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Pre-service teachers' mathematical work on a calculus task that connects school and university

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Abstract

This paper addresses the issue of the gap that exists between the mathematics studied in initial teacher training and the school mathematics that the in-service teacher must teach. We present empirical results on how pre-service teachers (PTs) activate and coordinate cognitive processes and epistemological aspects when solving a university mathematic task that can be solved in the calculus domain. This task can likewise be adapted to the school level. This study is based on the theory of mathematical working spaces (MWS). The results examine the differentiation of the MWS of the participants according to the dynamics present between the problem-solving processes used in the task. The study's conclusion explores the possibilities offered by the task in terms of enriching the MWS of PTs and contributing to the shrinking of the mathematical gap that exists between the university and the school.

Keywords: mathematical working spaces, initial teacher training, tasks design, calculus, university mathematics, school mathematics

INTRODUCTION

Diverse studies have reported the existence of a gap between the university mathematics that pre-service teachers (PTs) must learn and the school mathematics that they later need to teach (e.g., Wasserman et al., 2018; Weber et al., 2020; Winsløw & Grønbæk, 2014). In this respect, research indicates that there are both advantages and disadvantages to dedicating substantial space to advanced mathematics courses in initial teacher training (Weber et al., 2020). The present study considers the extensive mathematics education that students receive in initial teacher training in Chile according to the Standards for Programs in Mathematics Pedagogy (2021) and problematizes the need to find tasks that connect this knowledge with school mathematics.

Some research in mathematics education addresses the advantages of advanced mathematics courses in terms of the amount that the PT must learn. These are related to the encourage of abilities such as logical thinking, abstraction, generalization, and demonstration (Alfaro-Carvajal & Fonseca-Castro, 2018; Weber et al., 2020), which, it is hoped, the future teacher will proceed to encourage in their future students in the school. Meanwhile, regarding the disadvantages, some studies have demonstrated that there is no clear correlation between the number of advanced mathematics courses taken by the PT in the university and the performance of their students in the school (Ubah & Bansilal, 2018; Wasserman et al., 2018). For this reason, dedicating the majority of the plan of studies to mathematics courses could detract from other important topics in teacher education, such as teaching methods for specific contents, educational policy, or neurodevelopment, among others.

The study carried out by Ubah and Bansilal (2018), which involved 42 PTs, reports that despite the

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This paper included part of the a priori analysis of the task presented at the VI Symposium on Mathematical Work (Menares Espinoza, 2019), in Menares Espinoza, R. (2019). Planos dirigidos en el ETM personal de profesores en formación: Una herramienta metodológica [Directed plans in the personal ETM of teachers in training: A methodological tool]. In E. Montoya, & L. Vivier (Eds.), *Proceedings of 6th Symposium on Mathematical Work* (pp. 245-256). Pontificia Universidad Católica de Valparaíso.

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Contribution to the literature

- This study reveals the gap between the mathematics studied in initial teacher training and mathematics in secondary education, especially in the domain of calculus.
- One contribution of this research is the design and implementation of tasks that can be adapted to different contexts and educational levels, allowing for the construction of bridges between school and university.
- Another contribution of the study relates to the exhaustive analysis of the mathematical work of future teachers, considering epistemological and cognitive aspects, which can be extended to other mathematical domains of teacher training. This can contribute to improving specific aspects or making decisions regarding training.

participants taking various advanced mathematics courses, they showed difficulty in interpreting and solving school mathematics problems, specifically involving quadratic equations. Ubah and Bansilal (2018) attribute this problem, in part, to the way teacher education programs are structured. Furthermore, another study has shown that taking many advanced courses does not reduce difficulties in the development of recognized mathematical abilities, such as carrying out demonstrations (Stylianou et al., 2009).

Particularly in the calculus domain, teacher education programs devote a significant portion of the plan of studies to addressing mathematical objects such as real numbers or continuous functions–objects specific to this domain–which allows an understanding of the structure of the object in question, putting it into use in hypotheses and solving problems related to it. However, multiple studies point to deficiencies in this domain, which are largely attributed to the tasks presented and how they are undertaken (Ubah & Bansilal, 2018; Weber et al., 2020). This issue is linked to the fact that historically, the teaching of analysis has placed greater importance on algebraic treatment–in the sense of Duval (2006)–through logical connectives, or the use of symbolic instruments, relegating solutions to those that come from processes associated with the perceptual or visual (Presmeg, 1986; Tall & Vinner, 1981; Vinner & Dreyfus, 1989).

Learning Difficulties in the Calculus Domain

For decades, many studies have indicated difficulties in learning in the calculus domain (e.g., Harel & Sowder, 2005; Rupnow & Randazzo, 2023; Scheiner & Pinto, 2019; Weber et al., 2020). According to various research (e.g., Artigue, 1998; Harel & Sowder, 2005; Tall & Vinner, 1981), initial learning in calculus involves both epistemological and cognitive challenges and often the previous ideas that students bring with them from school are not sufficient or, indeed, can lead to difficulties in the construction, definition, and use of calculus objects. Added to these issues, the teaching of this domain generally "leaves the necessary reorganization of conceptions to the private work of students" (Artigue, 1998, p. 47).

One of the main difficulties for students in calculus is the transition from an intuitive idea to a formal definition (Artigue, 1998; Vinner & Dreyfus, 1989). In this sense, it is possible that in the act of creating or utilizing a definition, the necessity exists to deploy a set of carefully connected processes consciously or unconsciously. Tall and Vinner (1981) refer to two types of definition: *concept images*, definitions that are specific to the individual, including images, examples, properties, and associated mental processes; and *concept definitions*, which relate to those definitions that have been institutionalized by the mathematical community (Harel, 2006). Regarding the same idea, Cornu (1981) uses the term *spontaneous conceptions* to refer to the conceptual images that the individual has acquired through their experience, including daily experience. In doing so, Cornu (1981)in question has problematized the disconnections that occur when solving a mathematical task.

One line of research that utilizes this distinction (between concept and definition) details how students often reason based their own concept, even if they know or have had contact with the definition (e.g., Dahl, 2017; Elia et al., 2016; Sierpinska, 2000; Vinner & Dreyfus, 1989; Zandieh & Rasmussen, 2010). Therefore, the capacity of the PT to extract or select a definition to address a certain problem, and how this definition is put into action, emerges as a topic of interest.

In the literature on the teaching or learning of continuous functions, difficulties are reported mainly in relation to the difference between the intuitive idea and the formal definition of the concept (Tall & Vinner, 1981). This can be observed in work involving graphs (Baker et al., 2002; Dahl, 2017) or in problems related to movement (Sokolowski, 2019). These phenomena are linked to historical epistemological aspects of the concept–in the intuitive definition of Newton and Leibniz, as well as that of Euler, the continuous function is described as a freely guided curve, implying movement, flow, and change over time (Núñez et al., 1999).

Likewise, other studies establish difficulties with aspects of propositional logic in the use of open intervals in the definition of continuous functions (Ko & Knuth, 2009; Messias & Brandemberg, 2015). As a consequence, this definition does not allow for the analysis of atypical cases (Shipman, 2012), which in turn influences the treatment of other concepts that depend on continuity (Rodríguez-Nieto et al., 2021). Recently, aspects of propositional logic have been associated with difficulties of a linguistic variety in the definitions of concepts (Schüler-Meyer, 2022). All of these studies indicate that the tasks proposed are in a formal area of mathematics where the functions appear explicit; it is not clear that these tasks could work in the school environment, that is to say, they correspond more to higher education.

According to Weber et al. (2020), courses on analysis should be specifically designed for PTs, and the activities therein should meet the needs of mathematics teachers. Weber et al. (2020) propose activities that address trigonometric functions and trigonometric equations using theorems on continuity, injectivity, and monotony, considering that in mathematics teaching, these objects are treated algebraically and graphically, but without connecting the two approaches.

Considering the above, proposing tasks in teacher education that connect university mathematics with school mathematics, in topics in the calculus domain, could contribute to the formulation of didactic transpositions that future teachers will utilize in their school classrooms.

School Context in Chile

In the Chilean school curriculum, the continuity of functions is seen in one elective course for eleventh and twelfth grades (students aged 16 and 17, primarily) called "limits, derivatives, and integrals" (Ministerio de Educación de Chile [MINEDUC], 2021). The property is presented as closely linked with the concept of limits. Specifically, the concept of the continuous function is developed within the learning objective "argumentation about the existence of limits of functions at infinity and at a point to determine convergence and continuity in contexts of mathematics, science, and daily life, both by hand and using digital technological tools" (MINEDUC, 2021, p. 99).

The curriculum for this subject states, as a foundational idea, the necessity to develop work that provides opportunities to visualize concepts, make conjectures, experiment, and utilize software to learn mathematics (MINEDUC, 2021). This document also promotes working with various types of representations, along with indicating that it is advisable to present a structuring of the domains (such as algebra, geometry, or infinitesimal calculus). It also states that, beyond having students finish the course knowing all of the content, what is expected is for them to develop mathematical thinking, since the subject entails

> "their first approach to the explicit calculation of situations that involve reasoning about infinity […] [and] symbolic calculation alone does not guarantee that they will improve their intuition in this sense, which is why the use of representations, diagrams, analogies, and metaphors is suggested" (p. 25).

In general, the curricular documents in Chile offer examples of tasks and recommendations for teachers to address continuity (MINEDUC, 2021). In this regard, the tasks made available for the subject in question are varied and require a deep understanding of the calculus domain. This all suggests the need for a comprehensive preparation for the mathematics teacher, in order for them to be able to lead their future students to pose these questions and find answers by undertaking various types of work, from the visual or intuitive to the formal, articulating domains and planning work with digital tools.

Based on the above, considering the difficulties associated with learning calculus and the necessity for study tasks that help connect university mathematics and school mathematics in teacher education, the objective of this study is to *characterize the mathematical work of PTs when solving a task designed to connect school mathematics with university mathematics, referring to continuous functions.*

THEORETICAL FRAMEWORK

To characterize the synergy of processes put into play in the mathematical activity of PTs and understand their dynamics, by means of an open task that can be solved in both a school and university context, the theory of mathematical working spaces (MWS) (Kuzniak et al., 2016, 2022) is used as a basis. This theory identifies the conjunction of three processes (referred to as *geneses* in the theory): *semiotic genesis*, *instrumental genesis*, and *discursive genesis*.

The theory establishes a differentiation of MWS's according to the mathematical domain in which the work is framed. In this study, we focus on the MWS in the calculus domain, notwithstanding the possibility that work in other domains could appear in the analyses.

Mathematical Working Spaces

The MWS theory takes two aspects into consideration: an epistemological aspect that considers the objects and the signs with which they are represented, the artefacts used, and the mathematical model put into play; and a cognitive aspect that refers to the individual who visualizes, utilizes, and constructs discourses with available objects and tools. Thus, there are two planes defined as the basis for the MWS: the first plane–the *epistemological plane*–is composed of three

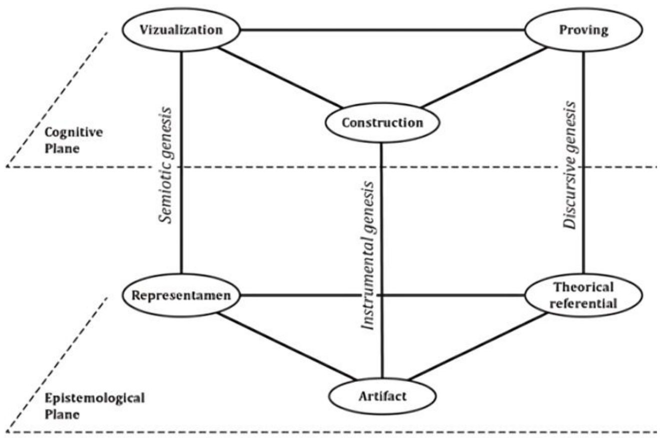

Figure 1. MWS diagram (Kuzniak et al., 2022, p. 61)

components: the *representamen*[1], the *artefacts*, and the *theoretical referential*. The second plane–the cognitive plane–is formed by the following components: *visualization*, *construction*, and *proof*.

The activation of a working space occurs by solving a *task*. We adopt the term *task* used by Kuzniak et al. (2022): "A 'mathematical task' refers to any type of mathematical exercise, question, or problem, with clearly formulated assumptions and questions, that is known to be solvable in a timely manner by students in a well-defined mathematical working space" (p. 8).

Within this context, the MWS is conceived of as an environment organized by and for the person solving mathematical tasks, an activity in which the cognitive and epistemological aspects are clearly inseparable units. The *geneses* manifest in the MWS when the epistemological and cognitive planes interact, each being articulated in turn based on the work being done. Thus, the *geneses* are defined as follows: *semiotic genesis*, which joins the representamen with visualization; *instrumental genesis*, which joins the artefact with construction; and *discursive genesis*, which joins the theoretical referential with proof (**Figure 1**).

In a mathematical activity, the geneses are not independent of each other, but rather are processes that influence one another mutually in an interrelation that can have varying degrees of closeness. A nutritive MWS in terms of the construction of knowledge or the confirmation that knowledge has been acquired is one in which all of the geneses converge, are joined with each other, and are each seen to have robust activity.

Semiotic Genesis

In the MWS theory, it is *semiotic genesis* that joins the *representamen* and *visualization* components. The *representamen* contains all class of signs and symbols belonging to a semiotic system (Duval, 2006). *Visualization* comes into play when an individual interprets such signs.

In terms of the activation of semiotic genesis in MWS, (Kuzniak & Richard, 2014) indicate that the process of visualization "can be considered as the process of structuring the information provided by diagrams and signs" (p. 4); *semiotic genesis* "gives meaning to the objects of the MWS and confers them their status as operational mathematical objects" (p. 5). Ultimately, different processes of treatment, conversion, or varied interpretations of signs indicate activations of semiotic genesis in MWS.

Instrumental Genesis

In MWS, *instrumental genesis* refers to the articulation between the *artefact*, from the epistemological plane, and the *construction*, from the cognitive plane, through an action. In MWS, a distinction is made between *material artefacts* (like a ruler or compass), *digital artefacts* (such as GeoGebra), and *symbolic artefacts* (e.g., an algorithm to solve an equation) (Flores Salazar et al., 2022).

Flores Salazar et al. (2022) give special attention to *digital artefacts*. A digital artefact for the teaching and learning of mathematics is defined as a set of propositions characterized by being executable by an electronic machine that possesses *historical intelligence* and *relative epistemological validity*. The former is related to the idea that artefacts are designed in an attempt to reproduce definitions, properties, and theorems that are the result of human constructions accumulated over the course of history. In contrast, *relative epistemological validity* is related to the impossibility of software representing all mathematical ideas or concepts in a totally faithful manner.

Discursive Genesis

Discursive genesis refers to the articulation between the *theoretical referential* elements, such as axioms, definitions, properties and theorems, from the epistemological plane, with the *proof*, from the cognitive plane. In MWS, we consider the *proof* from partial argumentations–as long as they consider elements of the theoretical referential–to a mathematical proof (Kuzniak et al., 2022).

Discursive genesis, as well as the other two MWS geneses, are not conceived in an independent manner; rather, in a mathematical activity, each genesis could be influenced by the other two. When it is not possible to distinguish which genesis is privileged, we refer to vertical planes: semiotic-instrumental [sem-ins], semiotic-discursive [sem-dis] and instrumental-discursive [ins-dis] (Coutat & Richard, 2011).

[1] We retain the original word (representamen) so as not to confuse this component with that defined by Duval (1995)–representative–in the records of semiotic representations, nor limit ourselves to the latter (which also has a cognitive component in its definition).

Mathematical Working Space Types

Lastly, the MWS theory is differentiated into three types: the *reference* MWS (Montoya-Delgadillo & Reyes Avendaño, 2022); the *idoine* MWS (Henríquez-Rivas et al., 2022); and the *personal* MWS (Menares Espinoza & Vivier, 2022). The present study focuses on the *personal* MWS of future mathematics teachers when solving a *task* proposed to help connect school and university mathematics. Likewise, a distinction is made between their planned work, termed *planned personal* MWS, and their *actual personal* MWS, alluding to the distinction made by Henríquez-Rivas et al. (2022).

METHODOLOGY

With the aim of characterizing the personal MWS of PTs when solving a task that is possible to develop in the calculus domain, a qualitative study is proposed (Flick, 2015). This is justified by the interest in analyzing and understanding how the mathematical work of the participants is carried out in order to solve a given task. Specifically, we undertook a collective instrumental case study (Stake, 2007), which considers as units of analysis the mathematical work and processes activated in the solving of a task (t) by PT from the *pedagogy in mathematics* undergraduate program at a public university in Chile.

Participants and Case Selection

PT correspond to six students who attended the module *profile 1 evaluation workshop*, a course in which aspects of mathematics and mathematics teaching that have failed in previous evaluations are covered. At the time of implementation, all of the PT students had passed the courses calculus I (where they initially cover contents including continuous functions, the intermediate value theorem (IVT), and Bolzano's theorem), calculus II, and calculus III. In the latter two courses, continuous functions are used as hypotheses to demonstrate other theorems. The application context was developed during class time, without prior preparation. The participants answered the proposed tasks voluntarily.

For the analysis, three instrumental cases (PT1, PT2, and PT3) were selected based on the fact that they are representative and revealing cases with respect to the mathematical content put into play and the processes activated (Yin, 2009). PT2 is representative of work that combines geometrics with the algebraic; PT3 represents a work of numerical approximations; and PT1 exhibits work closer to the calculus domain. These aspects, especially the representativeness regarding the use of certain content and the accessibility and clarity of the work displayed, supported the choice of cases.

Data Collection and Analysis

The techniques for data collection considered the written materials with the participants' productions when answering t, and the video recording and their respective transcriptions. For solving the task, the PT students had a sheet with instructions, construction paper with different-colored sides, a ruler, scissors, and the geometric software GeoGebra for those who opted to use it to answer.

In order to characterize the MWS of the PT participants according to the proposed task t and to specify the categories of the deductive analysis process, the study utilized the technique of content analysis (Leavy, 2014) for data reduction. The analyses were developed in two phases, according to the distinction proposed by Henríquez Rivas and Kuzniak (2021): *phase 1*, corresponding to the planned personal MWS, which refers to the study of the possible work of future teachers in relation to a given task; and *phase 2*, relating to the mathematical work on the activity produced by PT participants, corresponding to their *actual personal* MWS.

These two phases allow for a discussion regarding the gap between what is expected and the results, as well as the depth and richness of certain theoretical considerations, for questions of education and research that will be addressed in a subsequent section. For the analysis of the *planned personal* MWS and the *actual personal* MWS of the PT participants, elements of a methodology that has been used in MWS-based research were considered (e.g., Kuzniak & Nechache, 2021; Nechache & Gómez-Chacón, 2022) in order to describe the main actions taken when carrying out the task. The actions included the following:

1. Identification and description of work episodes (Es). The main Es of mathematical work were identified and described through a succession of mathematical actions carried out by the participant solving the problem.
2. Analysis of circulation. Based on the descriptions of the Es, each was analyzed and interpreted in terms of the circulation of MWS. These analyses included the use of theoretical categories related to the components, geneses, and vertical planes of MWS (**Table 1**).

Meanwhile, in relation to triangulation strategies (Denzin, 1978), triangulation of data was used based on the two-phased analyses described above. In addition, researcher triangulation was utilized, considering the experience and training of the authors (all of whom are PhDs in mathematics pedagogy specialized in the MWS framework with wide-ranging education and experience).

Presentation of the Task (t)

The proposed task can be developed using hypotheses related to continuous functions and the IVT,

Table 1. Theoretical categories for the analysis of the MWS of PT participants (adapted from Henríquez-Rivas & Verdugo-Hernández, 2023, p. 189)

Category	Components	Descriptor
Semiotic genesis	Representamen	Relates mathematical objects and their signifying elements and signs.
	Visualization	Interprets and relates mathematical objects according to cognitive activities linked with the register of semiotic representations (identification, treatments, and conversions) and symbols.
Instrumental genesis	Artefact	Utilizes material or symbolic artefacts, including algorithms or formulas, or technological artefacts.
	Construction	Based on the processes resulting from the actions triggered by the artefacts used and the associated usage techniques.
Discursive genesis	Referential	Utilizes definitions, properties, or theorems.
	Proof	Discursive reasoning is based on different forms of justification, argumentation, demonstrations, or types of proof.
Vertical plane and directed plane	[sem-ins]	Joins semiotic genesis with instrumental genesis in task-solving processes.
	[ins-dis]	Joins instrumental genesis with discursive genesis in task-solving processes.
	[sem-dis]	Joins semiotic genesis with discursive genesis in task-solving processes.

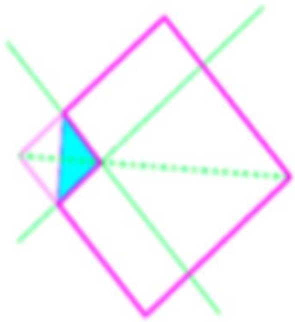

Figure 2. Task proposed to PTs (Source: Authors' own elaboration)

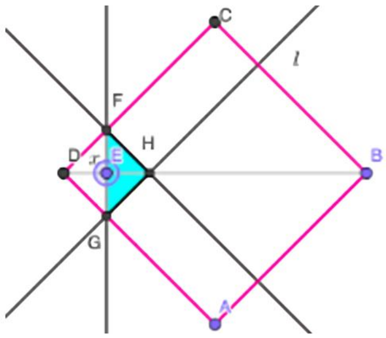

Figure 3. Geometric writing of the problem (Source: Authors' own elaboration)

Bolzano's theorem, or corollaries of these. Selection and adaptation of tasks was based on the degree of flexibility allowed by their possible solutions, which involved a diversity of mathematical knowledge and processes that could have been activated (Menares Espinoza & Vivier, 2022). In this context, this study is unique in presenting the possibility of connecting knowledge of school mathematics with university mathematics, and with the robust resources offered by the use of technological artefacts.

The task (t) was presented in a geometric context and had been adapted from a problem proposed by Carlson and Bloom (2005), which has been used and reformulated in different contexts (Kuzniak et al., 2011, 2013). Menares Espinoza (2016, 2019) proposes a new adaptation of t; the justification for this is based on the fact that the new formulation encourages answers in the calculus domain, in addition to the other resolution possibilities that t offers. In Menares Espinoza and Vivier (2022), a diversity of possible strategies is shown, as well as the problem's resolution by in-service teachers. We emphasize that the mathematical objects involved in the formulation of t are seen at the school level in Chile (MINEDUC, 2019), including squares, triangles, area of polygons, vertices, and diagonals.

Task (t)

A square of paper with different-colored sides is folded, forming a triangle whose vertex is on the same diagonal as that of the square. Is there a way to fold the paper so that the visible portions of the two different colors are equal in area? Explain your answer (**Figure 2**).

ANALYSIS OF THE TASK

To analyze the planned personal MWS, which responds to *phase 1* of this research, we considered five planned strategies. Some of these are reported in Menares Espinoza (2016, 2019) and Menares Espinoza and Vivier (2022), and others are original to this study, in particular those involving GeoGebra software.

Phase 1. Study of the Planned Personal MWS

Of the five planned strategies, the first three can be developed in two ways: in the first, the variable x is the height of the triangle FHG (segment DE); in the second, the variable x is a leg of the triangle FGH (segment FH) (**Figure 3**).

We present strategies 1, 2, and 3 utilizing the height of the triangle FGH as the independent variable. The procedures are analogous if we consider the segment *FH* as the independent variable.

Planned strategy 1. Analytical

This strategy entails five Es in the planned mathematical work.

E1. Identify the objects in the situation: This leads one to consider l on the side of the white square, then to consider two functions $f\colon \left[0, \frac{\sqrt{2}l}{2}\right] \to \left[0, \frac{l^2}{2}\right]$ and $g\colon \left[0, \frac{\sqrt{2}l}{2}\right] \to [0, l^2]$ that model the two areas of the different colors as the paper is folded. That is, f is the function that models the area of triangle FGH and g is the function that models the area of the polygon that joins the points A, B, C, F, H, and G. Thus, the functions are determined by $f(x) = x^2$ and $g(x) = l^2 - 2x^2$

E2. Consider the restrictions of the functions: The variable where both functions are evaluated corresponds to the height of the blue triangle. The area of the triangle is formed by folding over the diagonal and the remaining area are called the blue area and the white area, respectively.

E3. Identify the continuity of the functions on an interval: It should be noted that both functions are continuous on $\left[0, \frac{\sqrt{2}l}{2}\right]$, as they are polynomial functions.

E4. Analyze functions within the geometric context: Additionally, note that, when not folding anything: $f(0) = 0$ and $g(0) = l^2$, so $f(0) < g(0)$, and when the paper is completely folded: $f\left(\frac{\sqrt{2}l}{2}\right) = \frac{l^2}{2}$ and $g\left(\frac{\sqrt{2}l}{2}\right) = 0$, so $f\left(\frac{\sqrt{2}l}{2}\right) > g\left(\frac{\sqrt{2}l}{2}\right)$.

E5. Propose a conclusion based on analysis: Through the corollary of the IVT, there exists $x_0 \epsilon \left(0, \frac{\sqrt{2}l}{2}\right)$ such that $f(x_0) = g(x_0)$, that is, there exists a folding point at which the vertex is on the diagonal of the square, such that the two areas have the same value.

Planned strategy 2. Algebraic

This strategy considers three Es in the planned mathematical work.

E1. Set up an equation: An equation is formulated that makes the areas equal. With $A_T = x^2$ being the area of the blue triangle, and $A_R = l^2 - 2x^2$ the remaining white area, the following equation is established: $x^2 = l^2 - 2x^2$.

E2. Solve the equation: By performing *treatments* on the equation, the solutions obtained are $x_1 = \frac{\sqrt{3}l}{3}$ and $x_2 = -\frac{\sqrt{3}l}{3}$.

E3. Establish a solution: Considering the context of t, the negative value is discarded, so the solution is found when the height of the triangle is $\frac{\sqrt{3}l}{3}$.

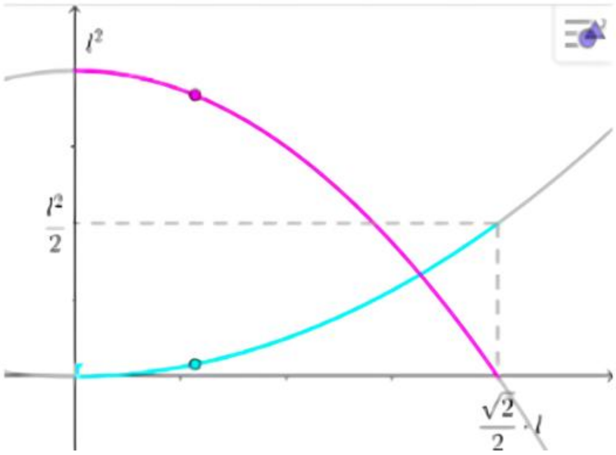

Figure 4. Graph of the functions associated with the functions from planned strategy 3 (Source: Authors' own elaboration)

Planned strategy 3. Dynamic graph

This strategy is centered on the use of GeoGebra and includes three Es.

E1. Establish the functions that are in play: Both functions that model the areas involved are formulated, as shown in planned strategy 1.

E2. Graph the functions: Both functions are graphed in GeoGebra using different colors (**Figure 4**), and the points of intersection with the x and y axes are established, along with their respective images, in terms of l.

E3. Draw a conclusion based on what is observed: In the graph in **Figure 4**, a point can be observed where the graphs of the functions intersect; therefore, there is a point where the functions are equal.

Planned strategy 4. Dynamic geometry: Independent variable is segment FH

This strategy once again entails the use of GeoGebra, but in a different manner than in planned strategy 3. Three Es are included in the development of this strategy.

E1. Construction of the figure: Using the GeoGebra tools *regular polygon* (to construct a square), *segment* (to construct the diagonal), *point* (to determine a point on the diagonal), and *line parallel to another that passes through a point*, the figure is built.

E2. Calculation of areas: In the GeoGebra sidebar, the Area command (Point, ..., Point) is used to determine the area of the triangle and remaining spaces.

E3. Use of slider to generate changes in the areas: The slider is used to move the point on the diagonal and observe the resulting changes in the areas. In this manner, one can determine whether there is a point where the areas coincide (**Figure 5**).

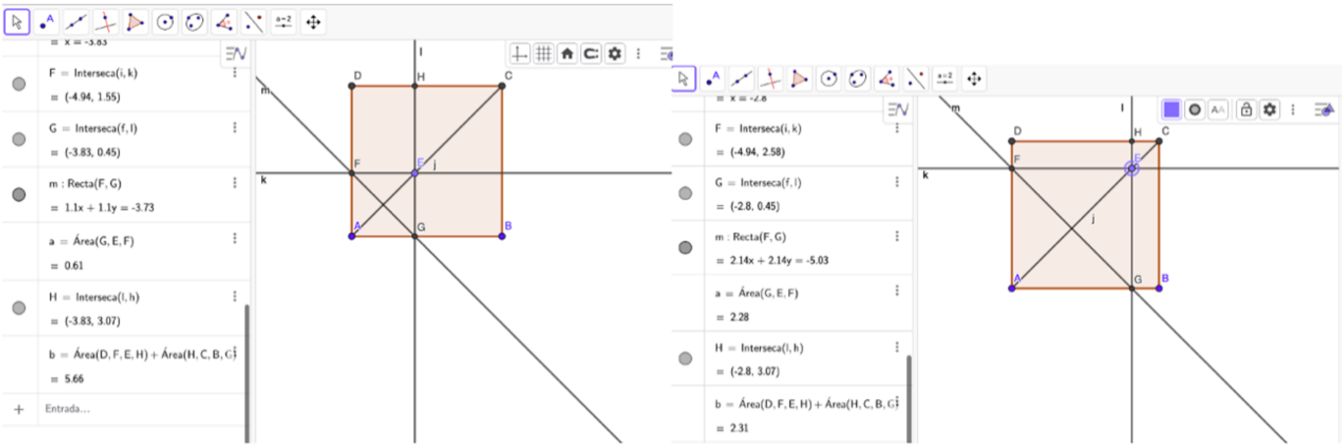

Figure 5. Strategy using GeoGebra (Source: Authors' own elaboration)

In this case, depending on the dimensions of the square, GeoGebra may not display a point where the areas are the same (it may differ by some decimal places), and if so, the student might respond that the point where the areas are equal does not exist.

Planned strategy 5. Analytical and graphical: Independent variable is the segment FH

This strategy involves three Es in the planned mathematical work.

E1. Determination of extreme points: Any two functions (f and g) are evaluated at the extreme points, according to the advancement of the diagonal of the square. It should be noted that in this strategy, functions are not explicitly established: $f(0) = 0$, $f(\sqrt{2}l) = \frac{l^2}{2}$, $g(0) = l^2$, $g(\sqrt{2}l) = 0$.

E2. Graphing the points and plotting the functions: With pencil and paper, the extreme points are graphed. Two functions are plotted freehand without following a given formula. In order to do this, it is important that the paper remains in place to avoid lifting the pencil. This step involves visual perception and is based on senses of physical movement in space.

E3. Establishing a conclusion: It is concluded that since the functions can be drawn without lifting the pencil, the graphs must intersect at a point, and therefore, there is a folding point at which the areas will be equal.

Circulations in the planned mathematical work

Considering the planned strategies presented, it can be said that the task in question generates a nutritive scenario for the development of the personal MWS, as it allows the problem to be solved using various tools, representations, and discursive reasoning, which is a first indicator of the activation and articulation of the MWS geneses.

In a more specific analysis, in the planned MWS for strategy 1 and strategy 2, we identify, firstly, an activation of the semiotic and instrumental geneses–the semiotic when moving from the figure to the algebraic statement of functions and equations; and the instrumental when the functions and equations are used as symbolic artefacts for the development of the task. Subsequently, planned strategy 1 displays a strong activation of discursive genesis, with the hypothesis of continuity, values of the functions at the extremes of the intervals, and the resulting use of a theorem from the analysis domain. The work for strategy 1 represents demonstration. Planned strategy 2, on the other hand, puts emphasis on instrumental genesis, solving the equation and establishing a conclusion based on this.

Planned strategy 3 and strategy 4 involve the use of GeoGebra as a technological artefact. However, the outlook for the mathematical work is conspicuously different. In planned strategy 3, the technological artefact is used to graph functions previously established algebraically, which implies semiotic-instrumental work. Once the functions have been graphed, the point of intersection can be found, which allows conclusions to be drawn and, thusly, activation of semiotic and discursive geneses. In planned strategy 4, on the other hand, GeoGebra is used as a sign and as a medium. By moving the points along the geometric figure, the point representing co-variation is moved. This shows the form of the function without knowing its algebraic expression a priori, activating the semiotic-instrumental plane, but in a different way than in planned strategy 3.

Finally, in planned strategy 5, primarily semiotic work is exhibited since visualization is predominant. This is combined with discursive genesis when notions of continuity are utilized, although the arguments come from experience and a physical sense of the problem, not necessarily from the theoretical referential. Here, the connected semiotic and discursive geneses can be identified; therefore, there is an activation of the [sem-dis] vertical plane.

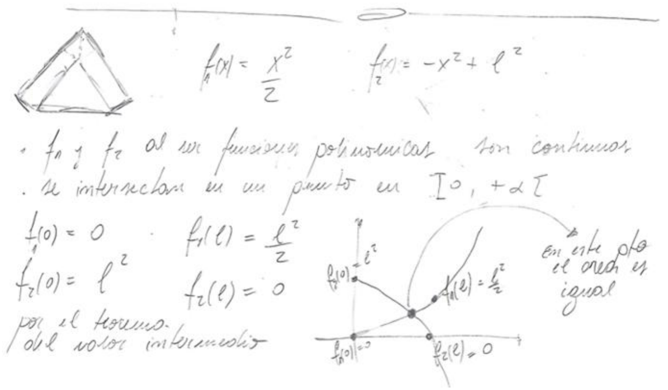

Figure 6. Excerpt of the work of PT1 (Source: Authors' own elaboration)

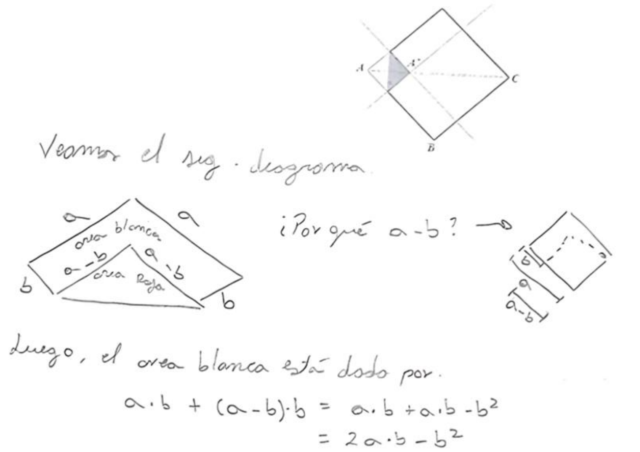

Figure 7. Excerpt 1 of the work of PT2 (Source: Authors' own elaboration)

RESULTS

Below are the results of the implementation of the task (t) with the PTs.

Phase 2: Personal MWS of Pre-Service Teachers

As indicated in the methodology section, we selected the work of three cases (PT1, PT2, and PT3).

Case 1 (PT1)

The strategy developed by PT1 is similar to the second way of approaching strategy 1, that of the planned analysis. PT1 begins by making a sketch, but it does not indicate which variables to consider (**Figure 6**). It can be inferred that they consider x to the be sides of the triangle, and l to be the sides of the initial square. Thus, they formulate two functions: $f_1(x) = \frac{x^2}{2}$, which we assume is the area of the triangle, and $f_2(x) = -x^2 + l^2$, which would represent the remaining area.

They then state the following:

- f_1 and f_2, being polynomial functions, are continuous.
- They intersect at a point on $[0, +\infty[$.

Then, PT1 evaluates for 0 and for l, resulting in the following: $f_1(0) = 0$, $f_1(l) = \frac{l^2}{2}$ and $f_2(0) = l^2$, $f_2(l) = 0$.

They then write the phrase, "through the IVT," which we interpret as meaning that with this theorem, it can be determined that there is a fold that makes the areas equal. This is supported by a graph that they construct. It is not clear if they construct the graph first and then draw this conclusion or vice versa. In any case, the graph can offer support to obtain the solution.

From the MWS perspective, an articulation of the three MWS geneses can be identified in this work. First, there is activation of instrumental genesis when the functions are formulated, acting as symbolic artefacts. From here, work is carried out using hypotheses (continuous functions and values of the function at key points), which demonstrates the activation of discursive genesis. However, the hypothesis of the inequalities of the values of the functions is not expressed, and it is initially established that the functions intersect at a point on $[0, +\infty[$, which is not false, but does lie outside the interval in question. Thus far, the work has a demonstrative character, but with certain inaccuracies, such as the mixing of hypotheses with theses and the absence of certain hypotheses.

In order to use the IVT, the hypothesis of inequality is represented in the graph, which shows coordination with semiotic genesis, but weakness in discursive genesis, given that the character of the demonstration is changed to a more pragmatic argument. Additionally, PT1 indicates the point in the graph where "*the area is equal*," which indicates a strong visualization component.

In conclusion, in the work exhibited by PT1, the activation of instrumental genesis is identified initially, and later the [sem-dis] plane, since the semiotic and instrumental geneses appear to be coordinated.

Case 2 (PT2)

PT2 names the variables and makes them depend on each other. To explain this dependence, they utilize a reconfiguration (**Figure 7**).

Then, they consider a as the side of the initial square and b as the part of the side of the square that remains after folding the paper. This configuration indicates work in semiotic genesis, as there is a strong visualization component. They calculate the white area considering the areas of two rectangles formed when folding the paper.

For the area of the other color ("red triangle"), the same letters (a and b) are maintained. PT2 indicates that "we must obtain its base and height," alluding to the calculation of the area of the triangle (**Figure 8**).

PT2 then indicates that they examine the diagram and redraw it, which shows support in the figure for visualizing the sides of the triangle.

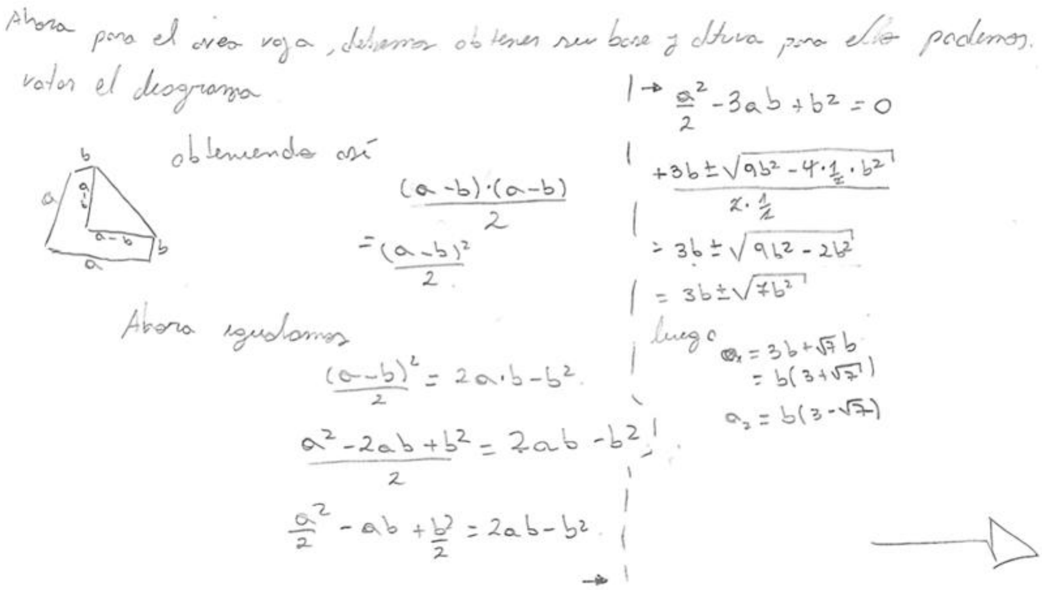

Figure 8. Excerpt 2 of the work of PT2 (Source: Authors' own elaboration)

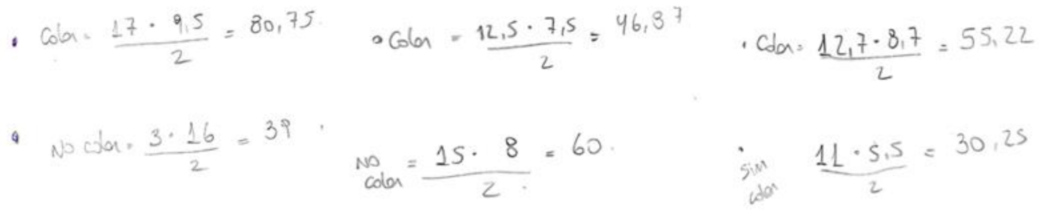

Figure 9. Excerpts from the work of PT3 (Source: Authors' own elaboration)

To obtain both areas, they utilize the formulas for the areas of rectangles and triangles. Then they make both areas equal, expressed with a and b, and consider a as the unknown they want to find. This indicates a problem in the theoretical referential, because according to PT2's configuration, a denotes a side of the initial square, which has a fixed length.

PT2 solves the equation using a formula for finding solutions to quadratic equations, so we can identify the use of a symbolic artefact. Ultimately, they obtain two possible values for a that depend on b, but they do not interpret the results, nor do they question the origin of a as the side of the initial square.

In PT2's work, we can identify activation of primarily semiotic and instrumental geneses. Semiotic genesis is identified by the initial treatment that they carry out on the figure to determine the unknowns that they are going to utilize. In addition, there is strong support in the figures for formulating the equations.

Throughout the work, fixed values are considered; therefore, this is presented as a static work. The rest of the work is mainly algebraic, with the use of symbolic artefacts. In this case, there is a problem with the theoretical referential because once the equation is formulated, it is solved considering the side of the initial square–a known value–as an unknown. The lack of questioning by PT2, along with the lack of interpretation of the results, indicates that discursive genesis is either not present in the work or only scarcely present.

Case 3 (PT3)

We consider the strategy of PT3 to be trial and error, and it does not fit into any planned strategy. The work of PT3 consists of making different folds in the paper and measuring (probably with a ruler) the sides to calculate the areas referred to as "color" and "non-color." In this manner, three pairs of areas are calculated (**Figure 9**).

Based on these values, three conclusions are drawn that are linked to one another:

1. "The non-color one is always smaller than the color one. But when you make the color triangle smaller, the non-color one is bigger."
2. "It depends on what fold is made."
3. "One gets bigger and the other gets smaller, so they are not the same size (at any time)."

This allows us to identify a work in which a dynamic is present on paper (the areas get smaller and larger). In addition, a dependence between the areas is expressed (**Figure 10**).

No variables are established, and only particular cases are tested; from this we conclude that, in Balacheff's terminology, it would correspond to a proof of the naïve empiricism type (1987). The notion of

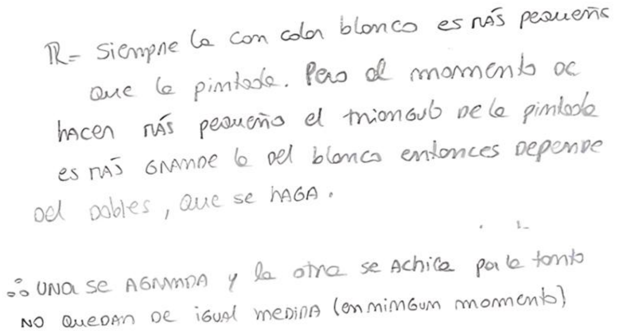

Figure 10. Excerpt from the conclusions of the work of PT3 (Source: Authors' own elaboration)

continuity is not present, and the absence of this property leads to an erroneous response.

The presence of the three geneses can be noted, but they are disjointed:

- Semiotic genesis is present when representing the areas numerically and visualizing the results.
- Instrumental genesis is present in the probable use of a ruler to measure the areas, the use of the paper, and the use of area formulas.
- Discursive genesis is present in the conclusions that PT3 obtains, although the theoretical referential is very weak and the conclusions are erroneous.

The varied strategies for developing the task, the different circulations in the MWS that are developed, and the domains involved point to the range of possibilities offered by the task to be addressed in initial teacher training and school contexts. This can be done with the purpose of working in a visual manner, utilizing technological artefacts, or working with the hypotheses of the IVT in a more formal way, among others, which will be discussed in the final section of this article.

DISCUSSION AND CONCLUSIONS

This study aims to characterize the mathematical work of PTs when solving a task designed to connect school mathematics with university mathematics, specifically focusing on continuous functions. In order to do this, a task was proposed, with the different ways in which it could be approached having been previously analyzed. We analyze three cases that show us the variety of possible solutions, but additionally that they are developed in different domains.

In terms of the PTs' solutions, the last case remains in the geometric domain, but its work is likewise mainly carried out through direct measurement; in the second case, we see a work that is mainly in the algebraic domain, focused on solving equations; only the first case is in the domain of functions and particularly that of continuous functions, with explicit use of some hypotheses of the IVT.

In this work, all of the MWS geneses have been identified; however, the final case presents problems with the relevant hypotheses. Meanwhile, the other two cases are more restricted to the elements of the theoretical referential to which each can resort.

Among the solutions provided, none makes use of dynamic software, which is notable considering that our society is experiencing increased technologization every day (Gaona & Menares, 2021).

Proposal for Work in Initial Teacher Training

Previous research indicates the existence of a decoupling of the teaching of calculus in teacher education and what is required to carry out work in the school (e.g., Weber et al., 2020), this in addition to the complexity of working with calculus objects in the university. According to Selden and Selden (1995), it is necessary for definitions to become operational for students, and it is desirable that they are not left with only intuitive notions.

According to our analysis, the task presented in this article allows mathematical work to be undertaken in different stages in which an evolution can take place from an intuitive notion to the use of IVT hypotheses to construct a formal mathematics that makes sense and is joined with previous intuitive and visual work. In this sense, the task proposed favors the idea of a *complete* MWS (Kuzniak et al., 2016), which can be used and adapted in different contexts and education levels so that different geneses and components of MWS are efficiently activated.

The latter point also offers a response to the requirements of the Chilean national curriculum, which establishes that work with objects of limits and continuity should contain a wealth of strategies, including visualization, the formation of conjectures and their validation, and the use of software that supports work by hand. Thus, working with the task in question can have a double role: developing the construction of formal knowledge about continuous functions and the IVT, and supporting approaches to school mathematics.

Menares Espinoza and Vivier (2022) report on the scarcity of dynamic responses presented when 13 PTs solved the task in question. In response to this, as a proposal, we suggest approaching the task first with work done by hand using paper, in which conjectures about the solution to the problem appear. Then, one can advance modeling the task with dynamic geometry software. University students will be able to set the variables and, in doing so, formulate the functions involved. It may be important that the applet generated shows the corresponding points on the graph of the function as the vertex of the triangle is moved, as shown in **Figure 11**.

In this manner, what is happening on the graph can be formally translated, granting importance to the fact

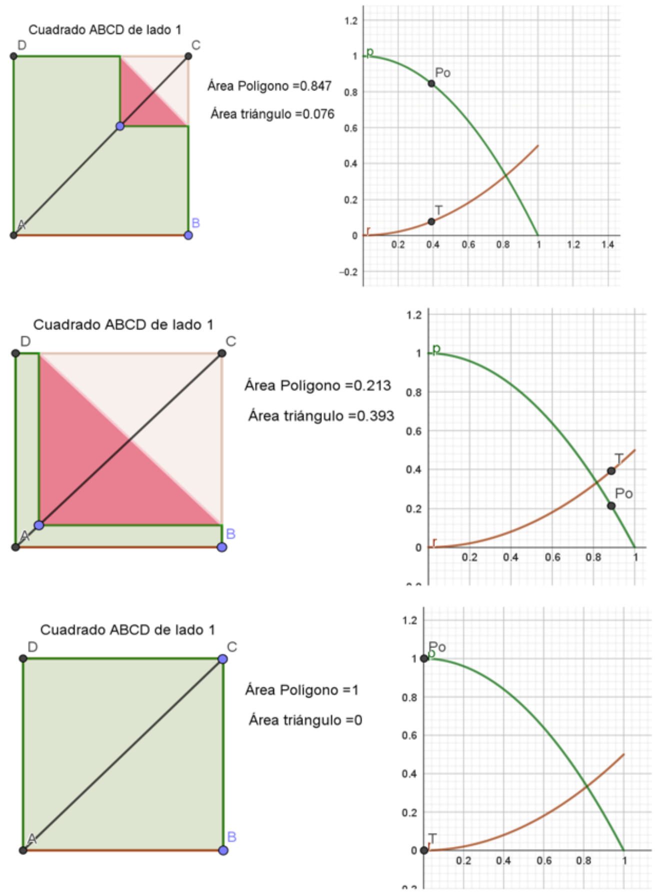

Figure 11. Applet showing the movement of the paper on the left, and the corresponding points on the graph of the functions on the right (Source: Authors' own elaboration)

that the function is continuous on the interval, and making the inequalities in the images of the functions at the extreme points of the interval explicit. Through this, the IVT hypotheses are obtained to conclude with the fact that the required point where the areas are equal must exist.

Proposal for Work in the School

According to school plans of study in Chile, an important objective is for students, beyond learning content, to develop mathematical thinking, understanding mathematics as being articulated among different domains (MINEDUC, 2021). This is especially important when working with calculus objects since students must connect what they know–which is mainly situated in geometric and algebraic contexts–with new objects that involve the idea of the infinitesimal.

In the case of working on the task at the school level, one strategy could be, once again, to begin by establishing conjectures through work on paper; students will be able to decide if a given point exists by simply making slides and measurements, as we observed in the case of PT3. Then the applet, which has been previously designed, can be provided, allowing the modeling of the situation on paper and the graphing of the functions in parallel. Students can discuss which variables are at play, establishing relationships between the figure and the graph, and thus make conversions from the figural register to the graphic register, which implies important semiotic work.

After this, students can make slightly more formal inferences and establish conclusions regarding the existence of the point where the areas are equal, encountering concepts of continuity, for example, by recognizing, "*there are no jumps in the graph*," and the inequality of the images of the functions, for example, by stating that "*it is necessary to cross the graph of one of the functions to go from one point to another*." This can prompt discursive work that can be strengthened progressively. It remains up to the teacher to formalize each of the hypotheses and arrive at the desired mathematical concepts.

Directed Vertical Planes Approach

Based on the analysis of the circulations, we can observe the prevalence of one genesis in each vertical plane of the MWS–not only as the one that takes on a greater role in the work, but as the one that motivates greater work in the other, as also noted by Menares Espinoza (2019) and Menares Espinoza and Vivier (2022). This allows us to differentiate two vertical planes that activate the same geneses, but in which the work is directed primarily by one of them.

Indeed, in the work of PT1, an articulation can be observed between semiotic and discursive geneses, and that semiotic processes are activated in order to provide the theoretical referential with elements (hypotheses and theses of the IVT) and thus enable the construction of a proof. Discursive genesis can be identified as the director of the personal MWS of PT1, which indicates the activation of a directed vertical plane, which we denote as [sem-**Dis***].

In the MWS of PT2, on the other hand, the semiotic processes that are carried out–with the reconfiguration of the figures–have the purpose of formulating and solving an equation. Thus, we identify instrumental genesis as the genesis directing the work, so there is once again a directed plane activated, which in this case we denote as [sem-**Ins***].

A clearer difference can be found in planned strategies 3 and 4, in which the activation of the [sem-ins] plane is recognized. However, in strategy 3, it is semiotic genesis that has the directing role, since the digital artefact is utilized to represent a function graphically that had previously been formulated algebraically. Meanwhile, in strategy 4, when working in GeoGebra, it is possible to visualize what is happening in the graph without having an algebraic formula for the function. Thus, we posit that in planned strategies 3 and 4, the directed planes activated are [**Sem***-ins] and [sem-**Ins***], respectively.

Lastly, based on the results presented, one of the limitations of this study is related to the number of tasks that have been considered, since the analyses are shown in the context of one specific task. However, in order to analyze mathematical work in depth, the decision was made to exhibit and analyze the personal MWS of future teachers for the single task proposed. Future work could consider other types of tasks in the calculus domain (or another domain) that connect school mathematics with university mathematics. Likewise, the task of this study could be further used or adapted for teacher education; in particular, it could be used in student-teaching courses in which future teachers take on the role of classroom teachers.

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Ethical statement: The authors stated that the study involved human participants (pre-service mathematics teachers) who voluntarily agreed to take part. Informed consent was obtained from all participants, and anonymity and confidentiality were guaranteed. The study was conducted in accordance with the ethical guidelines of the Universidad de Valparaíso and approved by the Comité de Ética Científica de la Universidad de Valparaíso on April 30, 2024 (Approval code: CEC-UV 293-24). Written informed consents were obtained from the participants.

AI statement: The authors stated that no generative AI or AI-based tools were used in the writing, analysis, or preparation of this manuscript.

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