was structured around key cognitive operations such as observation, comparison, analysis, and generalization.

In a subsequent stage, constant quantities in the initial examples were replaced by parameters, and the problems were reformulated as general mathematical models expressed in terms of variables and parameters. Within this framework, students could systematically analyze different cases of symmetry transformations.

By assigning specific values to the parameters, they generated particular instances from the general model, thereby deepening their understanding of the relationship between general mathematical structures and individual cases (see **Table 1**).

Sample

The study was carried out with the participation of 11th-grade students of the Academic Lyceum of Fergana State University (**Table 2**). A total of 57 male students took part in the research, of whom 28 were assigned to the experimental group and 29 to the control group. The participants, being only male students from a single institution, should be considered a context-specific convenience sample rather than a representative sample of all 11th-grade students.

Consequently, the study's findings must be understood within the institutional, gender-specific, and curricular framework of the intervention's implementation.

Table 1. Operational phases of the parameterization-based didactic framework

Phases	Anticipated cognitive processes	Instructor actions	Academic assignments	Demonstration of modelling proficiency
Particular cases	Examine and recognize known symmetry features	Present basic non-parametric reflective tasks	Reflect points and lines while delineating patterns	Accurate representation of certain instances
Parameter variation	Contrast and differentiate between invariant and variable elements	Systematically vary c, x_0, y_0, a, b, or k	Alter parameter values and document results	Identification of variables, constants, and invariants
General rule	Derive general principles from recurrent instances	Instruct students to develop formulae	State equations such as $x' = x_0$, $y' = 2c - y_0$	Accurate universal principle
Model construction	Symbolically or geometrically represent connections	Facilitate the construction and representation of models	Develop parametric transformation models	Model including parameters and conditions
Interpretation and validation	Examine, elucidate, and correlate algebra with geometry	Request for substitution, verification via GeoGebra, and elucidation	Validate the model using sample values and provide justification	Cogent elucidation and model explication

Table 2. Participant flow

Stage	Experimental group	Control group	Total
Eligible 11th-grade students invited	39	39	78
Students/parents providing consent	39	39	78
Completed pre-test	28	29	57
Participated in three-week instruction	28	29	57
Completed post-test	28	29	57
Included in final quantitative analysis	28	29	57
Interviewed in qualitative follow-up	8	0	8

The allocation of students to experimental and control groups according to a quasi-experimental matched-group methodology. Prior to the intervention, the students' mathematical preparedness, age, prior knowledge of symmetry transformations, coordinate geometry, functions, and algebraic expressions were evaluated to create two equivalent groups.

The examination of baseline equivalency utilized students' raw pre-test results instead of only depending on categorized performance levels. The comparison of pre-test scores revealed no significant differences between the two groups at the study's outset, indicating their general comparability prior to the intervention. Consequently, any variations seen post-test were evaluated with caution as indicative of the potential influence of the parameterization-based didactic framework, rather than as conclusive proof of widespread causal efficacy.

There was no loss of participants from the pre-test to the post-test; hence, all 57 students who finished the pre-test also engaged in the intervention and post-test and were incorporated into the final analysis. The selection of 11th-grade students was methodologically sound, as they had acquired the essential knowledge to engage with coordinate representations, algebraic expressions, functions, and geometric transformations. These conditions allowed them to undertake tasks including comparison, generalization, parameter change, and the development of algebraic and geometric models. However, due to the sample being restricted to male students from one institution, the applicability of the findings to mixed-gender, multi-institutional, or culturally diverse settings necessitates more research.

Instructional Process

This didactic framework is designed for three weeks of classroom instruction (total duration: 18 hours) within the 11th-grade algebra curriculum and serves to support the step-by-step study of symmetry transformations on the basis of the parameterization method (**Figure 1**). The framework consists of the following (see **Appendix A**) interrelated components, whose consistent and systematic mastery ensures the development of students' theoretical understanding, practical skills, and elements of intra-mathematical modeling related to this topic.

In the experimental group, a didactic design based on studying symmetry transformations through the parameterization method was implemented. Students in this group were guided not toward memorizing ready-made formulas, but toward analyzing particular examples, distinguishing constant and variable elements, introducing parameters, expressing relationships, generalizing them, and constructing a mathematical model. In the control group, instruction was conducted by the traditional method, mainly on the basis of ready-made rules, definitions, and standard examples. In this way, it became possible to carry out a comparative analysis of the effectiveness of the

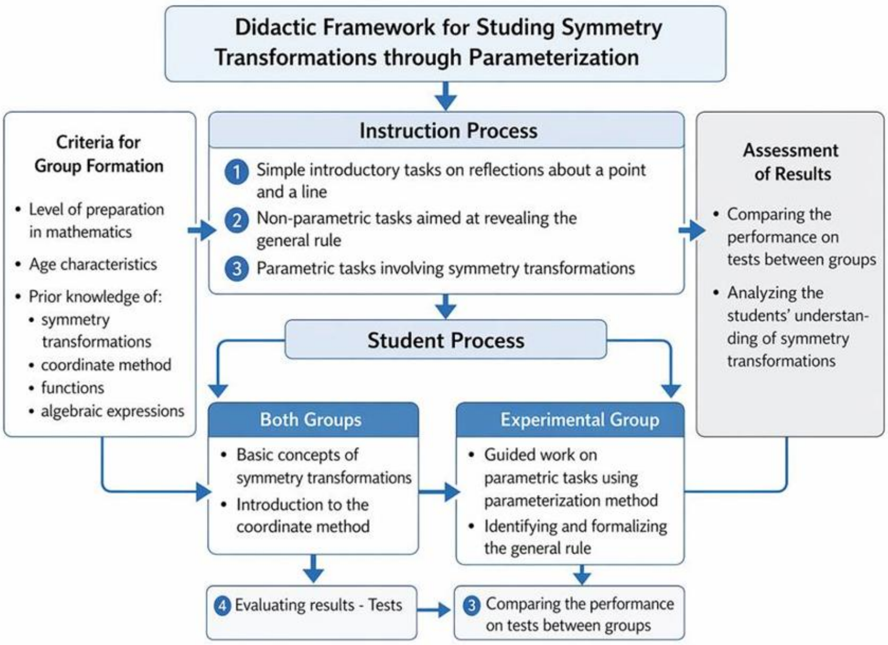

Figure 1. An overview of the study design and the selection of the sample (the authors' own elaboration)

Table 3. Comparing intervention and control group instruction

Characteristic	Experimental group	Control group
Duration	Three weeks, eighteen educational hours	Three weeks, eighteen educational hours
Educator	Same researcher	Same researcher
Theme	Symmetry transformations	Symmetry transformations
Main pedagogy	Directed investigation, case comparison, parameter modification, model development, validation	Standard elucidation of definitions, established guidelines, illustrative examples, and exercises
Learning sequence	Specific instances, parameter modification, overarching principle, model development, and analysis/verification	Definition, followed by formula elucidation, accompanied by practical examples, and concluding with regular exercises
Role of students	Dynamic investigation, juxtaposition, generalization, model development	Attending to elucidations, adhering to illustrations, implementing equations
Role of teacher	Instructor, interrogator, facilitator of abstraction	Clarifier, exhibitor, assessor of standard protocols
Utilization of parameters	Deliberate and methodical use of criteria to generalize instances	Parameters were not employed as the primary organizing instrument
Utilization of technology	GeoGebra/dynamic visualization is employed to investigate parameter modifications and validate models	Utilization is limited to ordinary visualization or demonstration; there is no systematic investigation based on parameterization
Anticipated evidence	Ability to establish and justify generic transformation models	Proficiency in accurately using established transformation formulae
Evaluation	Same pre-test/post-test	Same pre-test/post-test

parameterization-based approach in developing students' intra-mathematical modeling skills (see **Table 3**).

To prevent the teacher factor from becoming a confounding variable, the same instructor (one of the researchers) taught both groups. The lessons were conducted after regular school hours, and parents decided to pick up their children at a later time. The lessons were conducted in classroom environments equipped with smart boards and advanced technological facilities. While GeoGebra was used to help students visualize concepts and enhance their understanding, instructional practices also included class discussions, inquiry-based learning, exploratory activities, questioning, think-pair-share tasks, and group work. Both classes were carried out under the same conditions. Students solved sample problems in class by practicing applying these methods, and additional exercises were assigned as homework to reinforce the learned procedures. In the following week, students reviewed and discussed their homework solutions in class.

After the three-week intervention, both groups completed a post-test. The test lasted for two class periods (90 minutes), and two proctors were present in each group. The same teacher who graded the pre-test

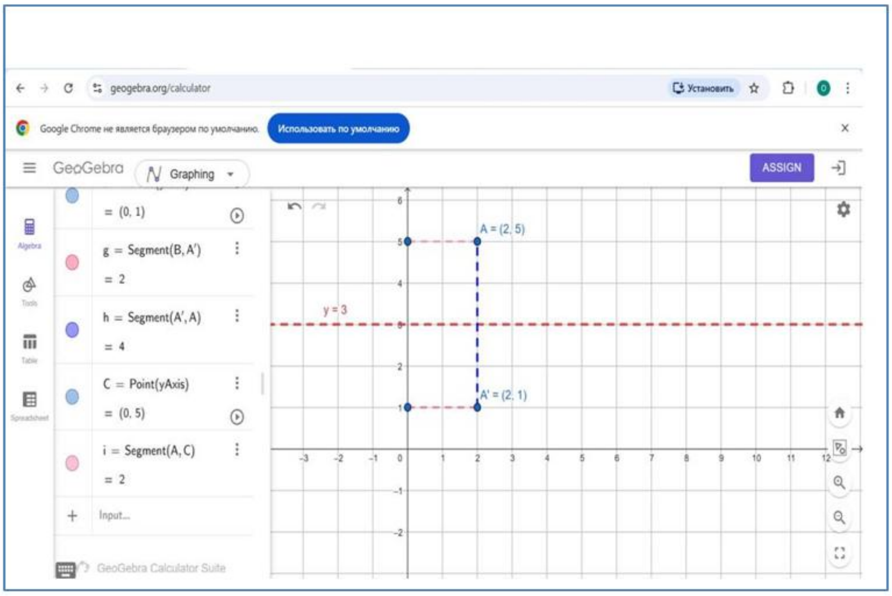

Figure 2. An example of the GeoGebra applets used for the study (the authors' own elaboration)

also evaluated and recorded the post-test papers. Following the analysis of the pre- and post-test scores, interviews were conducted with a small group of students to gather their perspectives on symmetry transformations through parameterization method. The interviews were carried out by a professional moderator and recorded using a digital recording device.

Figure 2 shows an example of the GeoGebra applets used for the study.

Instruments

To quantitatively evaluate the effects of the proposed method on students' performance, pre- and post-tests consisting of sample tasks (see **Appendix A**) were administered. These assessment tools were designed to measure participants' performance prior to and during the intervention by evaluating their comprehension of symmetry transformations and their capacity to formulate mathematical models using parameterization. These assessments were designed not only to determine students' foundational knowledge of symmetry transformations, but also to measure their intra-mathematical modeling skills in symmetry transformations through the parameterization method, as well as to identify and improve learning outcomes.

The evaluation was created as a customized educational assessment tailored to the study, in accordance with the intervention's goals. The progression of development encompassed construct definition, item creation, expert evaluation, pilot testing, rubric formulation, scorer instruction, and reliability assessments. Initially, mathematical modeling was characterized as the capacity of students to discern symmetry connections, distinguish between invariant and variable quantities, articulate relationships via parameters, develop generalized algebraic or geometric models, and substantiate or verify the resultant findings. Five open-ended questions were subsequently created to embody these aspects.

In the subsequent phase, the draft items were evaluated by eleven seasoned mathematics educators, each with a minimum of ten years of expertise instructing grade 11 mathematics within the Uzbekistan curriculum. These specialists assessed each element based on four criteria: mathematical precision, clarity of expression, pertinence to symmetry transformations, and conformity with the designated modeling dimensions.

Informed by expert comments, small adjustments were implemented to enhance the phrasing of the tasks, elucidate the anticipated reasoning stages, and guarantee improved alignment between the items and the scoring rubric. The modified measures were then tested with 21 grade 11 students who were excluded from the research. The pilot research aimed to assess participants' comprehension of task instructions, the appropriateness of the provided time, and the extent to which the items prompted the desired thinking processes. Subsequent to the pilot deployment, slight modifications were enacted to elucidate unclear statements and enhance the score descriptors. **Table 4** shows the alignment of assessment items with intra-mathematical modelling dimensions.

Each item was evaluated via analytic rubric (see **Table 5**). The rubric evaluated both the precision of the final result and the quality of the reasoning process. The total score was determined by aggregating the scores of the five items, with a maximum attainable score of 5 points. The raw scores served as the principal quantitative result, whereas low, medium, and high-performance levels functioned solely as extra descriptive indications.

Table 4. Alignment of assessment items with intra-mathematical modelling dimensions

Item	Mathematical emphasis	Main skill assessed	Modeling dimension
I1	A point's reflection under fixed circumstances	Understanding fundamental symmetry relationships	Conceptual comprehension
I2	Reflection with respect to $y = c$ or $x = c$	Determining invariant and variable quantities	Analytical representation and parameterization
I3	Reflection with respect to a point or general line	Using parameters to express relationships	Parameterization and generalization
I4	Reflection of a function or line	Formulating a universal transformation principle	Model development
I5	Analysis of a universal transformation formula	Elucidating and verifying the acquired model	Rationale and verification

Table 5. Scoring rubric

Score	Description
1	Accurate solution, suitable application of parameters, coherent model development, and sound mathematical rationale
0.75	Predominantly accurate solution with minor inaccuracies or a partially completed explanation
0.50	Partial comprehension of the symmetry relationship, but restricted parameterization or inadequate model development
0.25	Insufficient effort demonstrating a restricted conceptual comprehension and feeble association with the assignment
0	Inaccurate, empty, or unrelated reply

Fifth, the validity of scoring was enhanced by implementing scorer training and inter-rater reliability protocols. Prior to evaluating the primary dataset, the raters deliberated on the scoring criteria, assessed example replies, and reconciled differences in the understanding of the rubric definitions.

This research defines mathematical modeling as students' capacity to transition from specific scenarios to generalized algebraic and geometric representations within mathematics. The evaluations concentrated on conceptual comprehension, analytical articulation, parametrization, model development, and justification/verification.

A methodical approach was employed in the creation of assessment instruments. The procedure started with the delineation of the target construct inside the study's conceptual framework and proceeded with the formulation of five open-ended questions aimed at encapsulating the primary learning goals of the intervention. These enquiries pertained to the reflection of points, lines, and functions in relation to a point or a line. Participants were required to provide accurate responses and elucidate the thinking process that resulted in a generalized rule (see **Table 4**).

Reliability and Validity

This instrument was regarded as a regional evaluative tool for instruction instead of a completely standardized metric for mathematical modeling. A variety of evidential sources were evaluated to substantiate their legitimacy. A test-retest reliability method was utilized to assess score stability. The completed exam was conducted twice for the 21 pilot participants at a one-week interval. Throughout these administrations, the exam was utilized unchanged, and no explicit training on the intended content was delivered between the two assessments. The Pearson product-moment correlation between the two administrations was $r = .86$, signifying adequate temporal stability of the scores. Nonetheless, in this approach, test-retest reliability was not seen as adequate proof of concept validity. Conversely, it was seen indicative that the instrument yielded very consistent ratings throughout time. Content validity was established by expert review, wherein eleven seasoned mathematics educators assessed the items for clarity, mathematical precision, relevance, and conformity with the specified learning objectives. Evidence of the response process was gathered via pilot testing involving 21 students, which assisted in assessing whether the students comprehended the instructions and assignments as planned. Support for scoring validity was demonstrated via analytic rubric, rater training, dual scoring of a sample of responses, and inter-rater reliability assessment. Support for scoring validity was demonstrated through the analytic rubric, rater training, independent double scoring, and inter-rater reliability assessment. To estimate inter-rater reliability, all 114 written assessment papers were independently scored by two raters before consensus. This set included 57 pre- and 57 post-test scripts from both the experimental and control groups. Inter-rater reliability was calculated using a two-way random-effects intraclass correlation coefficient for absolute agreement, *ICC* (2, 1), because the assessment produced numerical rubric-based total scores. The obtained coefficient was *ICC* (2, 1) = .926, indicating excellent

Table 6. Summary of the generalized distribution of results

Total score	Level	Description
4.3-5.0 points	High level	Advanced analysis, accurate parameter selection, effective model development, and clear articulation
3.1-4.2 points	Medium level	Accurate execution of the majority of process phases, with some requirement for assistance
0-3.0 points	Low level	Challenges in modelling and generalization

agreement before consensus. The mean absolute difference between the two raters' scores was 0.20 points on the0-5 scale, and the maximum difference was 0.40 points. After the independent scoring phase, any remaining discrepancies were discussed and resolved through consensus before the final scores were entered into the dataset. Nonetheless, the instrument is considered a study-specific evaluation tool rather than a completely standardized psychometric measure. Nonetheless, the instrument is considered a study-specific evaluation tool rather than a completely standardized psychometric measure.

Data Collection and Analysis

The data collection for this study commenced following the acquisition of requisite ethical permission from the Human Research Ethics Committee of Fergana State University. Upon receiving authorization from the director of the Academic Lyceum, full information on the study's objective, scope, and research methodologies was provided to both students and their instructors. Students were not randomly allocated to the experimental and control groups. Instead, after consent and pre-test completion, the final 57 participants were organized into an experimental group of 28 students and a control group of 29 students using a quasi-experimental matched-group procedure. Group formation was guided by institutional conditions and by comparability criteria, including students' prior mathematical preparation and raw pre-test performance. The experimental group students were designated as S1, S2, …, S28, whereas the control group students were designated as P1, P2, …, P29.

Quantitative assessments provide a direct means of measuring students' abilities to construct models using a parametric approach in topics related to symmetry, as well as their generalization competencies and overall logical reasoning skills. However, to address concerns that quantitative measures alone may not fully capture the cognitive and conceptual dimensions of the learning process, qualitative methods—particularly student interviews and reflective discussions—were also incorporated into the research design. To provide an overall view of student performance in the experimental and control groups, comparative indicators and statistical analyses were calculated. Chi-square tests were employed to analyze disparities among category performance levels. The effect size was determined using Cramér's V (2017) to evaluate the extent of observed changes. During the assessment process, attention was paid not only to the final result but also to the stages of students' reasoning in solving the problem. In particular, the following aspects were evaluated: the student's analysis of a symmetry-related situation, identification of essential elements, expression of these elements through parameters, representation of the result of the symmetric transformation in the form of an algebraic or geometric model, verification of the result, and explanation of the obtained conclusion. On this basis, the following criteria were established: conceptual understanding, analytical expression, modeling, generalization, and justification-explanation. According to these criteria, students' intra-mathematical modeling activity was characterized at high, medium, and low levels. Students at the high level were able to independently analyze symmetry transformations, correctly choose parameters, construct a mathematical model, and logically explain the result. Students at the medium level completed the main stages successfully but required partial assistance in generalization and explanation. Students at the low-level experienced difficulties in moving to modeling, identifying relationships, and generalizing the result. This mixed-methods technique allowed the researchers to investigate the research phenomena from several angles, hence augmenting the dependability and rigor of the study's conclusions.

Assessment Criteria and Performance Levels

The evaluation procedure included both the students' final answers and the reasoning stages they exhibited during the solution process. The evaluated skills are as follows: recognizing essential components, analyzing symmetrical scenarios, articulating relationships via parameters, developing a model, validating outcomes, and offering explanations. Consequently, the following fundamental requirements were established: conceptual comprehension, analytical articulation, modelling, generalization, and justification-explanation. Based on these criteria, students were categorized into three performance tiers (see **Table 6**):

- **High level:** Advanced analysis, accurate parameter selection, effective model development, and clear articulation
- **Medium level:** Accurate execution of the majority of process phases, with some requirement for assistance

Table 7. Coding book for interview analysis

Student declaration/semantic unit	Preliminary code	Classification	Final theme
"A single principle applied to numerous distinct instances."	Identifying prevalent regulations	Generalization	Comprehension via generalization
"I could derive the formula independently rather than committing it to memory."	Formulating a formula	Model development	Shift to modeling
"Initially, I was perplexed when the lettering altered."	Challenges with variables	Challenges of abstraction	Preliminary challenges in abstraction
"GeoGebra enhanced my understanding of the alterations in reflection."	Visual assistance	Comprehension facilitated by technology	Visualization assistance
"Upon altering the parameter's value, the figure subsequently modified."	Variation of parameters	Connecting algebra and geometry	Reasoning based on parameters

- **Low level:** Challenges in modelling and generalization

Qualitative Data Collection and Analysis

After analyzing the pre- and post-test data quantitatively, semi-structured interviews were conducted with eight students from the experimental group. The interview subjects were chosen by purposive sampling to reflect various levels of post-test performance. Students from 3 high, 3 medium, and 2 low-performance tiers were specifically chosen to ensure that the qualitative data accurately represented learning experiences across various accomplishment levels. Participation in the interviews was on a voluntary basis. Each interview was facilitated by a competent moderator promptly following the intervention and lasted around 20 minutes. The interviews were facilitated by a moderator who did not evaluate the students' written assignments. The interviews were audio-recorded with the participants' consent using digital equipment and subsequently transcribed for analysis. The interview technique comprised a primary open-ended inquiry and many follow-up enquiries aimed at extracting students' experiences about learning symmetry transformations via parametrization (see **Appendix A**). The interview data underwent thematic analysis. The researchers began by reviewing the transcripts many times to familiarize themselves with the data. Subsequently, significant units pertaining to the participants' experiences, challenges, generalization processes, and model development were discerned. Preliminary codes were subsequently developed and classified into overarching groups. Ultimately, these categories were evaluated and categorized into themes. The coding approach included both deductive and inductive components: the primary analytical emphasis was guided by the study's conceptual framework, while supplementary codes were permitted to arise from the participants' replies. To enhance dependability, two researchers independently coded a sample of the transcripts and subsequently compared their coding judgements. To clarify coder positionality, the first coder was one of the researchers involved in the instructional intervention, whereas the second coder was not involved in delivering the lessons or assessing students' written test responses. The interviews were conducted by a professional moderator who did not evaluate students' written assignments. Before consensus discussion, the two coders independently coded the eight interview transcripts at the student-theme level, producing 32 binary coding decisions. Cohen's kappa indicated substantial initial inter-coder agreement, $\kappa = .79$. Disagreements were subsequently discussed until consensus was reached, and the final coding book was revised accordingly. Any differences that arose were deliberated until a consensus was achieved. A coding book was created to record the correlation among raw student statements, initial codes, categories, and final themes (see **Table 7**). In the findings section, exemplary quotations were chosen to exemplify each subject.

Mixed-Methods Integration

The quantitative and qualitative components were integrated at three phases. During the planning phase, the research adhered to an explanatory sequential approach: quantitative data from pre-tests and post-tests were initially gathered, succeeded by qualitative interviews to elucidate the impact of parameterization-based education on learners. During the analytical phase, interview subjects were chosen based on their post-test performance metrics to guarantee representation of students exhibiting high, medium, and poor rubric-based results. Furthermore, qualitative coding concentrated on issues that could elucidate the quantitative trends, including generalization, model development, challenges with abstraction, and assistance from visualization. During the interpretation phase, the quantitative and qualitative results were linked via a combined presentation and a cohesive narrative. The aim of integration was not just to validate the numerical results, but also to elucidate students' perceptions of the instructional sequence and to discern the factors that could have facilitated or hindered their modeling performance.

FINDINGS

The results are delineated in two halves. The initial segment evaluates students' performance on pre- and

Table 8. Descriptive statistics for raw pre- and post-test scores

Group	Test	N	M	SD	Minimum	Maximum
Experimental	Pre-test	28	2.41	0.32	1.90	3.10
	Post-test		3.72	0.38	3.00	4.60
Control	Pre-test	29	2.36	0.26	1.80	2.80
	Post-test		2.70	0.23	2.10	3.20

Table 9. Pre-test-adjusted primary model for post-test raw scores

Predictor	B	SE	t	df	Exact p	95% CI
Intercept	0.91	0.24	3.77	54	.0004	[0.43, 1.40]
Experimental vs. control	0.98	0.06	16.93	54	< .001	[0.86, 1.10]
Pre-test raw score	0.76	0.10	7.49	54	< .001	[0.55, 0.96]

post-tests to see if there were significant score differences before and following the intervention.

Subsequently, student response corroborates the experimental findings. At the end of the experimental study, the effect of the didactic design based on teaching symmetry transformations through the parametrization method on students' intra-mathematical modeling skills was evaluated according to the pre- and post-test results, and it revealed that there were significant differences in student performance between the experimental and control groups. Although both groups showed improvement, the experimental group, which was taught using the parametrization method, demonstrated substantially higher gains in modeling performance compared to the control group.

The quantitative data were initially analyzed using students' raw pre- and post-test scores, which served as the primary outcome variable. This approach was predicated on the premise that raw scores retain greater information on students' performance and facilitate the detection of minor variations in success with greater sensitivity than wide category levels. The baseline equality of the two groups was assessed using unadjusted pre-test results.

The findings indicated that the two groups were largely equivalent prior to the intervention. Following the intervention, post-test performance was evaluated using raw scores, and when applicable, a pre-test-adjusted comparison was performed to address baseline disparities between the groups. The findings indicated that students in the experimental group demonstrated more enhancement compared to those in the control group.

Consequently, the low, medium, and high performance categories served just as supplementary descriptive markers to depict the distribution of students' modelling performance. The data indicated a significant rise in the proportion of high-performing children within the experimental group, accompanied by a drop in the number of low-performing students. The assessment of the intervention's efficacy was predominantly grounded in raw score analyses instead of category comparisons.

Raw-Score Descriptive Statistics

The fundamental quantitative assessment relied on students' raw scores on the 0-5 scale. Raw scores used as the primary quantitative outcome due to their capacity to preserve more information than general performance categories. This aligns with the evaluation segment of the article, where the overall score was established as the aggregate of five components, each capable of earning a maximum of 5 points, whereas performance tiers were regarded only as ancillary descriptive metrics (**Table 8**).

The experimental and control groups were generally comparable at pre-test. An independent-samples t-test indicated no statistically significant baseline difference between the groups, $t(55) = 0.59$, $p = .561$, Hedges' $g = 0.15$. The experimental group increased from a pre-test mean (M) of 2.41 to a post-test $M = 3.72$, whereas the control group increased from $M = 2.36$ to $M = 2.70$. These descriptive results indicate a larger raw-score gain in the experimental group; however, the primary between-group comparison was conducted using the pre-test-adjusted model reported below.

Primary Pre-Test-Adjusted Model

A pre-test-adjusted linear model was utilized to assess post-test performance while considering baseline performance (**Table 9**). The raw score from the post-test served as the dependent variable, the instructional group acted as the independent variable, and the raw score from the pre-test was utilized as the covariate. This adhered to the analytical framework outlined in the article, where the post-test score was designated as the dependent variable, the group as the independent variable, and the pre-test score as the covariate.

Upon adjusting for pre-test performance, the analysis revealed that the experimental group attained a superior adjusted post-test mean compared to the control group, $B = 0.98$, $SE = 0.06$, $t(54) = 16.93$, $p < .001$, 95% confidence interval [0.86, 1.10], partial $\eta^2 = .841$. The modified post-test average was 3.70 for the experimental cohort and 2.72 for the control cohort (**Table 10**). This outcome demonstrates superior rubric-based mathematical modeling efficacy in the experimental cohort within the confines of this research. Assumption verifications

Table 10. Adjusted post-test means

Group	Adjusted post-test mean	SE	95% CI
Experimental	3.70	0.04	[3.62, 3.78]
Control	2.72	0.04	[2.64, 2.80]
Adjusted mean difference	0.98	0.06	[0.86, 1.10]

Table 11. Supplementary distribution of performance levels

Group	N	Test	Low level: n (%)	Medium level: n (%)	High level: n (%)	Gain in high %
Experimental	28	Pre-test	27 (96.4)	1 (3.6)	0 (0.0)	-
		Post-test	1 (3.6)	24 (85.7)	3 (10.7)	+10.7
Control	29	Pre-test	29 (100)	0 (0.0)	0 (0.0)	-
		Post-test	27 (93.1)	2 (6.9)	0 (0.0)	0.0

Table 12. Exploratory categorical comparisons between groups

Comparison	Observed distribution	Statistical test	Test statistic	Exact p value	Effect size	Interpretation
Pre-test categorical distribution	(Experimental: L = 27, M = 1, H = 0); (Control: L = 29, M = 0, H = 0)	Fisher's exact test/ Pearson χ^2 as descriptive	$\chi^2(1, N = 57) = 1.054$	Fisher p = .491	Cramér's V = .136	No meaningful baseline categorical difference
Post-test categorical distribution	(Experimental: L = 1, M = 24, H = 3); (Control: L = 27, M = 2, H = 0)	Fisher-F.-Halton exact test/Pearson χ^2 as exploratory	$\chi^2(2, N = 57) = 45.755$	Exact p < .001	Cramér's V = .896	Strong exploratory association; interpret cautiously because sparse cells were present

Note. L: Low; M: Medium; & H: High

corroborate the understanding of the pre-test-modified model. The group × pre-test interaction did not reach statistical significance, $F(1, 53) = 2.91$, $p = .094$, suggesting that the assumption of uniformity in regression slopes was upheld. The residuals exhibited an approximate normal distribution, Shapiro-Wilk $W = .992$, $p = .960$, and Levene's test confirmed the homogeneity of residual variance, $F(1,55) = 0.62$, $p = .436$. Consequently, the pre-test-adjusted model was deemed suitable for analyzing post-test group disparities.

Supplementary Categorical Distribution

Alongside the raw-score evaluation, students' scores were classified into low, medium, and high-performance tiers for descriptive analysis. The thresholds adhered to the evaluation standards: low = 0-3.0, medium = 3.1-4.2, and high = 4.3-5.0 (**Table 11**).

The categorical distribution exhibits a distinct descriptive shift in the experimental group. At pre-test, almost all students in the experimental group were at a low-performance level, and after the intervention only 1 student remained at this level. Most of the students in the experimental group went to the moderate-performance level and three students moved to the high-performance level. However, most students in the control group stayed at the low-performance level at post-test. These categorical results are useful for describing the distribution of performance levels; however, they are not treated as primary evidence because categorization reduces information from the original raw scores.

The statistical analysis not only confirmed that the observed difference was significant but also indicated that the improvement observed in the experimental group was unlikely to be due to chance. More importantly, beyond statistical significance, the magnitude of the difference demonstrates a practically meaningful educational effect.

Exploratory Chi-Square Comparison

Categorical comparisons were employed solely as additional exploratory examinations (**Table 12**). Due to the fact that within-group pre-/post-test categorical alterations consist of repeated measurements from identical students, thus breaching the independence assumption, they were excluded from analysis via Pearson's Chi-square test (Cramér's V, 2017). Conversely, alterations within the group were articulated utilizing the distribution table provided before.

In the post-test categorical analysis, the anticipated counts were 13.75, 12.77, and 1.47 for the experimental cohort, while the control group had expected counts of 14.25, 13.23, and 1.53 across the low, medium, and high classifications, respectively. Due to the anticipated counts for the elevated-level cells falling below 5, the subsequent categorical analysis was regarded as exploratory.

Consequently, the primary quantitative analysis is grounded on the raw-score pre-test-adjusted framework instead of the categorical Chi-square evaluation.

Table 13. Number of students who mentioned each qualitative theme

Theme	Frequency	Illustrative transcript identifier/example
Generalization	6/8	S3/one rule for many cases
Modeling	5/8	S5/I built formulas myself
Difficulty	3/8	S2/confusing at first
Visualization	4/8	S7/technology facilitation

Cramér's V (2017) was computed to assess the strength of relationship between group membership and performance level in the post-test categorical analysis. The post-test comparison between the experimental and control groups revealed a Pearson's Chi-square test result of $\chi^2(2, n = 57) = 45.755$, $p < .001$. The associated Cramér's V was .896. According to Cohen's (1988) established guidelines for Cramér's V/Cohen's *w* this number signifies a moderate-to-large correlation between instructional group and post-test performance level.

The results indicate a significant difference in the distribution of low, medium, and high performance levels between the experimental and control groups at post-test. From a practical standpoint, the proportion of students achieving high performance was greater in the experimental group at post-test (10.7%) compared to the control group (0.0%). The experimental group exhibited a more significant enhancement in high-level performance from pre-test to post-test (+10.7% points) compared to the control group (0.0% points).

Given that the group effect was statistically significant in the pre-test-adjusted model, the H_0 was dismissed. The findings corroborated the H_1 suggesting that the experimental and control groups exhibited differences in post-test intra-mathematical modeling performance after adjusting for pre-test performance. Nonetheless, due to the quasi-experimental approach employed in the study, this result needs to be seen as indicative of a modified group disparity rather than as conclusive causative evidence.

The combined pre- and post-test results indicate a good impact of the parameterization-based didactic framework on students' intra-mathematical modeling performance for symmetry transformations. The categorical comparisons and effect-size indications indicate that students in the experimental group were more inclined to achieve elevated rubric-based performance levels post-intervention compared to those in the control group. Nonetheless, due to the limited, single-site sample and the quasi-experimental methodology, these findings should be approached with caution. The findings indicate that teaching symmetry transformations via a structured parameterization sequence may enhance students' capacity to generalize transformation rules, link algebraic and geometric representations, and develop mathematical models in this particular instructional context, rather than statistically validating overall effectiveness.

From an academic standpoint, these notable distinctions may enhance elements of rubric-oriented modelling performance instead of simply procedural proficiency. Students exposed to this didactic framework based on the parametrization method were more successful in generalizing transformations, relating algebraic representations to geometric interpretations, and demonstrating flexible problem-solving strategies. Overall, the findings were associated with higher rubric-based post-test performance in this sample.

Qualitative Findings

The qualitative data were obtained via semi-structured interviews with eight students from the experimental group, designated S1-S8. The interviews were performed following the quantitative analysis and were designed to offer explanatory insights into students' experiences with the parameterization-based educational sequence. Themes were derived from a thematic analysis of the interview transcripts. Two researchers initially discerned significant units within the students' replies and categorized them employing a hybrid deductive-inductive methodology. Deductive codes were derived from study's conceptual framework, encompassing generalization, parameterization, model development, and visualization, whereas inductive codes documented unforeseen or student-specific events, including initial uncertainty with variables and challenges in abstraction. The programmers evaluated their coding choices, addressed discrepancies, and achieved consensus prior to categorizing the codes into overarching themes. Four primary themes were identified: comprehension via generalization, transition to modelling, challenges in abstraction, and assistance for visualization (see **Table 7** and **Table 13**).

Theme 1. Understanding through generalization

Six of the eight students examined indicated that the teaching they received facilitated their transition from solving isolated cases to identifying broader principles applicable to other symmetry contexts. This transition signifies that individuals have cultivated a more profound comprehension of the functioning of mathematical systems. A student (S3) reflected:

> "Prior to this series of classes, I addressed each symmetry problem individually. I examined the figures presented in the question and attempted to ascertain the solution just for that particular

> instance. Upon examining the parameters, I noticed that the same principle may be used in several contexts. When the parameter's value alters, the example modifies, although the rule's structure stays unchanged."

This quotation illustrates a shift from case-based procedural reasoning to structural generalization. The learner no longer perceives each transformation as an isolated assignment but starts to recognize symmetry transformations as constituents of a cohesive family of connected mathematical scenarios. This corroborates the notion that parameterization assisted students in recognizing invariant linkages amongst changing situations.

Theme 2. Transition to modeling

Likewise, five students articulated a transition from rote memorization of formulas to independently creating mathematical representations. Utilizing parameters allowed students to construct their own computations and demonstrate the interrelations among variables. A student (S5) elucidated:

> "In prior lessons, I often awaited the teacher's provision of the formula. I subsequently attempted to apply it to analogous enquiries. In this approach, we initially examined cases, subsequently modified the values, and ultimately formulated the equation independently. Utilizing parameters facilitated my comprehension of the formula's origin rather than merely committing it to memory."

This answer signifies a crucial cognitive transition towards model development. The student's focus on "the origin of the formula" indicates that the parameterization sequence facilitated the shift from utilizing established principles to developing generalized algebraic models. The qualitative evidence corresponds with the model-construction aspect of the evaluation rubric, wherein students were required to articulate transformation connections via parameters and substantiate the resultant rule.

Theme 3. Difficulty in abstraction

Despite the generally favorable interview replies, three students indicated that they first encountered challenges when variables and parameters assumed various functions. A student (S2) remarked:

> "Initially, I was perplexed due to the excessive number of letters. Occasionally, the identical letter appeared as a coordinate, while at other times, a different letter served as a parameter. Upon resolving several examples and verifying them in GeoGebra, I started to comprehend that the parameter governs the scenario, whilst the variables delineate the point or the function."

This remark suggests that parameterization may initially provide obstacles in abstraction, and while the strategy promotes higher-level thinking, it necessitates cognitive adaption during the initial phases of learning. Nonetheless, it illustrates that students' challenges were not only impediments; instead, they were integral to the learning process. The student's differentiation between a parameter that "regulates the situation" and variables that "characterize the point or the function" reflects a developing comprehension of the purpose of parameters in mathematical modelling.

Theme 4. Visualization support

Four students highlighted that the GeoGebra-enhanced exercises, specifically, facilitated their connection of algebraic formulae to geometric transformations and enhanced their comprehension of the subjects. A student (S7) articulated:

> "Upon initially observing the formula, it was challenging to comprehend the alterations occurring. Upon altering the parameter in GeoGebra, I saw the point or graph in motion. Subsequently, the formula got more comprehensible as I was able to correlate the letters with the motion on the coordinate plane."

This response indicates that visualization helped students' understanding of parameter fluctuation. The dynamic depiction facilitated the student's association of symbolic expressions with geometric motion, which is essential for comprehending symmetry changes as both algebraic and geometric entities. Consequently, the application of visual technology in education seems to have facilitated the interpretation and validation stages of the modelling process.

Overall, the qualitative findings are consistent with the quantitative results (see **Table 14**), indicating that students perceived the parameterization-based sequence as a progression from isolated instances to general principles, from rote memorization to formula development, and from static symbolic representations to dynamic visual comprehension. Students' observed challenges with abstraction suggest that the methodology needs meticulous scaffolding, particularly when they initially confront variables and parameters in various capacities.

DISCUSSION

This study's findings offer the first evidence that a parameterization-based didactic framework may enhance students' mathematical modeling skills regarding symmetry transformations. The discourse is structured around the subsequent principal results.

Table 14. Joint display of quantitative results and qualitative themes

Quantitative pattern	Qualitative theme	Illustrative qualitative evidence	Interpretation
In the experimental group, the percentage of high-level students rose from 0 % in pre-test to 10.7% in post-test	Comprehension via generalization	Six out of eight interviewed students indicated that the strategy assisted them in identifying "one rule applicable to several scenarios"	The increase in advanced performance corresponds with students' accounts of transitioning from isolated instances to generalized transformation principles
The experimental group had a greater improvement in high-level performance compared to the control group	Transition to modeling	Five students said that they starting formulating equations instead of memorizing pre-established norms	The enhancement shown in the post-test aligns with students' accounts of advancing model-construction techniques via parameterization
The category distribution predominantly transitioned from low to medium performance within the experimental cohort	Difficulty in abstraction	Three students expressed initial perplexity when variables and parameters assumed distinct functions	The qualitative findings substantiate the quantitative enhancement by demonstrating that parameterization can facilitate modeling while necessitating cognitive adaptation and scaffolding
The intervention employed dynamic visualization to facilitate parameter variation	Visualization support	Four students indicated that GeoGebra facilitated their understanding of how variations in parameters influenced points, lines, and graphs	Visualization may have facilitated students' connection of algebraic formulas to geometric significance, hence aiding in the understanding and validation of models

The quantitative results indicate that students in the experimental group had superior post-test performance compared to those in the control group. This discovery corroborates other studies that see parameter-based tasks as a method for enhancing multiple-case analysis, generalization, and structural reasoning in mathematical education (Prus & Lenchuk, 2024). This paper advances the existing research by applying parameterization especially to symmetry transformations, rather than to algebraic equations, inequalities, or generic parameter issues. The study indicates that parameterization may assist students in viewing symmetry transformations not as discrete operations, but as a cohesive set of connected mathematical scenarios guided by overarching principles.

Secondly, the results augment prior studies on mathematical modelling proficiency by concentrating on intra-mathematical modeling. A significant portion of the modelling literature focuses on the representation of real-world scenarios. This research investigated modelling processes inherent to mathematics and demonstrated students' progression from particular transformation instances to generalized algebraic and geometric representations. This reinforces the perspective that modelling competency encompasses the integration of representation, transformation, interpretation, and validation (Kurudirek et al., 2026). In the current study, these processes were elucidated as students recognized invariant and variable elements, established parameters, devised transformation methods, and validated the resultant models with specified values.

Third, the study enhances the literature by defining parameterization as a sequential instructional framework. The framework extended beyond merely requiring students to incorporate parameters in final formulae. It directed students through a sequence from particular instances to comparative analysis, parameter modification, general principle formulation, model development, and interpretation/validation. This structured sequence seems to have facilitated students' association of specific instances with abstract mathematical frameworks (Adamchuk & Samsikova, 2020). Consequently, the study's contribution encompasses not just the use of parameterization, but also the formulation of an educational framework whereby parameterization serves as a mechanism for mathematical modelling (Koryanov & Prokofiev, 2011).

Collectively, the results yield first, context-specific evidence suggesting that a sequenced parameterization approach might effectively structure training on symmetry transformations to enhance rubric-based mathematical modeling outcomes. The study's significance is not in showcasing widespread efficacy, but in creating and analyzing a systematic instructional framework that links specific cases, parameter variation, general rule development, model construction, and interpretation/validation.

Fourth, students' interview replies indicated that they started to identify "one rule for many cases," formulate equations instead of simply memorizing them, and associate algebraic principles with geometric representations (Xu et al., 2021). The findings substantiate the hypothesis that the intervention facilitated a transition from procedural application to relational and structural comprehension (Hernández-Zavala et al., 2025). The early challenges associated with abstraction simultaneously temper the encouraging outcomes. The strategy may facilitate advanced cognitive processes; yet it necessitates that students adjust to manipulating variables and parameters in a more adaptable manner. This indicates that parameterization-based education must be implemented incrementally and supplemented by examples, guided enquiries, and visual aids.

The qualitative results do not indicate the intervention's efficacy; instead, they offer explanatory insights into the experiences of the chosen students about the parameterization-based sequence. Their replies underscore potential processes, including the identification of overarching principles, the formulation of equations, and the linkage of algebraic and geometric representations, while also exposing challenges with abstraction that temper the optimistic assessment. These challenges indicate that parameterization might not be readily attainable for all learners, necessitating meticulous instructional support.

The GeoGebra-supported visualization seems to facilitate the teaching process. The findings do not suggest that GeoGebra independently induced improvement; instead, the program served as a representational instrument that enabled students to see the impact of parameter alterations on geometric transformations. This corroborates prior research on the significance of dynamic geometry environments (Benning et al., 2023), while the current study expands upon this by linking visualization explicitly with parameter change and mathematical modelling in symmetry transformations.

The noted differences in the post-test may have been affected by elements beyond the parameterization sequence. Despite both cohorts receiving instruction from the identical educator, the instructor's participation in the research may have led to the emergence of expectation effects. The implementation of GeoGebra and dynamic visualization in the experimental setting may have also enhanced students' involvement and representational comprehension. Furthermore, the backdrop of the after-school program and the circumstance that the same educator evaluated both the pre- and post-test may have impacted either student outcomes or the evaluation procedure.

The main contribution of this paper may therefore be articulated with greater precision. The innovation is not parameterization as such, not symmetry transformations in isolation. The study enhances understanding by integrating four components:

- instructing symmetry transformations via parameterization
- structuring teaching as a progressive sequence from specific instances to model development
- analyzing students' learning through the framework of mathematical modelling
- providing classroom-based comparative evidence within an 11th-grade algebra setting.

The study's findings provide evidence consistent with a possible positive contribution to structuring symmetry-transformation education in a more conceptual and modeling-focused manner.

Limitations

The findings offer first evidence that the parameterization-based didactic framework may enhance students' mathematical modelling abilities regarding symmetry transformations; nevertheless, some limitations must be acknowledged when interpreting the results.

- The sample size was modest (n = 57) including exclusively 11th-grade male students from a singular academic high school. Consequently, the results cannot be readily extrapolated to mixed-gender classrooms, other school types, or wider cultural and educational settings.
- The study utilized a quasi-experimental design including two comparable groups instead of a completely randomized experimental approach. While baseline equivalence was assessed by pre-test results, the assignment method did not eradicate all potential selection biases. Consequently, the observed post-test changes should be regarded not as conclusive causal evidence, but as indicative of the possible impact of the intervention.
- Despite the same instructor teaching both the experimental and control groups to minimize teacher-related variance, the influence of the instructor cannot be entirely eliminated. Following the implementation of the lesson by one of the researchers, variables such as pedagogical approach, familiarity with the intervention, expectations, and excitement may have affected students' involvement and performance.
- The intervention persisted for only three teaching weeks. The brevity of this length constrains the study's ability to assess the long-term evolution of students' innate mathematical modelling skills. Furthermore, no delayed post-test was conducted; hence, the study is unable to ascertain if students retained the observed improvements over time.
- The assessment tool was particularly created for this study. Despite the utilization of expert review, pilot implementation, rubric-based grading, and reliability evidence to enhance the instrument, it should not be considered a completely standardized measure of intrinsic mathematical modelling.

- Students' raw scores were classified into low, moderate, and high performance levels for descriptive reasons. While these categories facilitate the presentation of student performance distribution, they may result in information loss and obfuscate minor advancements within the categories. Consequently, raw scores were regarded as the principal quantitative result, whereas category levels served solely as ancillary descriptive indications.
- Finally, the qualitative component relied on interviews with a limited number of students from the experimental group. While the interviews yielded valuable insights into students' perspectives of generalization, model development, and visualization, the small participant pool and narrowly defined interview technique limited the depth of qualitative analysis.

Collectively, these limitations suggest that the findings have to be regarded as context-dependent and provisional. The study indicates that parameterization-based education may be beneficial for facilitating mathematical modeling in symmetry transformations; nevertheless, further research is required before making larger assertions regarding its efficacy.

CONCLUSION AND RECOMMENDATIONS

This study analyzed the instruction of symmetry transformations within a parameterization-based didactic framework and explored its possible impact on participants' mathematical modelling abilities. The results offer first evidence that the suggested framework may facilitate students' transfer from particular symmetry instances to generalized algebraic and geometric representations.

The findings suggest that directing engaged students through particular situations, parameter variation, general rule formulation, model development, and interpretation may enhance their organized comprehension of symmetry transformations. Specifically, students in the experimental group exhibited superior post-test performance compared to those in the control group, and the interview results indicated that numerous students regarded the method as beneficial for generalization, formula development, and linking algebraic representations with geometric significance. Nonetheless, these findings should not be construed as proof of widespread efficacy across diverse student demographics or educational contexts.

Consequently, the parameterization-based framework need to be considered and assessed as a potential pedagogical method rather than a well-tested model. The study's potential worth resides in its organization of symmetry transformations, which facilitates student comparisons of examples, identification of invariant and variable elements, expression of connections through parameters, and construction of generalized mathematical models.

The findings indicate to educators the potential benefits of integrating parameterization as a pedagogical approach in geometry and transformation subjects. Rather than offering transformation rules as pre-formulated equations, a parameterization-based sequence may assist students in systematically organizing mathematical concepts by progressing from specific instances to parameter variation, general rule development, model creation, and interpretation. The research further enhances the literature by proposing that parameter-based representations might connect visual geometry and symbolic mathematics, thereby facilitating the modelling cycle of representation, transformation, and interpretation. Cognitive load was not explicitly assessed in this study; so, any potential correlation between parameterization and students' cognitive processes should be seen as a theoretical inference rather than an empirical conclusion.

The research should not be construed as endorsing a universally successful teaching framework. Its main contribution is the formulation and preliminary analysis of a systematic design process for instructing symmetry transformations. This sequence includes four verifiable design principles: structured variation, guided generalization, model development, and visual validation. Structured variation denotes the methodical alteration of parameters to enable students to analyze comparable transformation instances. Guided generalization pertains to assisting learners in transitioning from particular instances to overarching transformation principles. Model development entails articulating these principles in algebraic or geometric terms. Visual validation pertains to the examination and interpretation of resultant models using diagrams, coordinate representations, or interactive tools like GeoGebra, which can assist learners in cultivating a more profound comprehension of the foundational principles.

The findings suggest that structured variation, guided generalization, model construction, and visual validation are encouraging design concepts warranting more investigation in further research. Subsequent investigations ought to assess the durability of these results using more stringent study methodologies. Future research ought to encompass independent replications carried out by educators or researchers uninvolved in the intervention's development, postponed post-tests to assess retention, blind scoring methodologies, validated assessments of mathematical modeling, and sufficiently powered multisite samples that reflect diverse educational environments and coeducational classrooms. Subsequent research ought to differentiate the impacts of parameterization from those of dynamic visualization instruments to enable a clearer assessment of each educational element's contribution.

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APPENDIX A

The mathematical explanations and practical examples in this appendix serve as illustrative teaching artefacts designed to elucidate the structure and pedagogical rationale of the proposed paradigm. These resources are excluded from empirical data analysis but are presented to enhance transparency, replicability, and instructional transferability of intervention.

A1. Find the points symmetric to $A(x_0, y_0)$ with respect to the following lines:

a) $y = c$

b) $x = c$

c) $y = x$

d) $ax + by + c = 0$

A1a. In finding the point symmetric to $A(x_0,\ y_0)$ with respect to the line $y = c$, students' intra-mathematical modeling skills are developed through these tasks by means of the parameterization method.

a) *Identifying the pattern through a coordinate table*

Example 1. Reflect the following points with respect to $y = 3$: $A_1(2, 5)$, $A_2(-1, 1)$, $A_3(1, 5)$.

Students are given the following tasks:

– Reflect each point $(x_0,\ y_0)$ with respect to $y = 3$, find $A'(x', y')$, and construct a table. Then, for each pair, determine the relationship between x' and x_0.

– Compute the following expression: $\frac{y_0+y\prime}{2}$. What is this number equal to? Based on the pattern you observe, write a general formula for $A(x_0,\ y_0) \to A'(x', y')$.

– Using GeoGebra, illustrate the symmetry of the given points.

Note. Symmetry with respect to a point is a rotation by 180°(central symmetry); that is, if the point $A(x_0, y_0)$ is mapped to the point $A'(x', y')$, symmetric to the point $A(x, y)$, then the point A is the midpoint of the segment A_0A':

$$x = \frac{x_0+x\prime}{2}, y = \frac{y_0+y\prime}{2}. \tag{1}$$

Explanation. By completing the tasks, students identify the following pattern. In all cases, the abscissa of the point remains unchanged: $x' = x_0$. While for the ordinate, $\frac{y\prime+y_0}{2} = 3, y' + y_0 = 6 = 2c, y' = 2c - y_0$. Hence, the point symmetric to $A(x_0, y_0)$ with respect to the line $y = c$ is $A'(x_0,\ 2c - y_0)$. **Figure A1** illustrates the symmetry of the points given in the problem statement.

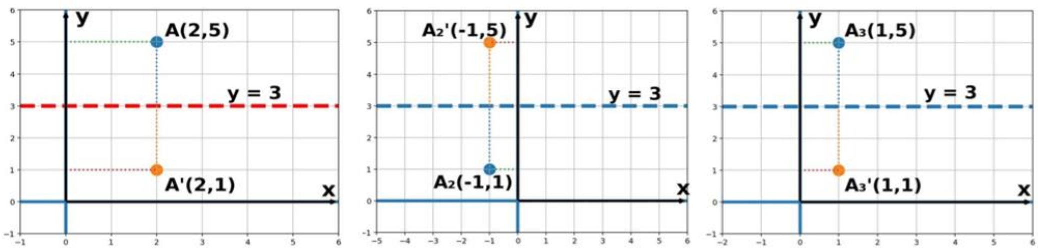

Figure A1. Symmetry of the given points-1 (the authors' own elaboration)

b) *Identifying the pattern through equality of distances (parametric problem)*

Example 2. Find the point symmetric to $A(x_0,\ y_0)$ with respect to the line $y = c$ (here at least one of x_0, y_0, and c is treated as a parameter).

Students are given the following tasks:

– Since $y = c$ is a horizontal line, find which coordinate of the point remains unchanged under reflection: $x' = ?$

– According to the definition of symmetry, the line $y = c$ is the perpendicular bisector of the segment A_0A'. Therefore, determine the vertical distances from the points A_0 and A' to the line $y = c$, namely $|y_0 - c|$ and $|y' - c|$, and justify that these distances are equal.

– Taking into account that the points lie on opposite sides of the line under reflection, solve the resulting equation, find y', and write the final formula.

Explanation. Since $y = c$ is a horizontal line, students determine by moving the point that the abscissa remains unchanged under reflection. According to the definition of symmetry, the line $y = c$ is the perpendicular bisector of the segment A_0A'. Hence, the distances are equal: $|y_0 - c| = |y' - c|$.

Since the reflected points lie on opposite sides of the line, $y' - c = -(y_0 - c)$. Solving the equation, we obtain $y' = 2c - y_0$. Therefore, the final formula is

$$\begin{cases} x' = x_0, \\ y' = 2c - y_0 \end{cases}. \quad (2)$$

Hence, the point symmetric to $A(x_0, y_0)$ with respect to the line $y = c$ is $A'(x_0,\ 2c - y_0)$.

A1b. In finding the point symmetric to $A(x_0, y_0)$ with respect to the line $x = c$, students' intra-mathematical modeling skills are developed through the above tasks by means of the parameterization method.

a) *Identifying the pattern through a coordinate table*

Example 3. Reflect the following points with respect to $x = 2$ and find $A'(x', y')$: $A_1(5,1), A_2(4,-2), A_3(-1,2)$. Students are given the following tasks:

– Find $A'(x', y')$ for each point $(x_0,\ y_0)$ and represent it on the coordinate plane;

– For each pair, examine the relationship between y' and y_0;

– Compute the following expression: $\frac{x_0 + x\prime}{2}$. What is this number equal to?

– Based on the pattern you observe, write the general formula: $A(x_0,\ y_0) \rightarrow A'(x', y')$.

– Using GeoGebra, illustrate the symmetry of the given points.

Explanation. By completing the tasks, students determine that $y' = y_0, \frac{x\prime + x_0}{2} = 2$ that is, $x' + x_0 = 4 = 2c$. Hence, the point symmetric to $A(x_0, y_0)$ with respect to the line $x = c$ is $A'(2c - x_0,\ y_0)$. **Figure A2** illustrates the symmetry of the points given in the problem statement.

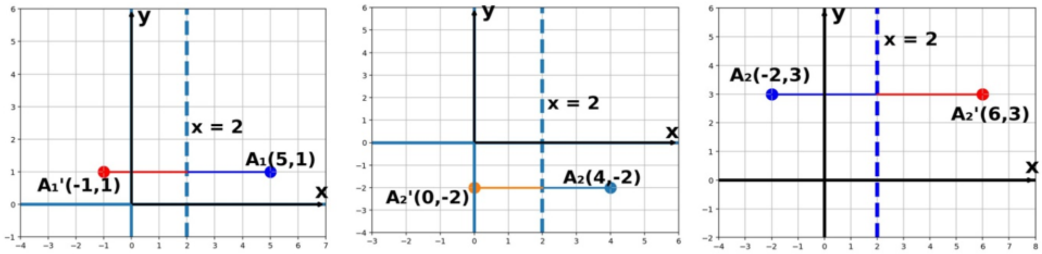

Figure A2. Symmetry of the given points-2 (the authors' own elaboration)

b) *Identifying the pattern through equality of distances (parametric problem)*

Example 4. Find the point symmetric to $A(x_0, y_0)$ with respect to the line $x = c$ (here at least one of x_0, y_0, and c is treated as a parameter).

Students are given the following tasks:

– Since $x = c$ is a vertical line, determine which coordinate remains unchanged under reflection: $y' = ?$

– Using the fact that the reflected points lie on opposite sides of the line and that the line $x = c$ is the perpendicular bisector of the segment A_0A', form an equation. Solve the equation and write the final formula.

Explanation. Students verify that $y' = y_0$. Taking into account the perpendicular bisector property in reflection, the horizontal distances to the line $x = c$ are equal: $|x_0 - c| = |x' - c|$. Since the points lie on opposite sides of the line, $-(x_0 - c) = x' - c$, hence $x' = 2c - x_0$. Therefore, the point symmetric to $A(x_0, y_0)$ with respect to the line $x = c$ is $A'(2c - x_0,\ y_0)$, which is given by the formula

$$\begin{cases} x' = 2c - x_0, \\ y' = y_0 \end{cases}. \quad (3)$$

A1c. In finding the point symmetric to $A(x_0, y_0)$ with respect to the line $y = x$, students' intra-mathematical modeling skills are developed through the following tasks by means of the parameterization method.

a) *Identifying the pattern through a table and observation*

Example 5. Reflect the following points with respect to $y = x$ and find $A'(x', y')$: $A_1(4,1), A_2(-2,3),\ A_3(3,-4)$.

Students are given the following tasks:

– For each point $(\ x_0,\ y_0)$, find $A'(x', y')$ and represent it on the coordinate plane:$A(x_0, y_0) \rightarrow A'(x', y')$.

– For each pair, determine what values $x'$and y' take.

– State the pattern in words and write it in the form of a general formula.

– Using GeoGebra, illustrate the symmetry of the given points.

Explanation. Students observe that under each reflection the coordinates interchange, and thus they write the final rule: $A(x_0,\ y_0) \rightarrow A'(y_0,\ x_0)$. **Figure A3** illustrates the symmetry of the points given in the problem statement.

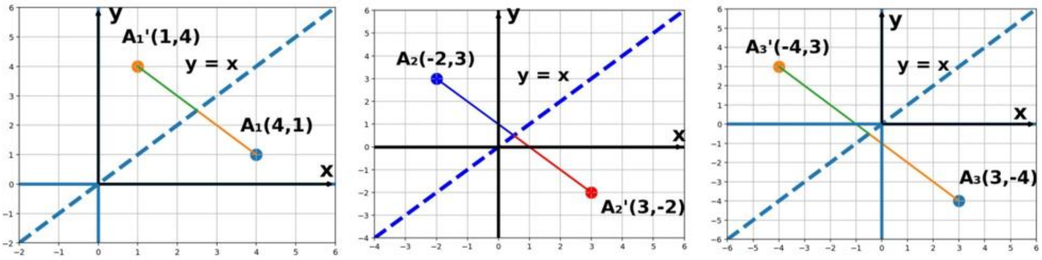

Figure A3. Symmetry of the given points-3 (the authors' own elaboration)

b) *Deriving the formula through geometric justification (parametric problem)*

Example 6. Find the point $A'(x', y')$ symmetric to the point $A(x_0,\ y_0)$ with respect to the given line $y = x$ (here one of x_0 and y_0 is treated as a parameter).

Students are given the following tasks:

– The segment A_0A' joining the points A_0 and A' must be perpendicular to the line $y = x$. If the direction vector of $y = x$ is $(1, 1)$, determine a perpendicular vector of the same length (this vector is $(1, -1)$). Using this, write how the direction of the segment A_0A' should look.

– Justify that under symmetry the line $y = x$ is the perpendicular bisector of the segment A_0A', and find the coordinates of the midpoint of this segment.

– Taking into account that the midpoint lies on the line $y = x$, compare its coordinates and write the formula for the reflection.

Explanation. Since the vectors $\vec{a} = (1, 1)$ and $\vec{b} = (x, y)$ are perpendicular and have equal lengths, their dot product must be zero, and their lengths must both be equal to $\sqrt{2}$: $\begin{cases} x + y = 0 \\ x^2 + y^2 = 2 \end{cases}$.

Solving the system gives $(x, y) = (1, -1)$. The midpoint of the segment A_0A' is $B\left(\frac{x_0+x'}{2}, \frac{y_0+y'}{2}\right)$, and it lies on the line $x = y$. Therefore, $\frac{x_0+x'}{2} = \frac{y_0+y'}{2}$, hence $x_0 + x' = y_0 + y'$. From this, derive the result $x' = y_0, y' = x_0$. Therefore, the final formula is $A(x_0,\ y_0) \rightarrow A'(y_0,\ x_0)$.

A1d. In finding the point symmetric to $A(x_0, y_0)$ with respect to the line $ax + by + c = 0$, students' intra-mathematical modeling skills are developed through these tasks by means of the parameterization method.

a) *Finding the image point through calculations*

Example 7. Reflect the following points with respect to the line $3x + 4y - 10 = 0$ and find the resulting points $A'(x', y')$: $A_1(-1, 2), A_2(-3, 1)$.

Students are given the following tasks:

– From each point $A(x_0,\ y_0)$, draw a perpendicular to the line and denote the point of intersection by $B(x_h,\ y_h)$. Then find the vector $\overrightarrow{A_0B}$.

– Since the normal vector of the line $3x + 4y - 10 = 0$ is $(a, b) = (3, 4)$, the vector A_0B is collinear with it $\overrightarrow{A_0B} = \vec{a}t,\ t \in R$. Using this, find the point $B(x_h,\ y_h)$.

– Since the point $B(x_h,\ y_h)$ lies on the line $3x + 4y - 10 = 0$, substitute it into $3x_h + 4y_h - 10 = 0$ and find t.

– Under reflection, the point $B(x_h,\ y_h)$ is the midpoint of the segment A_0A'. Therefore, $x' = 2\,x_h - \ x_0, y' = 2\,y_h - y_0$. Using this, find the point $A'(x', y')$.

– Using GeoGebra, illustrate the symmetry.

In the course of completing the task, students are asked the following questions to help them identify the pattern:

– In each result, how is A' obtained when passing from $A_0 \rightarrow A'$?

– Which operations are repeated each time?

– When finding t, which expression always appears?

– How does the quantity $3x_0 + 4y_0 - 10$ enter into the results?

Explanation. Students complete the above tasks step by step and then seek answers to the given questions. As a result, they identify the pattern. This process is completed by deriving the general formula through solving the following parameterized problem. **Figure A4** illustrates the symmetry of the points given in the problem statement.

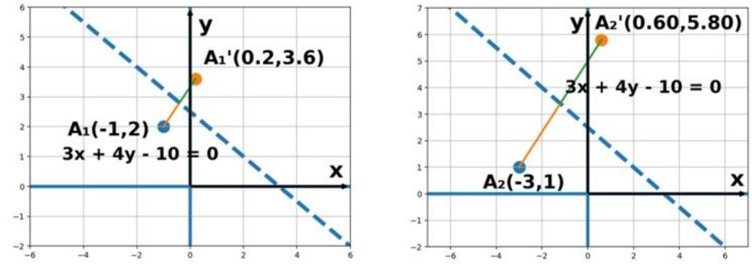

Figure A4. Symmetry of the given points-4 (the authors' own elaboration)

b) *Leading to the general formula through parameterization (parametric problem)*

Example 8. Given the line $ax + by + c = 0$, find the point $A'(x', y')$ symmetric to the point $A(x_0,\ y_0)$ with respect to this line (here $a, b, c,\ x_0,\ y_0$ are parameters, while $x,\ y$ are variables).

Solution. The normal vector of the line is $\vec{n} = (a, b)$. According to the property of reflection, the segment A_0A' is perpendicular to the line, and the line is the perpendicular bisector of the segment A_0A'. Therefore, the foot of the perpendicular dropped from A_0 to the line, denoted by $B(x_h,\ y_h)$, is the midpoint of the segment A_0A'. Since $\overrightarrow{A_0B} \parallel \vec{n}$, we have $\overrightarrow{A_0B} = t\vec{n}$. Hence,

$$x_h = x_0 + at, y_h = y_0 + bt. \tag{4}$$

Since the point $B(x_h,\ y_h)$ lies on the line, we have $ax_h + by_h + c = 0$.

Substituting the expressions in Eq. (4), we obtain $a(x_0 + at) + b(y_0 + bt) + c = 0$, that is, $ax_0 + a^2t + by_0 + b^2t + c = 0$, or $(a^2 + b^2)t + (ax_0 + by_0 + c) = 0$.

From this, $t = -\frac{ax_0+by_0+c}{a^2+b^2}$. Substituting this value into Eq. (4), we find the coordinates of the point $B(x_h, y_h)$: $x_h = x_0 + a\left(-\frac{ax_0+by_0+c}{a^2+b^2}\right) = x_0 - \frac{a(ax_0+by_0+c)}{a^2+b^2}$ and $y_h = y_0 + b\left(-\frac{ax_0+by_0+c}{a^2+b^2}\right) = y_0 - \frac{b(ax_0+by_0+c)}{a^2+b^2}$.

Since the point $B(x_h,\ y_h)$ is the midpoint of the segment A_0A', we have $\frac{x\prime+x_0}{2} = x_h, \frac{y\prime+y_0}{2} = y_h$.

Substituting these values, we obtain the formulas for the coordinates of the point $A'(x', y')$, which is symmetric to $A(x_0,\ y_0)$ with respect to the line $ax + by + c = 0$:

$$\begin{cases} x' = 2\left(x_0 - \frac{a(ax_0+by_0+c)}{a^2+b^2}\right) - x_0 = x_0 - \frac{2a(ax_0+by_0+c)}{a^2+b^2} \\ y' = 2\left(y_0 - \frac{b(ax_0+by_0+c)}{a^2+b^2}\right) - y_0 = y_0 - \frac{2b(ax_0+by_0+c)}{a^2+b^2}. \end{cases} \tag{5}$$

Remark. If $y = c$, then $(a = 0,\ b = 1)$, and $y' = 2c - y_0, x' = x_0$. If $x = c$, then $(a = 1,\ b = 0)$, and $x' = 2c - x_0, y' = y_0$. If $y = x$, then the coordinates are interchanged.

A2. Construct the equation of the line obtained by reflecting the line $y = kx + b$ with respect to the point $A(x_0,\ y_0)$.

In finding the line symmetric to the line $y = kx + b$ with respect to the point $A(x_0, y_0)$, students' intra-mathematical modeling skills are developed through the following tasks by means of the parameterization method. By solving them, students independently discover the rule of reflection (central symmetry) of the line $y = kx + b$ with respect to the point $A(x_0, y_0)$ and derive the general formula.

a) *Guiding students toward generalization and identification of the pattern*

Example 9. Find the reflections of the following lines with respect to the point $A(3, 2)$: $y = 2x - 1, y = -4x + 3$.

Students are given the following tasks:

– First, choose any three convenient points lying on a line $y = 2x - 1$ (for example, $A_1(0, -1), A_2(1, 1),\ A_3\left(\frac{3}{2}, 2\right)$. Reflect them with respect to the point $A(3, 2)$.

– Find the equation of the line passing through the obtained points $A_{1'}$ and $A_{2'}$.

– Write the result in the form $y = kx + b$ and determine how the slope k and the intercept b change.

– Using GeoGebra, illustrate the symmetry.

Students should notice that under central symmetry the direction of the line is preserved: the slope does not change, only the position of the line changes.

Explanation. First, reflect on the points $A_1(0,-1)$, $A_2(1,1)$ with respect to $A(3,2)$. Then the points $A_{1'}(6,5)$, $A_{2'}(5,3)$ are obtained. According to the rule of symmetry, these points are found from $x' = 2\cdot 3 - x = 6 - x, y' = 2\cdot 2 - y = 4 - y$ by substituting the selected points of the given line. Next, the equation of the line passing through $A_{1'}$ and $A_{2'}$ is found. Its slope is $k = \frac{5-3}{6-5} = 2$. Recall that for a line passing through the points $(x_1,\ y_1)$ and (x_2, y_2), the slope is given by $k = \frac{y_1 - y_2}{x_1 - x_2}$. Hence, the new line is sought in the form $y = 2x + b'$. Substituting one of the points, say $A_{2'}(5,3)$, we get $3 = 2\cdot 5 + b'$. Therefore, $y = 2x - 7$. If students repeat the same procedure for the points $(1,1)$ and $\left(\frac{3}{2}, 2\right)$, they again obtain the same reflected line. Thus, the lines $y = 2x - 7$ and $y = 2x - 1$ differ only in their constant terms.

Students identify the following patterns:

- -If the line is $y = kx + b$ and the center of symmetry is $A(x_0, y_0)$, then each point on the line transforms as $(x, y) \mapsto (x', y') = (2x_0 - x,\ 2y_0 - y)$.
- -In the equation of the line passing through any two reflected points $A_{1'}$ and $A_{2'}$ the slope coefficient k remains unchanged, and only the constant term may change.

Hence, after reflection, the line is sought in the form $y = kx + b'$. Here, the new intercept \$b'\$ is found by substituting one of the reflected points $A_{1'}$ or $A_{2'}$ into the equation $y = kx + b'$. **Figure A5** illustrates the symmetry of the given functions with respect to a point.

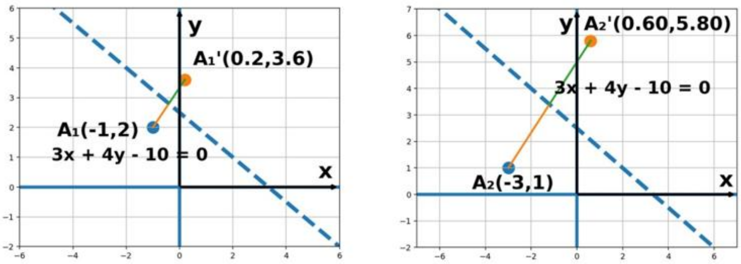

Figure A5. Symmetry of the given points-5 (the authors' own elaboration)

b) Deriving the general formula through parameterization (parametric problem)

Example 10. Construct the equation of the line obtained by reflecting the line $y = kx + b$ with respect to the point $A(x_0, y_0)$.

Solution. We use the first pattern identified in example 9. Since the point $A(x_0, y_0)$ is the center of symmetry for the points $(x, y) \to (x', y') = (2x_0 - x, 2y_0 - y)$ and (x', y'), we have $x = 2x_0 - x', y = 2y_0 - y'$. Here, recall that (x, y) are points lying on the given line $y = kx + b$, while (x', y') are their corresponding symmetric images. Since the points $x = 2x_0 - x', y = 2y_0 - y'$belong to the given line, we substitute them into the equation $y = kx + b$: $2y_0 - y' = k(2x_0 - x') + b$. Simplifying, we obtain

$$y' = kx' + (2y_0 - 2kx_0 - b). \tag{6}$$

We now use the second pattern identified in example 9. According to it, under central symmetry the slope of the reflected line remains unchanged, while only the intercept changes. Therefore, in Eq. (6) the new intercept is $b' = 2y_0 - 2kx_0 - b$. Hence, the equation of the line obtained by reflecting $y = kx + b$ with respect to the point $A(x_0, y_0)$ is

$$y = kx + (2y_0 - 2kx_0 - b). \tag{7}$$

Eq. (7) may also be remembered in the equivalent form

$$2y_0 - y = k(2x_0 - x) + b. \tag{8}$$

In analytic geometry at the university level, the problem of reflecting a line of the form $y = kx + b$ with respect to a point $A(x_0, y_0)$ is regarded as a particular case of central symmetry. Therefore, it is natural that the symmetric image of the line $y = kx + b$ should again have the form $y = kx + b'$. However, deriving this result not directly from the equation, but through parametric representation, is theoretically and methodologically important.

Introducing a parameter, we write the line as $\begin{cases} x = t, \\ y = kt + b \end{cases}$.

This representation makes it possible to interpret the line as a family of points. Now, according to the formula for central symmetry, each parametric point is transformed as $\begin{cases} x' = 2x_0 - t, \\ y' = 2y_0 - (kt + b) \end{cases}$.

From this system, we find $t = 2x_0 - x'$. Substituting into the second equation, we get $y' = 2y_0 - (k(2x_0 - x') + b) = kx' + (2y_0 - 2kx_0 - b)$.

As a result, the equation of the reflected line takes the form $y = kx + (2y_0 - 2kx_0 - b)$. Thus, it is established that under reflection the slope coefficient k of the line is preserved, while only the constant term changes. The methodological significance of this parametric approach lies in the fact that it serves as a bridge for expressing symmetry not merely as a sequence of particular algebraic transformations for a line, but also for functions in general. In this way, a transition is made from the particular case (a linear function) to the general case (an arbitrary function), making it possible to formulate general rules for the symmetry of the function $y = f(x)$.

A3. Rules for reflecting a function $y = f(x)$:

1) with respect to a point $A(x_0, y_0)$
2) with respect to the line $y = c$
3) with respect to the line $x = c$
4) with respect to the line $y = x$
5) with respect to the line $y = -x$

A3a. In finding the function obtained by reflecting $y = f(x)$ with respect to the point $A(x_0, y_0)$, students' intra-mathematical modeling skills are developed through these tasks by means of the parameterization method.

a) *Guiding students to identify the pattern through a table and points*.

Example 11. Find the function obtained by reflecting the function $y = f(x) = x^2$ with respect to the point $A(1, 2)$.

Tasks for students:

– Compute the values of the function at the following points $x = -1,\ 0,\ 1,\ 2,\ 3$ and construct a table.
– Reflect each point with respect to $A(1, 2)$.
– Plot the resulting points (x', y') on the coordinate plane and draw the new graph.
– Now observe the following: how are the values $x'$and y' related to x and y? Try to express y' in terms of x', so that the result is obtained in the form of a function $y' = g(x')$.
– Determine the rule for obtaining the function $y' = g(x')$.
– Using GeoGebra, illustrate the symmetry.

Explanation. Students should notice that the new function follows from the transformation $(x, y) \mapsto (x', y') = (2x_0 - x,\ 2y_0 - y)$. From this, x and y are expressed in terms of $x'$and y', and by substituting into $y = f(x)$, the new function is obtained. Part a in **Figure A6** illustrates the problem.

b) *Finding the general pattern and the reflected function through parameterization (parametric problem)*

Example 12. Find the function obtained by reflecting $y = f(x)$ with respect to the point $A(x_0, y_0)$.

Solution. Each point $(x, f(x))$ of the function, after reflection, takes the image $(x', y') = (2x_0 - x,\ 2y_0 - f(x))$.

Therefore, according to the rule identified in example 11, $x = 2x_0 - x'$. Substituting this into the original function gives $y' = 2y_0 - f(2x_0 - x')$. Hence, under point symmetry, since each point of the graph transforms according to the relation above, the equation of the reflected function takes the form

$$y = 2y_0 - f(2x_0 - x). \tag{9}$$

A3b. In finding the function obtained by reflecting $y = f(x)$with respect to the line $y = c$, students' intra-mathematical modeling skills are developed through these tasks by means of the parameterization method.

a) *Guiding students to identify the pattern through a table*.

Example 13. Find the function obtained by reflecting $f(x) = x^2$ with respect to the line $y = 2$.

Tasks for students:

– Find the values of the function at the following points, $x = -1, 0, 1, 2, 3$ and construct a table in which the points $A(x, f(x)) = (x, x^2)$ are listed.
– Reflect each point with respect to the line $y = 2$.
– Now observe the following: how are the values x' and y' related to x and y?
– Identify the pattern from the table, express y' in terms of y and c, and then write the equation of new graph.

– Using GeoGebra, illustrate the symmetry.

Explanation. Students should notice that the new function follows from the relations $x' = x, \frac{y+y'}{2} = 2, y' = 4 - y$. Hence, in general, the new function is of the form $y' = 2c - f(x)$.

Part b in **Figure A6** illustrates the problem.

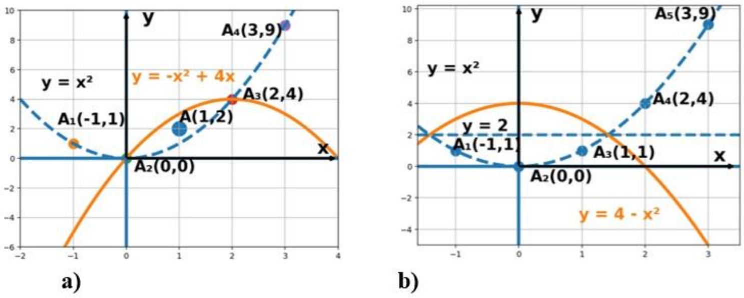

Figure A6. Symmetry of the given points-6 (the authors' own elaboration)

b) *Finding the general pattern and the reflected function through parameterization (parametric problem)*

Example 14. Find the function obtained by reflecting $y = f(x)$ with respect to the line $y = c$.

Solution. As observed in example 13, if the function $y = f(x)$ is reflected with respect to the line $y = c$, each of its points transforms according to the rule $(x, y) \mapsto (x,\ 2c - y)$. Therefore, the new function $g(x)$ is given by

$$g(x) = 2c - f(x). \tag{10}$$

Thus, under reflection with respect to the line $y = c$, the function values are reflected about the level c: each value y is replaced by $2c - y$, while x remains unchanged. As a special case, if $f(x) = kx + b$ and $y = 0$, then the graph of the linear function is reflected with respect to the x-axis, yielding the function $y = -f(x)$.

A3c. In finding the function obtained by reflecting $y = f(x)$ with respect to the line $x = c$, students' intra-mathematical modeling skills are developed through these tasks by means of the parameterization method.

a) *Guiding students to identify the pattern through a table.*

By solving the following examples, students independently determine the rule for finding the function obtained by reflecting $y = f(x)$ with respect to the line $x = c$, and derive the general formula themselves. As a result, they establish that under symmetry with respect to the vertical line $x = c$, the y-coordinate remains unchanged, while the x-coordinate is reflected about c.

Example 15. Find the function obtained by reflecting $y = f(x) = \sqrt{x}$ with respect to the line $x = 4$.

Tasks for students:

– Select convenient points from the graph, for example $(0, 0), (1, 1), (4, 2), (9, 3)$. Reflect each point with respect to $x = 4$.

– Verify that $y' = y, \frac{x+x'}{2} = 4$, and determine x' from this relation.

– Now generalize the results and determine what form the function $y = f(x)$ takes after reflection.

– Using GeoGebra, illustrate the symmetry.

Explanation. From the relations $y' = y, \frac{x+x'}{2} = 4$, students should determine that $x = 8 - x'$, and, in the general case, $y' = y, x = 2c - x'$. From this, they should recognize that after reflection the new function is given by $y' = f(2c - x')$. Part a in **Figure A7** illustrates in the problem.

b) *Finding the general pattern and the reflected function through parameterization (parametric problem)*

Example 16. Find the function obtained by reflecting $y = f(x)$ with respect to the line $x = c$.

Solution. If the function $y = f(x)$ is reflected with respect to the line $x = c$, then, according to the conclusion reached in example 15, each of its points transforms according to $(x, y) \mapsto (2c - x,\ y)$. Thus, the new function $g(x)$ is given by

$$g(x) = f(2c - x). \tag{11}$$

As a special case, if $f(x) = kx + b$ and $x = 0$, then the graph of the linear function is reflected with respect to the y-axis, yielding the function $y = f(-x)$.

A3d. In finding the function obtained by reflecting $y = f(x)$ with respect to the line $y = x$, students' intra-mathematical modeling skills are developed through these tasks by means of the parameterization method.

a) *Tasks aimed at discovering the pattern*.

The content of these tasks consists of posing a problem, conducting an experiment (graphical observation), identifying the pattern, and making a generalization. The problem-posing stage begins with the following question to students:

– If a graph is reflected across the line $y = x$, how does the new graph change?

First, students are asked to make initial conjectures. Then the following tasks are assigned:

– Draw the graphs of the functions $y = x^2$ and $y = x$. Mark the points $(-2,4), (-1,1), (0,0), (1,1), (2,4)$ on the graph of the function $y = x^2$.

– Reflect each point symmetrically with respect to the line $y = x$.

At this stage, students should notice the rule $(x, y) \mapsto (y, x)$.

– Write the points obtained after reflection, $(4,-2), (1,-1), (0,0), (1,1), (4,2)$, and draw corresponding graph.

Explanation. To help students identify the rule $(x, y) \mapsto (y, x)$, the teacher may ask the following questions: What was the equation of the original graph? How did the points change after the reflection? Therefore, what will the new equation be? Students should arrive at the following result: $x = y^2$. Part b in **Figure A7** illustrates the reflection in the problem.

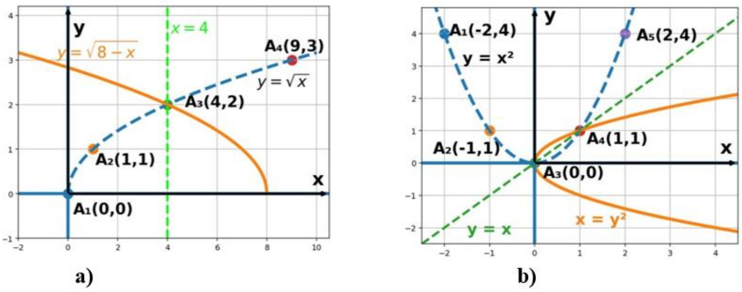

Figure A7. Symmetry of the given points-7 (the authors' own elaboration)

To generalize the results, another function is then given to students, for example:

– Investigate the function $y = 2x + 1$ using the same method as above.

– Take the points $(0, 1)$, $(1, 3)$, $(2, 5)$ from the graph of the function and determine how they change after reflection across the line $y = x$.

– After reflection, do the points $(1, 0)$, $(3, 1)$, $(5, 2)$ belong to the inverse function, since the transformation rule is $(x, y) \mapsto (y, x)$? Hence, does the reflected graph satisfy $x = 2y + 1, y = \frac{x-1}{2}$?

– Using GeoGebra, illustrate the symmetry.

b) *Determining the general pattern and finding the function through parameterization (parametric problem)*

Example 17. Find the function obtained by reflecting the graph of $y = f(x)$ across the line $y = x$.

After completing the above tasks, students arrive at the following conclusion: one needs to find the inverse function of $y = f(x)$. If the function $y = f(x)$ has an inverse, then the function obtained by reflecting it across the line $y = x$ is given by $x = f(y)$ or $y = f^{-1}(x)$ In particular, if $y = f(x)$ is a linear function, then for $y = kx + b$, the function obtained by reflecting it across the line $y = x$ is $y = \frac{1}{k}x - \frac{b}{k}$.

A3e. In finding the function obtained by reflecting $y = f(x)$ with respect to the line $y = -x$, students' intra-mathematical modeling skills are developed through these tasks by means of the parameterization method.

a) *Tasks aimed at discovering the pattern*.

Through the following examples, students independently identify the symmetry rule of the function $y = f(x) = x^2$ with respect to the line $y = -x$, and derive the general formula on their own.

Example 18. Find the function obtained by reflecting the graph of $y = f(x) = x^2$ across the line $y = -x$.

Students are given the following tasks:

– Find several points of the function, for example, $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(2, 4)$. Reflect each point across the line $y = -x$. Observe how the points change under symmetry with respect to $y = -x$, for example, $(-2, 4) \mapsto (-4, 2), (1, 1) \mapsto (-1, -1), (2, 4) \mapsto (-4, -2)$. Draw a general conclusion from this observation.

– Write down the resulting points, for example, $(-4, 2)$, $(-1, 1)$, $(0, 0)$, $(-1, -1)$, $(-4, -2)$.

– Try to write an equation for these points.
– Do these points satisfy the equation $x = -y^2$? Verify that the following observation follows from this: $-x = f(-y)$. Part a in **Figure A8** illustrates the reflection of the points given in the problem.
– Using GeoGebra, illustrate the symmetry. Part a in **Figure A8** illustrates the reflection in the problem.

Example 19. Find the function obtained by reflecting the graph of $y = 2x + 1$ across the line $y = -x$.

Tasks for students:

– Take several points belonging to the function $y = 2x + 1$, for example, $(-1,-1), (0,1), (1,3), (2,5)$. Reflect each point across the line $y = -x$. For the sample points, $(-1,-1) \mapsto (1,1), (0,1) \mapsto (-1,0), (1,3) \mapsto (-3,-1), (2,5) \mapsto (-5,-2)$.
– Reflect the sample points across the line $y = -x$ and verify that $(x, y) \mapsto (-y, -x)$.
– Determine the new function.
– Using GeoGebra, illustrate the symmetry.

Explanation. The given function is $y = 2x + 1$. After reflection, $-x = 2(-y) + 1$. Simplifying this equation, we obtain $x = 2y - 1, y = \frac{1}{2}x + \frac{1}{2}$. Part b in **Figure A8** illustrates the reflection in the problem.

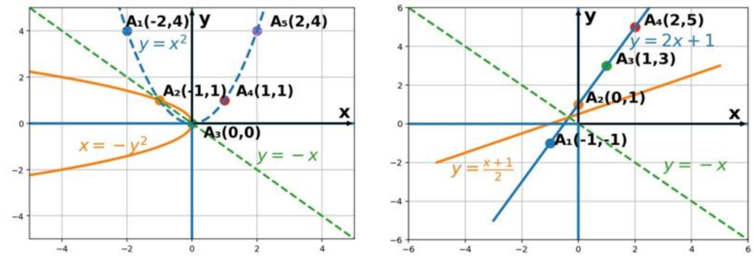

Figure A8. Symmetry of the given points-8 (the authors' own elaboration)

b) Determining the pattern through parameterization (parametric problem)

Example 20. If the graph is given by $y = f(x)$,what is the analytic form of the function obtained after reflection with respect to the line $y = -x$?

Solution. From solving the first and second examples, students verify through observation that the transformation rule is $(x, y) \mapsto (-y, -x)$. Therefore, the reflected function is written in the form $-x = f(-y)$. If the function $y = f(x)$ has an inverse, then its symmetric image is $y = -f^{-1}(-x)$. In particular case when $y = f(x)$ is a linear function, that is, $y = kx + b$, the function obtained by reflecting it with respect to the line $y = -x$ is $y = \frac{1}{k}x + \frac{b}{k}$.

Interview Main Question

Tell us about your experiences learning how to use the parameterization approach to do symmetry transformations over the course of three weeks on the project.

Follow-up prompts

- In what manner does this strategy differ from your familiar approach to learning symmetry transformations?
- Did the use of parameters enhance your comprehension of general principles? Kindly provide an example.
- Were you capable of independently formulating models or formulas? In what manner?
- What challenges or ambiguities were encountered initially?
- In what manner did GeoGebra or visual representations enhance your comprehension?
- Did this strategy facilitate your connection between algebraic formulae and geometric transformations?
- Which activity or task was most beneficial to you? What is the reason?