

Preservice Teachers' Argumentation and Some Relationships to Didactic-Mathematical Knowledge Features

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Abstract

This paper presents research on the argumentation that preservice teachers perform when designing and teaching geometry. Argumentation is associated with speech acts carrying didactical intentions. This research study features of preservice teachers' argumentation when explaining geometry tasks both to peers, during preparation and discussion of designed activities, and to students in the classroom. This is qualitative research and the results support establishing relationships between the didactical dimension of the didactic-mathematical knowledge model and some characteristics of the argumentation that preservice mathematics teachers exhibit during their planning and teaching.

Keywords: argumentation, teacher knowledge, logic, rhetoric, dialectic

INTRODUCTION

Several authors explore the argumentation, not only as an activity performed by teachers during their classes but also as a factor in student education in science (Duschl & Osborne, 2002; Erduran, 2007; Jimenez-Alexandre & Erduran, 2007; Soysal, 2015). In Mathematics Education, several papers have explored argumentation and its relationships with proof (Boero et al., 2010; Bussi et al., 2006; Campbell et al., 2020; Miyakawa, 2017; Pedemonte, 2007; Shinno et al., 2018; Turiano & Boero, 2019).

The complexity of the teacher verbal and didactical activity (Muller et al., 2009) do not allow 'only' considering the deduction rules of Aristotelian logic but requires the use of 'persuasion' (Manghi, 2010; Perelman, 1997) and dialogue (Duschl & Osborne, 2002; Muller et al., 2009) for it to be studied. The mathematics classes are based, mostly, on verbalization of instructions; thus, the teacher uses, either consciously or unconsciously, argumentation theory resources. The Toulmin' model (1958) has been used for the analysis of arguments in mathematics education as it deals with logic-substantive attributes of the arguments of both preservice teachers (Arzarello & Sabena, 2011; Erkek & Bostan, 2019) and elementary schoolers (Cervantes-Barraza et al., 2019; Douek & Scali, 2000; Goizueta &

Planas, 2013); warrants (Rumsey et al., 2019; Tristani et al., 2015). Even though the Toulmin' model (1958) has been widely used to study the structure of arguments, many studies report its limitations (Conner et al., 2014; Harada, 2009; Metaxas et al., 2016; Molina et al., 2019; Nielsen, 2011; Simpson, 2015) to study the dialogical and dialectical elements of verbal interactions that take place in the classroom.

Due to the complex nature of the argumentation process for both students and teachers, in this study we have set ourselves the objective of characterizing the arguments of in service teachers (based on their dialectical and rhetorical attributes) when they design and teach Euclidean geometry. The structure of the argument refers to its components: data, conclusions, guarantees, supports, modal qualifiers, and rebuttals. In order to do this, we use the Toulmin's model because it helps us to understand the argument's structure. However, to understand the characteristics of the arguments we must go beyond the structure (Harada, 2009; Nielsen, 2011), for which reason we complement Toulmin's model with the inclusion of rhetorical (Perelman, 1997) and dialectical (van Eemeren et al., 2006) attributes of the argumentation.

Contribution to the literature

- The paper proposes a relationship between preservice teacher didactic-mathematic knowledge and logical, rhetorical, and dialectical qualities.
- The paper uses elements of argumentation to study the teaching of mathematical concepts and uses them to understand the complexity of teaching mathematics.
- The paper proposes to use the DMK model of teacher knowledge to study geometry teaching activities.

In this paper the complexity of argumentation is manifested by a process "...that includes all the assumptions (initial data and warrants) of the full argumentation, but which hides the relationships between these assumptions." (Knipping & Reid, 2015, p. 90). In our case more features are considered (data, warrants, claim, and rhetorical resources) than supposes paying attention to more variables and nuances in the argumentation process.

The results of our study show evidence that the proper use of the structures and attributes of the arguments promote suitable didactic-mathematical practices by future teachers, which impacts the generation of learning for their future students.

THE ROLE OF THE ARGUMENTATION IN THE INSTRUCTIONAL PROCESSES

When argumentation is studied from a logical perspective, the subject which arguments, is not considered, while rhetoric and dialectic consider a communicative context where a sender and a receiver, who is part of an audience, take on protagonist or antagonism roles. The classroom is an environment where argumentation, guided by deductive logical rules, is not enough to reach neither the students' participation nor the learning of mathematics (Crespo et al., 2010), other resources must certainly be used. Argumentation in the classroom is an activity that can be performed both in oral and in written form, that is to say, argumentation is a social activity where teacher addresses students with didactical intentions. Johnson (2000, p. 12), defines argumentation as "the socio-cultural activity of constructing, presenting, interpreting, criticizing and revising arguments". He considers that proper work has to be done towards a better theory of argument in order to have a balanced theory of argumentation.

Duschl and Osborne (2002, p. 41), defines 'argumentation' as a "social and collaborative process necessary to solve problems and advance knowledge", through acts of communication, where teachers request and offer arguments (Habermas, 1999) based on questions and answers to both their peers and students. In this paper argumentation is defined as a social, collaborative, rational and verbal process where a subject addresses another; it is social as it is used in the classroom by students and teachers alike to discuss mathematics; it is collaborative because an agreement is

reached with the help of students and teachers, it is rational as it aims to present or to defend a point of view, so a critic accepts it with a reasonable attitude (van Eemeren et al., 2006), and verbal as it is deployed either in oral or written form (van Eemeren et al., 2006). Argumentation is well recognized in science teaching as a competency that should be promoted in students, but argumentation is limited to proof in mathematics education. Argumentation as knowledge and competency is disconnected from teacher knowledge and from its use in the classroom, so the objective is to find a link between features of argumentation and teacher knowledge.

Teachers, in general, may feel unprepared to teach argumentation (Gabel & Dreyfus, 2013; Reid & Zack, 2011). Teachers and students alike face similar difficulties in dealing with argumentation. One of the most challenging goals for mathematics teachers refers to helping students in the development of argumentation "[...] to design means to support teachers in developing forms of classroom mathematics practice that foster mathematics as reasoning and that can be carried out successfully on large scale" (Yackel & Hanna, 2003, p. 234).

In science education, many researchers have reported positive results on developing conceptual learning and skills related to argumentation (Jiménez-Aleixandre & Pereiro Muñoz, 2002; Mendonça & Justi, 2013; Sampson et al., 2013). The literature reports evidence of teachers' importance in promoting students' engagement in argumentation and developing their knowledge related to this practice in the classroom. McNeill and Pimentel (2010) report that there is a relationship between teachers' practice working with instructional material explicitly designed towards argumentative practice and the development of students' writing and argumentation. The latter implies that teachers need help to gain experience or knowledge about the teaching of argumentation and, to help teachers, it is necessary to investigate the use they make of argumentation while teaching mathematics. According to Fielding-Wells (2014, p. 28), "...if argumentation as a pedagogical practice has demonstrated potential for deepening discipline-specific understandings in science education, a sister' science with mathematics, it is a worthwhile endeavor to consider its potential for similar affordances in mathematics education." There is no much research in mathematics educations dealing with the teacher

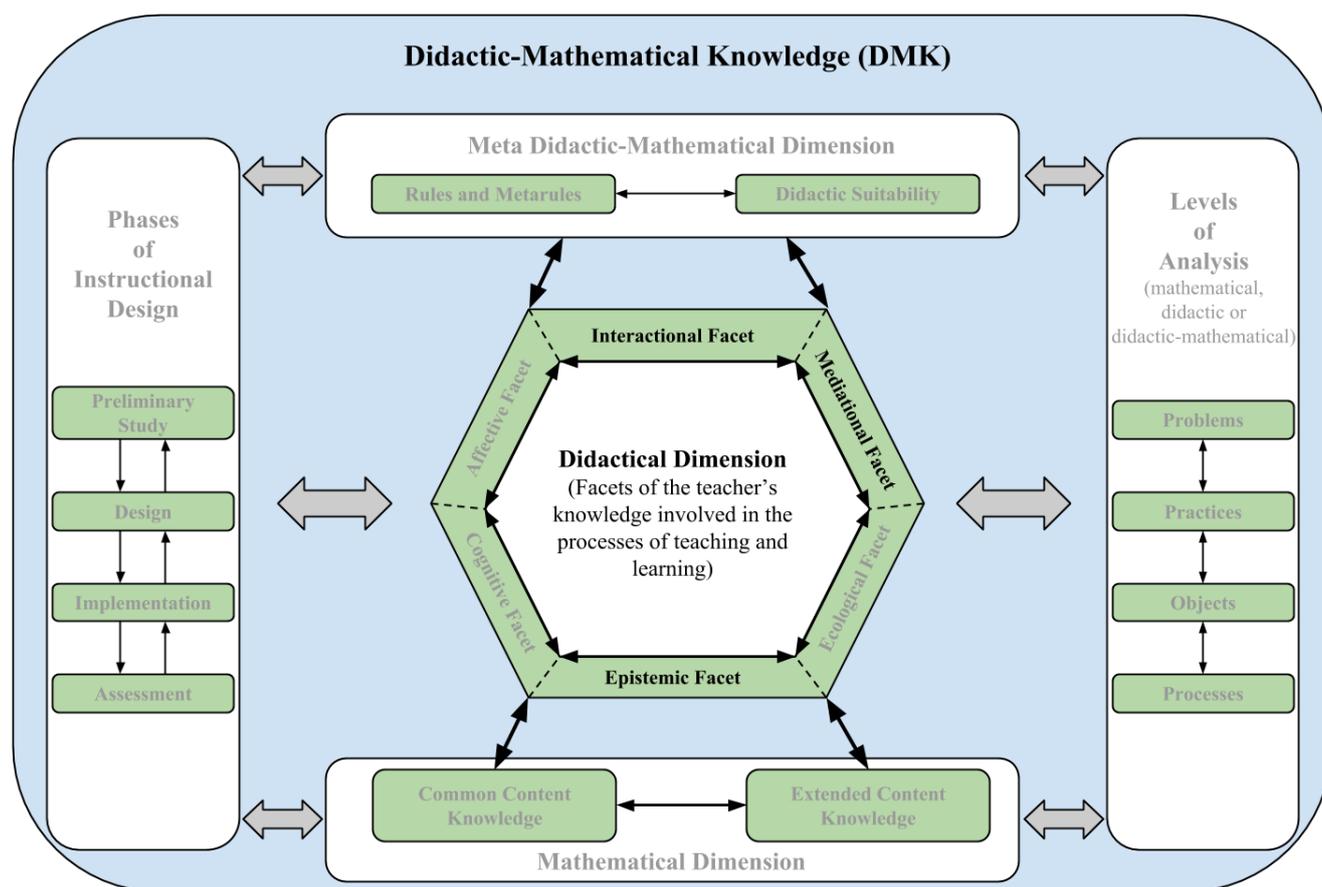


Figure 1. Dimensions and components of Didactic-Mathematical Knowledge (Pino-Fan, et al., 2015, p. 1433)

preparedness to teach mathematics while stressing the benefits of argumentation.

In this paper, we study mathematics pre-service teacher argumentation features that includes not only distinctive elements of mathematics (logical-deductive argumentation) but also elements from argumentation theory (Harada, 2009). We assume a pragmatic perspective of argumentation and study argumentation indicators, rhetorical resources and argumentative features identified in the argumentations of pre-service teachers while teaching. The research question deal with the link between argumentation features in our chosen argumentation definition and some characteristics of a model for the mathematics' teacher knowledge.

TEACHERS' DIDACTIC-MATHEMATICAL KNOWLEDGE

The international research has led to conceptualizations (and models) on teachers' knowledge to teach mathematics. However, even though some (Hill et al., 2008; Rowland et al., 2005; Schoenfeld & Kilpatrick, 2008; Shulman, 1986), none offer tools to analyze the didactic knowledge. The Didactic-Mathematical Knowledge Model (DMK) (Pino-Fan, Assis, & Castro, 2015; Pino-Fan, Godino, & Font, 2016) offers epistemic, interactional features mediational facets allow studying the didactical dimension for teaching. The features can

be linked to comprehending the paper of argumentation in mathematics teaching.

The Didactic-Mathematical Knowledge model (Pino-Fan, Godino, & Font, 2016) interprets and characterizes the teacher's knowledge from three wide dimensions: mathematical, didactical and meta didactic-mathematical (Figure 1). This model has been widely addressed in Pino-Fan, Assis, and Castro (2015). In this paper, we only explore epistemic, interactional and mediational facets. According to Pino-Fan, Assis, and Castro (2015, p. 1434-1436):

- *Epistemic facet* refers to specialized knowledge of the mathematical dimension. The teacher, apart from the mathematics that allow him solving problems which require him mobilize his common and extended knowledge, must have a certain amount of mathematical knowledge 'shaped' for teaching; that is to say, the teacher must be able to mobilize several representations of a mathematical object, to solve a task through different procedures, to link mathematical objects with other mathematical objects taught at a certain educational level or from previous or upcoming levels, to comprehend and mobilize the diversity of partial meanings for a single mathematical object, to provide several justifications and argumentations, and to identify the knowledge at

play during the process of solving a mathematical task.

- *Interactional facet* refers to the knowledge of the interactions that occur within a classroom. This subcategory involves the required knowledge to foresee, implement and evaluate sequences of interaction, among the agents that participate of the process of teaching and learning, oriented towards the fixation and negotiation of meanings (learning) of students. These interactions do not only occur between the teacher and the students (teacher-student), but also can occur between students (student-student), student-resources, and teacher-resources-students.
- *Mediational facet* refers to the knowledge of resources and means that might foster the students' learning process. It deals with the knowledge that a teacher should have to assess the pertinence of the use of materials and technological resources to foster the learning of a specific mathematical object, and also the assigning of time for the diverse learning actions and processes.

The three facets are highlighted in Figure 1, and they are used in this study because we want to investigate the integration the teacher does among features related to mathematical wealth (representations, concepts, definitions, properties, procedures), the wealth of interactions that occur in the classroom and the resources and means that it uses to manage the classes, and how this integration uses it to carry out argumentation processes with suitable use of dialectical and rhetorical elements. This would allow teachers to provide arguments adjusted to the educational level in which they work, without detracting from their mathematical wealth. The model takes into consideration (Pino-Fan et al., 2016): 1) the contribution and development of the theoretical framework known as Onto-Semiotic Approach (OSA) to cognition and mathematical instruction, which has been developed in several research studies by Godino et al. (Godino & Batanero, 1994; Godino et al., 2007); 2) the development and contribution of Godino's research (2009) where the foundations and basis of DMK are presented; 3) the findings and contribution of the several models that currently exist in Mathematics Education Research – Shulman (1986); Grossman (1990); Ball et al. (2008); Hill et al. (2008); Schoenfeld and Kilpatrick (2008); Rowland et al. (2005) –; and 4) the results obtained in several empiric studies (Pino-Fan et al., 2011, 2013).

CONTEXT OF THE STUDY AND DATA ANALYSIS

The research was conducted in the School of Education, University of Antioquia, during pre-service teachers teaching internship. During it, they design

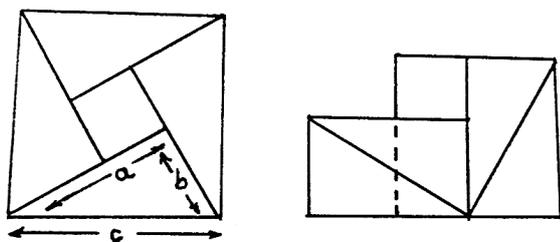
classes and teach elementary schoolers. This research considers two scenarios, of one-semester duration each: one for class design and discussion, and another one for teaching. Pre-service teachers chose to design and to present classes on Euclidian geometry in elementary school. They had attended Euclidian geometry and geometry methods courses. In Colombia, education is ruled by curriculum guidelines issued by the Ministry of Education that acknowledge features related to school geometry (MEN, 1998).

Three pre-service teachers participated in this research, by the pseudonyms Carlos, Helena, and Maria, who enrolled in the Seminar during a year and a half. Two pre-service teachers attended the course, but did not enroll; they will be called Peer 1 and Peer 2, their argumentations were not analyzed, but they participated during the teaching planning sessions. In what follows, we offer evidence of features of pre-service teachers' argumentations identified, either in the seminar scenario or in the classroom scenario. All the video and audio segments were transcribed and analyzed by the three authors, using progressive coding, which helped determine the segments analyzed in detail. The segments were analyzed using the theoretical framework composed by Toulmin's proposal, the facets of mathematical didactic knowledge, and rhetoric elements such as metaphors, models, and examples. With these elements and with an analysis rubric, the arguments were reconstructed.

ARGUMENTATIVE STRUCTURE: DATA, WARRANTS AND CLAIM

Toulmin (1958), proposes that an argument is based on three components: data, warrants, and claim. The *claim*, refers to a conclusion or to a point view expressed by someone, and the data support assertion when it is challenged, and the warrant presents the incidence of the data in the assertion when it is challenged how the data can be connected (or support) the assertion. So, the warrant can be expressed by a rule that acts as a bridge between the data and the claim; in other words, the warrant is the transition from data to claim. Warrants are used not only to guarantee that the relationship (implication) is valid but also are to be taken as knowledge about definitions, theorems, properties and statements based on the experiences of those who participate in the argumentation process, therefore, we place greater emphasis on this study. In geometry teaching, the equivalent of warrants is the theorems or properties, while the backing usually is referred to as the explanations, where students resource to examples or particular cases. We decided to study the epistemic facet of the DMK model, which can be assimilated to the warrants in Toulmin theory.

Even though the seminar preceded the classroom teaching to schoolers, we will comment on the fragments



$$c^2 = 4\left(\frac{1}{2} ab\right) + (a-b)^2 = a^2 + b^2$$

Figure 2. Pythagorean Theorem according to Bhaskara
Source: Designed by a pre-service teacher

in accordance with the features we want to stress. Next segment corresponds to a design session, presented by Carlos.

A Priori Warrants with the Use of Tools

A priori warrant refers to ‘epistemic’ or theoretical statements known beforehand for those who participate in the argumentation, meanwhile, tools refer to both software and tangible materials to support explanations (Baccaglioni-Frank & Mariotti, 2010). This type of warrant appeared several times both in the seminar and in classes. The use of tools required knowledge that falls in the mediational facet of the DMK model. This knowledge can be discussed during teacher training (Ebby, 2000) but most of the time it should be acquired through the teaching experience (Eisenhart et al., 1993). The bold format is used for argumentative indicators (Van Eemeren et al., 2007) that are important throughout the discussion of the argumentative segments. Numbers on the left stand for each participant’s turn to speak and are referred back later in the text; they are not sequential because they refer to fragments taken from interviews.

- 1 Researcher: How would you teach the Pythagorean Theorem to ninth graders?
- 2 Helena: I would like to explain **it using the Bhaskara puzzle**, who was an Indian mathematician that created a puzzle... with the square formed here and with the other one formed here, to show that the sum of these two squares equals this other square (Figure 2).

The teacher proposes to use paper sheets to introduce the theorem through Bhaskara’s verification (Figure 2); and then proceeds to verify the theorem, by coupling the triangles. The teacher coupled the triangles but did not discuss the algebraic equalities. Asked by the researcher about the purpose of writing down those equalities and do not refer to them during his teaching, the pre-service teacher said that she did not know how to explain the equalities geometrically, though she establishes that she

does know how to prove the validity of the algebraic equalities.

This segment shows that without proper mathematical knowledge teachers are unable to advance their students in learning certain mathematics subjects, the epistemic facet affects the good teacher intentions when taking illustrations to class. The presence of algebraic equalities, whose relationship to the geometric graphics are not known to the teacher, stresses how important for the pre-service teachers is to relate the mediational and the epistemic facet of the DMK model. The use of resources requires mathematics knowledge in order to achieve adherence (Perelman & Olbrechts-Tyteca, 2006). According to Conner et al. (2014) the teacher support for collective argumentation “allows an explicit focus on the warrants provided in a class as an indication of the reasoning that is being made public” (p. 424).

Another example of warrants with the use of tools, during the seminar, happens when Carlos uses GeoGebra, the calculator and Figure 3, [33] to discuss the Pythagorean Theorem in the seminar. The tool here has several layers: GeoGebra, calculator, and figure.

- 33 Carlos: What relationship can I establish out based on the graphic? **May I use the calculator?** [Pre-service teacher refers to graphic shown in Figure 3]
- 34 Carlos: 6,51 – 6,45 equals 0,06 It is not equal, it is not, it is different. 1,63 + 6,45, is obviously not 6,51. I don’t see any relationship. Maybe there is one, but I don’t see any. [Numbers refer to those that are shown by GeoGebra, Figure 3, when increasing and decreasing the legs of the triangle].
- 35 Researcher: So, we would have to move... what?
- 36 Carlos: We would **have to move...** We could move... [Pre-service teacher refers to the length of legs and hypotenuse]
I don’t know... If I go a little further... [Referring to increase the length]
- 37 Peer 1: There, you’re onto something!
- 38 Carlos: Let’s say it’s not a right triangle, **as I make it a right triangle**; the two legs get closer and closer to the sum of the hypotenuse.
- 39 Carlos: **I want to see something** [He explores several triangles].

The exploration is made using the software in order to design what Carlos “wants to see”.

The pre-service teacher uses GeoGebra to explore both the right triangle and the rectangles built on its sides and tries to use this dynamic exploration as a guarantee to conclude the validity of the Theorem. He notes that there is no conjecture [34]; and raises, through

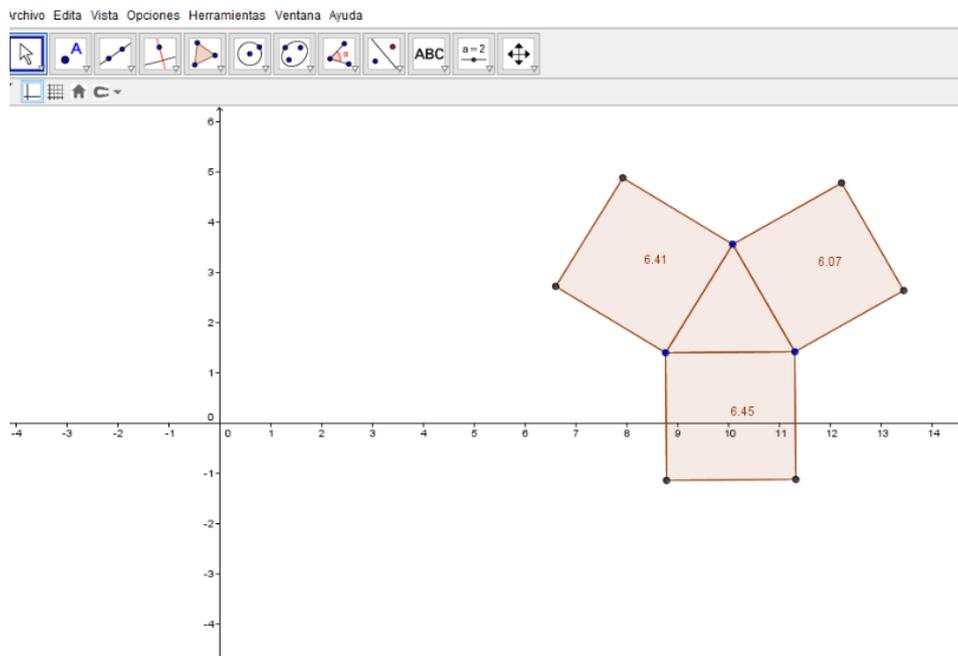


Figure 3. Theorem for any triangle
Source: Designed by Carlos

the modal qualifier ‘may,’ the possibility of the existence of conjectures. On this episode, the pre-service teacher uses a mediational tool to discuss his class, and with the help of the software manages to stress how precise is the relationship among triangle side lengths and the ‘numeric’ relationship among them. This episode illustrates how the use of appropriate tools can help the teacher develop his class. During his teaching in the classroom, with real pupils, he developed his class attached to his planning. The use of a priori warrants with the use of tools presents a nuance to the concept of warrant, where teachers use a resource to help convince their students of the validity of the teachers’ mathematics statements.

A Priori-Epistemological Warrant

A priori-epistemological warrant refers to the use of definitions or theorems, used by pre-service teachers. These warrants are related to the Mathematical Dimension of the knowledge a teacher must possess in order to teach mathematics. According to Nardi et al. (2012), these warrants are ‘a priori’ because they are previously known by the person who is discussing. Even though all warrants should be known beforehand during teaching, both teacher and student, use data presented during the argumentation, no matter if the data was unknown in advanced by those discussing.

Next segment corresponds to a design session, presented by Carlos. Numbers on the left stand for each participant’s turn to speak and are referred back later in the text. Numbers are not sequential because they refer to fragments taken from interviews. The bold format is used for argumentative indicators (Van Eemeren et al.,

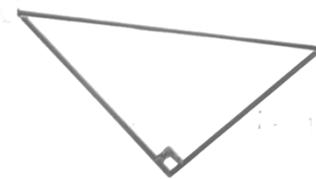


Figure 4. A tilted right triangle

2007) that are important throughout the discussion of the argumentative segments.

The following segments correspond to a seminar session where pre-service teachers presented their planning to teach the Pythagoras theorem. The researcher asks the pre-service teachers ‘how would you teach the Pythagorean theorem in ninth grade?’ Carlos, take the lead:

- 10 Carlos: I would do it using segments... I would draw a segment.
Then I trace the perpendicular line, trace a ratio three, *then* a ratio four *and* join the intersections.
- 11 Peer 2: Could we do it? [Draw a right triangle in a different position, Figure 4].
- 12 Carlos: But I would use the standard right triangle position...I think students could misunderstand...
- 13 Peer 1: I think you are right...it is much better to use the standard position [Draw a right triangle in a standard position- Figure 5]

The phrase ‘I would do it’ implies a possible action related to a definition for ‘perpendicular’, ‘ratio’ and

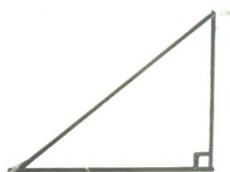


Figure 5. A right triangle

Source: Drawn by pre-service teacher

'intersections' consistent with a priori-epistemological warrant [10]. Even though they lack teaching experience, pre-service teachers choose to draw the right triangle to ease a pupil's comprehension. This competence is linked to the mediational and interactional facet of the DMK model.

Next segment is taken from a session where Pythagoras theorem is taught to schoolers for the first time. Helena is the pre-service teacher who presents her planning.

Helena used a powerpoint -Figure 2- to teach her class.

21 Helena: *Then* the point is to reconcile it [pointing at the graphic] [with] the algebraic part...

We could say that there's a triangle with an 'a' side, *and* 'b' side would be the other leg *and* the hypotenuse is, let's say, 'c' side; *then* how do we manage to figure that out? *I mean*, how could they get to that *very* algorithm to demonstrate that it can be the case for *any* triangle? ... *but* now I know that it happens for every right triangle (Figure 2).

Helena refers to legs and hypotenuse, proposes the theorem' formalization, that presents the use of a priori-epistemological warrant [21]; the teacher does not explain the meaning of the leg, hypotenuse, algebraic part, and right triangle. The researcher, after the class, questions Helena about the terms leg and hypotenuse and her not referring to them; Helena says that students are supposed to know the meaning, which is required to understand the explanation.

The lack of mathematical knowledge affects the mediational and interactional facet. It seems that the pre-service teacher fails to manage the epistemic and the mediational facet as well.

Empiric-Professional Warrant

Empiric-professional warrants refer to the use of previous teaching experience (Nardi et al., 2011) to solve instructional situations. These warrants refers to the knowledge that teachers should have about students learning (Hill et al., 2008; Shulman, 1986). Although pre-service teachers have little teaching experience, they claimed that have gained experience during their teaching practice. In one seminar design session, the



Figure 6. A right triangle

Source: Drawn by pre-service teacher

teacher educator proposed the following hypothetical situation to the pre-service teacher: 'a student may ask about the position of the tracing in the drawing of a right triangle' and raises the question [13], so Helena defends her stance (McClain, 2009). Carlos, Helena's partner, participating in the session, asks her a question, then she answers with an empiric-professional warrant related to the positions of a right triangle [17] (Figure 2). This warrant uses the teacher's experience to support teaching actions. The role of the teacher's experience in this segment is appreciated.

In the following episode, Helena discusses along with colleagues about the possible positions for a right triangle. Helena draws a triangle, Figure 6, where the hypotenuse is drawn horizontally.

12 Helena: This is right triangle... you see.... [Pointing at the right angle]

13 Teacher: Helena! A student could say: Teacher, you're wrong! Look, there you're doing it in that position [Shown in Figure 6], *and* here the triangle has been drawn in this way? [On the position of the right triangle] (Figure 2).

14 Teacher: What would you reply?

15 Helena: Well, I would tell them that those triangles are congruent and are the same, just, from other perspectives, from the center of the square.

16 Carlos: *I mean*, when you get the half ...there you're going to trace...

17 Helena: The thing is the student *tends* to say that if I do this [she refers to the right triangle in standard position], then it is *no longer* a right triangle.

INTENTION OF THE ARGUMENTS

The argument' intention has been studied by several authors (Perelman, 1997, 2007; Perelman & Olbrechts-Tyteca, 2006). In mathematics education, several studies have reported about argument intentions shown by elementary schoolers or by pre-service teachers, for example, arguments to validate (Balacheff, 1999, 2000; Harel & Sowder, 1998; Hoyles & Küchermann, 2002); arguments to justify (McClain, 2009); arguments to rebut (Balacheff, 1999, 2000; McClain, 2009; Reid et al., 2011); arguments to defend (McClain, 2009); arguments to

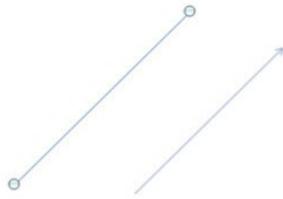


Figure 7. Diagonal line
Source: Pre-service teacher

explain (Balacheff, 2000; De Villiers, 1993) and arguments to persuade (Arzarello & Sabena, 2011; Crespo et al., 2010; De Villiers, 1993; Goizueta & Planas, 2013a, 2013b; Reiss & Renkl, 2002). In pragma dialectics, the role of the intention is critical (C, 1999), but we do not take that perspective in this work. In this paper we assume the definition "Intention is the impulse to persuade others is a constructive and valuable aspect of human symbolic interaction" (Jørgensen, 2007, p. 165).

The intention of the speech act tends to be following the curriculum objectives, which derivates to secondary intentions (Reed & Long, 1997) that we associate with "auxiliary" intentions which are: to validate, justify, rebut, defend, explain and persuade. In this paper, the arguments to validate are considered with epistemic reasoning; the arguments to justify, rebut, explain, persuade and defend, with teleological reasoning. We will discuss episodes taken from both class design and classes with kids.

To Validate

The speech acts in the classroom are interactive, and can be of confirmation, if the speaker agrees or invalidation if he is not. The teacher acts of confirmation, on a student idea, are assumed as validation no matter if the idea is mathematically incorrect. Sometimes the teacher accepts students' ideas to promote participation and class discussion. Candela (1991) proposes social classroom knowledge construction based on the oral discussion that promotes students scientific training. Validation refers to the checking or proving the accuracy of a statement.

The following segment is taken from the seminar, where Carlos presented his intended class about diagonal lines.

- 1 Carlos: Can anyone tell me what a diagonal is?
- 2 Peer 1: A diagonal is a line or a segment.... [Describe a tilted line with the hand]
- 3 Carlos: What is that...what does it mean? [Referring to his classmate gesture]
- 4 Peer: A diagonal...
- 5 Carlos: A diagonal is always like the one in [power point] slide? [Figure 7]
- 6 Carlos: Please students [addressing his classmates] discuss with your

classmates and then propose an idea of what a diagonal line is...

After a while:

- 7 Carlos: Ok, let's resume...what did we say?
- 8 Peer 2: We say it's a straight line we trace when we're going to do something.
- 9 Carlos: Always?
- 10 Student: It doesn't always have to be straight.
- 11 Carlos: No, I'm talking about: "always..."
- 12 Student: We always trace it? No, also when you walk in a diagonal
- 13 Carlos: An imaginary line!?
- 14 Student: Yes! That too!

Line 9 and Line 13 presents Carlos' intention to validate the information. Carlos asks about diagonal drawing with the intention to highlight the relation between a diagonal line and a horizontal line. He presents question [2] with the intention of validating statement [1].

The intention to validate occurs when the teacher tries to relate students' points of views and mathematics knowledge, it is to say, teacher establish relationships between students' ideas and what is accepted as mathematically valid. The pre-service teachers attempted to improve students' concept of diagonal or to institutionalize it, based on the student personal knowledge (Godino et al., 2007) teacher action is needed. Teachers try to scaffold the development of argumentation based in norms (Makar et al., 2015).

In the following segment, two pre-service teachers, Carlos and Helena, teach together and discuss the concept of diagonal line with schoolers using students' ideas about the diagonal. The teachers raise questions both to motivate the participation of students [2, 3, 5 and 6] and to question the definition of the diagonal line [4].

- 1 Carlos: What else?
- 2 Student 1: When they take a ruler, *and* say
- 3 Student 2: They say, trace a line, vertical, horizontal, diagonal.
- 4 Carlos: *So*, a diagonal *can be* horizontal or vertical?

The pre-service teacher shows students a power point slide [Figure 8. A Street is Calle is Spanish, and Carrera is a wider street]. In line 4 the pre-service teacher question that a diagonal line could be horizontal or vertical. The after-class interview let the researchers know that the pre-service intention was to question what he believed was a typical student idea: a diagonal line must be tilted. This episode draws on personal student knowledge and how the pre-service teacher tries to adapt that knowledge to a more precise one. Findings reported by (Brown, 2017, p. 198) suggest that aspects of collective argumentation such as students "...explaining

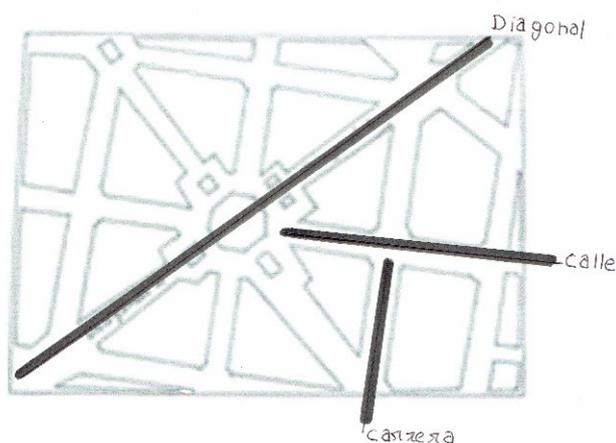
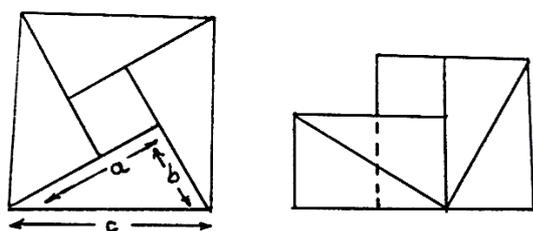


Figure 8. A diagonal shown to students
Source: Designed by a pre-service teacher



$$c^2 = \left(\frac{1}{2} ab\right) + (a-b)^2 = a^2 + b^2$$

Figure 9. Pythagorean Theorem according to Bhaskara
Source: Designed by a pre-service teacher

and justifying their ideas to others and presenting ideas to the whole class for discussion and validation can be used by teachers to promote behavioural, emotional and cognitive engagement with mathematics”.

- 5 Student 2: It’s from corner to corner, it’s like this [makes a diagonal hand gesture]
- 6 Helena: That’s why they [students] made this gesture with their hands.

To Rebut

Rebuttal is understood as opposing points of view presented by others. For example Helena presents her students with a puzzle intended to introduce the Pythagorean Theorem [Figure 9]. Maria considers that a puzzle is not a good way to introduce the activity. Helena [17] questions Maria who did not consider the construction of the puzzle on the paper (Figure 9). Helena’s statement [18] intends to rebut Maria, emphasizing the need to revise her class preparation.

- 16 Helena: In my class I would introduce the theorem with a puzzle... [Show the puzzle, Figure 9]
- 17 Maria: A puzzle? I think it is not a good idea...
- 18 Helena: No! If we’re going to do it in a certain way, the thing is you think I would do

it with the construction. Right?... but a puzzle is a construction... not need to use classical geometric construction with ruler and protractor...

- 19 Maria: Much better...
- 20 Helena: I thought it was an interesting sketch of the idea per se, that will always be practical, something to construct... so the youngsters [her students] has the opportunity to reach that formal statement. Yes, interesting!

In another class, Carlos proposes a question about diagonals.

- 1 Carlos: You tell me...where diagonals are used?
- 2 Student 2: Diagonal is used in house addresses...some addresses such as streets, avenues, diagonal, parks, alleys...
- 3 Carlos: Who has more...ideas?
- 4 Carlos: Show me!
- 5 Student 1: I think that it’s useful for art, and technology class, giving addresses and tracing maps.
- 6 Carlos: How do you use it when tracing maps? I didn’t know that one!
- 7 Carlos: Can you tell me is the geometric idea of diagonal is the same idea used in addresses as our colleague-student 1- has said?
- 8 Carlos: How do you use it when tracing maps? I didn’t know that one!

In [6 and 7] Carlos has the intention of rebutting the relationship between diagonal lines and mail addresses [2], proposed by student 2. Van Eemeren et al. (2006, p. 46), consider that argumentation “is always an attempt to justify or rebut something”.

To Justify

Justification refers to presenting reasons so that a previously presented idea can be considered as admissible within the dialogical argumentation. In other segment, Maria asks one student [43] about the importance of geometry. The student claims that “the column can support a house” and compares it with cylindrical geometric figures [46].

- 43 Maria: *And* is geometry important or not?
- 44 Student 1: Well, I think it’s important.
- 45 Maria: *Why?*
- 46 Student 1: Well! *Because* geometry studies geometrical figures, which allow to make things; for example, let’s say a cylinder, as..., let’s say we’re building

a house and I need a rod for a column, thanks to the cylinder I have that rod to build the house.

In both cases, the arguments, expressed through questioning, have the intention of justifying features of the mathematical objects been studied. In this paper, justify is tantamount to give a good reason. Mercer (2009) reports that “teacher’s contributions include “reasoning words” such as “what,” “how,” “if” and “why” as the children are lead through the activity” (p. 188); the author reports that “children’s individual reasoning capabilities appeared to have been improved by taking part in the group experience of explicit, rational, argumentation and collaborative problem solving” (p. 191).

To Explain

Explanation refers to allowing a participant to help others understand a specific idea. In a segment that corresponds to a class where ‘diagonal’ is discussed, Carlos asks for arguments in order to explain ([9] and [11]).

- 7 Carlos: How do you use it [diagonal] when tracing maps! I didn’t know that one!
- 8 Student 1: Of course! With a compass! I mean, no! Not with a compass! With a...
- 9 Carlos: When do you use it in a map?
- 10 Student 1: When you have to trace and angle, *let’s suppose* it’s an obtuse angle, you’d have to trace a diagonal from the middle point up to that angle.
- 11 Carlos: In that moment, is it a diagonal?
In a Seminar session, Carlos and a peer discuss a task.
- 29 Peer 1: **What is the relationship between** the sum of the areas of the drawn squares in *each leg* and the area of the square drawn on the hypotenuse?
- 30 Carlos: *Then* $16+9$ equals 25. Just as the big square! That would be the relationship, *or* if I compare this one with this other one. i.e., $25-9$ equals 16 *or* $25-16$ equals 9 (Figure 10).

Peer 1 asks Carlos about the relationship between the areas of the squares on the legs and the hypotenuse of the right triangle. Carlos states the mathematical relationship through the indicator ‘then’ [30], which expresses an intention to explain (De Villiers, 1993). This numerical relationship matches the sum of the squares of the length of the legs [30]. Carlos replies using the argumentative indicator ‘then’ [30], which intends to explain.

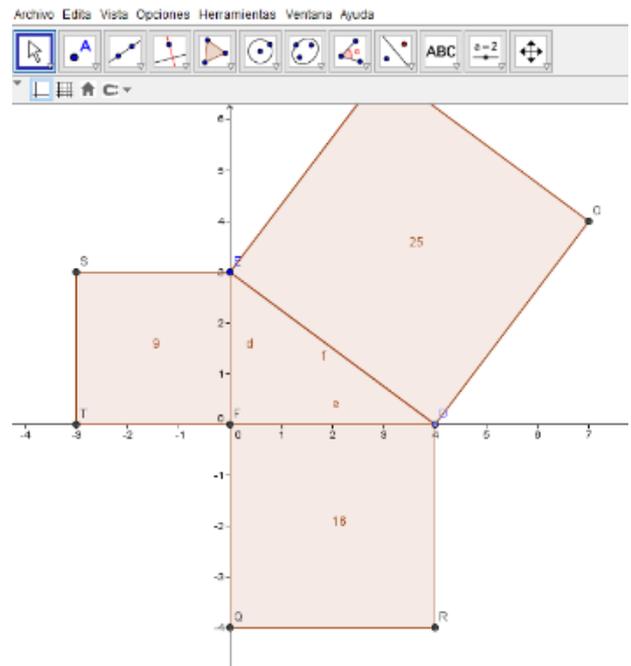


Figure 10. Pythagorean Theorem by Carlos
Source: Designed by Carlos

RHETORICAL RESOURCES

Argumentation considers rhetorical resources as an essential instrument to study the speaker argumentative intention. Several authors acknowledge the importance to involve students on actual science practices, among them, to argue and to use rhetorical resources to communicate findings and perspectives (Duschl et al., 2007). Reid et al. (2011) consider rhetorical resources used by mathematics teachers as mediating tools. Among rhetorical resources, we find: example, illustration, model and metaphor (Perelman & Olbrechts-Tyteca, 2006). The model of didactic-mathematical knowledge considers the mediational aspect, which refers to different resources’ teacher uses to conduct the class (Bartolini & Mariotti, 2008; Pino-Fan et al., 2015). Even though some researchers (Bartolini Bussi & Boni, 2003; Durand-Guerrier et al., 2011) have studied the use of tools to study proof, our work study the use mathematics knowledge, tools and interaction by teachers to teach mathematics, but considering argumentation as the background of teacher’s didactical work.

Among the linguistic resources used by pre-service teachers, we find: example, illustration, model and metaphor. The teacher uses these resources to support the comprehension of the mathematical objects under study. Here, we illustrate some uses, by the teachers, of these resources.

Example

It is a rhetorical resource based on a rule, and it supposes “the existence of regularities that examples can

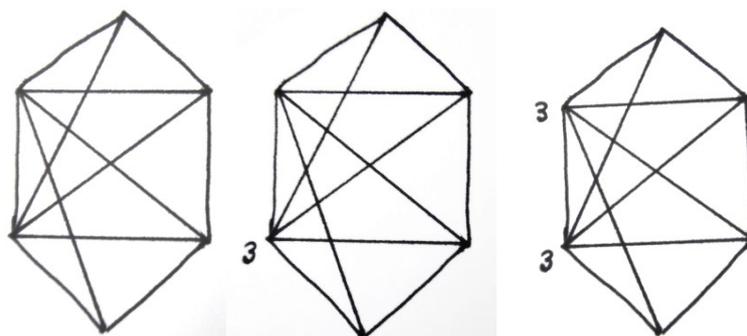


Figure 11. Tracing of the diagonals from the second vertex
Source: Designed by one student

attest". Examples are used to persuade through empirical arguments (Harel & Sowder, 1998). The example's scope is the generalization of the particular case, but not the very principle of the generalization (Perelman, 1997). Examples have the virtue of being representative of a broader category, and it is an element that possesses the most general characteristics.

The following two episodes correspond to seminar sessions where pre-service teachers discussed their classes previous to real teaching. In the first episode Helena presents an example of the Pythagorean theorem, while the second episode deals with the discussion about the number of diagonals that can be drawn in a hexagon, the teacher wanted to discuss the general case of the number of diagonals in any polygon, and also wanted to relate this diagonal meaning to the one discussed before.

During the discussion, Helena proposed the theorem's formalization and used the definitions of both, leg and hypotenuse, which is related to the use of a priori-epistemological category [21]. This argument corresponds with a generalization based on a particular case; the teacher wants to draw a general conclusion out of a particular case. According to Helena, the use of this resource will make the students to generalize the theorem out from a particular case [21]. The objective for this class was 'the students will explore the number of diagonals in a polygon.'

21 Helena: **Then** the point is to reconcile it [with] the algebraic part, so **we could say** that there's a triangle with an 'a' side, **and** 'b' side would be the other legs **and** the hypotenuse is, let's say, 'c' side; **then** how do we manage to figure that out? **I mean**, how could they get to that **right** algorithm to demonstrate that it can be the case for **any** triangle? **And not** build several pieces, **but** now I know that it happens for every right triangle (Figure 2).

In the following segment, whose objective is to discuss about the number of diagonals that can be drawn in a polygon (Figure 11), taken from a class taught to

schoolers, the teacher gives students a hand-out with copies of the same polygon and ask them to draw segments or diagonals from every vertex, and count them.

3 Student 2: What we have done is to draw lines from every vertex and count them... **some** of them are repeated.... See... We do with this one, **and** this one [pointing out the first three vertices and the segments], right? **So**, I go to this one, and **then** to this one **and** to this one, how many do we have there?

4 Helena: Three, **three, as well!**

5 Helena: Now, can anyone tell how do we find the number of diagonals in any polygon?

6 Student: Drawing diagonals and counting them...we should not count twice!

7 Helena: Well...any one?

8 Helena: But if I tell you the numbers of sides in a polygon...can you tell me how many diagonals? [Students kept silent]

Helena uses the example as a rhetorical resource and utters the expression 'the same' to indicate that the process is repeated. Even though she establishes that the objective is "to explore," during the interview says that she wanted the students to find the rule to find the number of diagonals in any polygon. She intends the students, based on this example, to perform a generalization. Unfortunately, the students were not able to 'jump' to the general expression for the number of diagonals in a polygon.

Asked about her intended objective and how the students fail to reach it, Helena admitted that an example based on a single polygon was not enough. In this case, the objective is clearly stated, and the material well designed, the interaction was not, because the students needed more time and more discussion to reach, at least, a verbal statement of the expression to find the number of diagonals. In this segment, the teacher knows how to solve the problem- epistemic facet-, has designed the task- mediational facet- but has failed in considering that

the activity is difficult for the schoolers and more interaction was needed in order to reach her objective.

Illustration

Offering or demanding an illustration “serves to reinforce the adherence to a known and accepted rule, providing particular cases that make the general statement clearer, show its interest in the variety of possible applications, and increase its presence on conscience” (Perelman & Olbrechts-Tyteca, 2006, p. 546). The illustration can be described as the repeated use of an example whose purpose is both to show different aspects of the mathematical object in question and to reinforce students' adherence.

A student is asked to find the total number of diagonals that can be traced from each vertex of a convex polygon, Peer 2 traces diagonals from the second vertex (Figure 6); and appeals to illustration (Perelman, 1997); then uses both the expression ‘the same’ to indicate that the process is repeated using an example (Perelman, 1997; Perelman & Olbrechts-Tyteca, 2006). The argumentation indicators ‘and’ [3], and ‘then’ [3] indicates that the argumentation progresses (van Eemeren et al., 2006, 2007). This segment ends with the question that Peer 2 raises to the auditorium, how many did we do there? to which Helena answers three [4]. This answer uses the indicator ‘as well’ referring to the last vertex.

3 Peer 2: We continue with this one, the same for this *and* this one [indicating the first three vertices and the diagonals], right? *So*, I go to this one, and then to this one *and* to this one, how many do we have there?

4 Helena: Three, *three, as well!*

Model

It means “the particular case [that] can be presented as a model to follow instead of an example or an illustration; but not any action is meant to be imitated: only those people who are admired, or who have some authority or social prestige, should be imitated, because of their competence, their functions or the position they have in society” (Perelman, 1997, p. 148).

Peer 2 claims that from first and second vertex, three diagonals may be traced, but not so from the third, this may be inferred through the use of words like ‘assume’ and ‘as well’, which respectively mean, ‘suppose or guess’ and ‘likewise or similarly’ [5]. The use of ‘assume’ warns the auditorium that there will be a change in the number of diagonals that are traced from the third vertex. So, to the question worded by Peer 2 to the auditorium, Helena replies that only two diagonals are traced [8]. Once more, Peer 2 uses the illustration (Figure 7), drew on the board as a persuasive mechanism

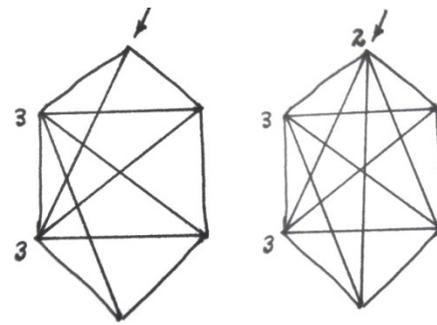


Figure 12. Tracing of diagonals from the third vertex
Source: Pre-service teachers

(Perelman, 1997; Perelman & Olbrechts-Tyteca, 2006; Manghi, 2010).

5 Peer 2: This one has three *and* this one has three, one would assume that for this one we'd have three *as well*, just like this one, this one and this one. What happens when I get here?

6 Helena: We already had a diagonal...

7 Peer 2: *So*, what is left to link? [Pointing the third vertex] (Figure 12).

8 Helena: Two diagonals.

9 Peer 2: This one, *and* I'm missing this one, *so*, how many did I do? Two.

3 Maria: **Initially**, as Carlos did, I would begin with a little bit of history, although the students are ninth graders, **I could make** up a story for them about the history of the Pythagorean Theorem **because** it is believed that Pythagoras invented it. Babylonian and Egyptian civilizations also had ideas of Pythagorean numbers making approximations with square roots and arithmetical measurements.

Moving on with her arguments, Maria claims that she considered Carlos' answer [3], which shows the use of the model as a rhetorical resource (Perelman, 1997; Perelman & Olbrechts-Tyteca, 2006).

Metaphor

It is discussed through the offering or request of metaphor when there is “an accurate change of meaning of a word or phrase” (Perelman & Olbrechts-Tyteca, 2006, p. 610). Lakoff and Johnson (1999), consider the metaphor and the thinking that turns to it, as the interpretation of a field of experiences concerning another one already known. Several authors have studied metaphor (e.g., Perelman, 1997, 2007; Perelman & Olbrechts-Tyteca, 2006), and its role in mathematics (English, 1997; Lakoff & Núñez, 2000; Núñez, 2000; Núñez & Lakoff, 1998; Van Dormolen, 1991). “Metaphors are notable for creating, between a source



Figure 13. Geometry in daily life
Source: school boy

realm and a target realm, a conceptual bridge that allows the transmission of properties of the source realm into the target realm” (Font et al., 2003, p. 406).

In another segment, the teacher raises a question [14] about the identification of examples where geometry is used in daily life. Students use the metaphor [15] when comparing geometric figures with objects in their environs. It is remarkable the teacher’ use of metaphor as a rhetorical resource, both in the question and in the answer.

- 14 Maria: Where can we find geometry in everyday life?
- 1 Maria: Can anyone draw examples... on the blackboard? [Figure 13 shows what a student draw on the blackboard]
- 15 Student 1: In the TV, the washing machine, in the computer, of course.
- 16 Maria: *And why* in those... [Things]? In those appliances?
- 17 Student 1: Well! They are three-dimensional geometric figures.
- 18 Student 2: **That is it**, it **also** has the vertices **and** the edges, **and** that makes them geometric figures.

Here we appreciate the use of metaphor when comparing everyday objects with geometric objects in order to persuade. In another segment, Carlos uses an a priori warrant, with the intention to persuade the auditorium, through the use of GeoGebra, the calculator, and the figure is explicitly drawn to discuss the theorem. He uses the indicator ‘no’ to express that he does not see any conjecture [34]; he also uses the modal qualifier ‘may’ to show the possibility of the existence of some conjectures.

- 31 Peer 1: To verify for triangles that are not right triangles! (Figure 14).
- 32 Carlos: *That is*, do I trace *any* triangle?
- 33 Carlos: What relationship can I get from it? Using the calculator?
- 34 Carlos: 6.51–6.45 equals 0.06. It’s *doesn’t* equal, it *doesn’t* correspond, it’s different. 1.63 + 6.45, obviously *doesn’t*

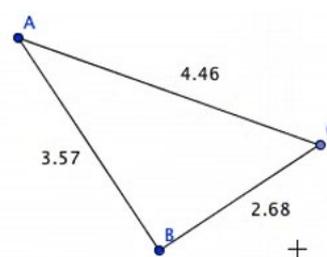


Figure 14. Triangle GeoGebra generated
Source: Pre-service teacher

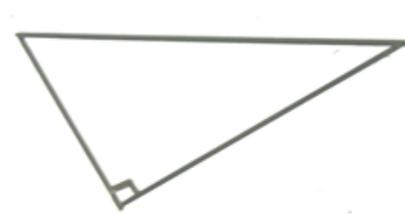


Figure 15. A right triangle
Source: Drawn by pre-service teacher

equal 6.51. I *don’t* see *any* relationship. They *may* be one, *but I don’t* see *anyone*. [Numbers refer to those shown by GeoGebra]

Carlos draws a triangle and verifies if it is a right triangle using the Pythagorean Theorem. For Perelman & Olbrechts-Tyteca (2006, p. 67) the difference between persuasive and convincing is the effect of the argumentation over the audience; they claim: “We call an argumentation persuasive when it only intends to serve a particular audience, and convincing when it is supposed to obtain adherence from every reasonable being”, this is, a universal audience.

Lakatos (1978) connects argumentation with rebutting and validation. This association has implications in mathematical education when rebutting is assumed as a critical argumentation component. Reid, Knipping and Crosby (2011), have studied rebuttal and logic in the classroom.

The rebuttal appeared simultaneously with modal qualifiers.

- 13 Researcher: Helena! A student might say: Teacher, you made a mistake! You are putting it in this position *and* here in this position? [Regarding the position of the right triangle] (Figure 2).
- 14 Researcher: What would you answer?
- 15 Helena: Well, I would tell them that those four triangles are still congruent *and* the same, just, from other perspectives, from the center of the square.

In a segment, taken from a class taught to schoolers, Maria takes up the question posed at the beginning [23] requesting the use of the metaphor [23], the students reply that geometry is in a mattress, in a door, in a fridge



Figure 16. Examples of geometric shapes in daily life

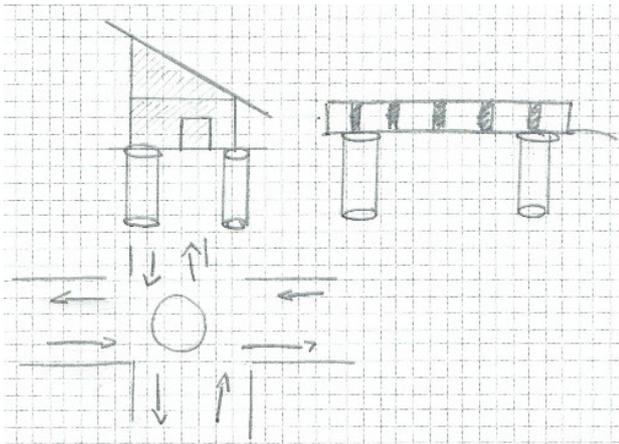


Figure 17. Geometry in daily life
Source: Student

and that these figures are rectangular prisms [24]. Another student also uses metaphor to compare everyday objects to geometric figures [26] and concludes, due to the request for a priori-epistemological warrant by Maria [25] that they are both, rectangular prisms and three-dimensional figures [26].

- 23 Maria: Student 3! Where can we find geometry?
- Maria: can anyone give us some examples... on the blackboard?
- 24 Student 3: In a mattress, in a door, in a fridge which are rectangular prisms. [Figure 16]
- 25 Maria: Are they rectangular prisms? *And why* are they rectangular prisms?
- 26 Student 3: *Because* they're rectangles, *but* they're three-dimensional figures.

This student recognizes the importance of geometry by using the *example* and the *metaphor* [46] when pointing the column that can support a house and compares it with a cylindrical shape [46].

- 43 Maria: *And* is geometry important or not?
- 44 Student 1: Well, I think it's important.
- 45 Maria: *Why?*
- 46 Student 1: Well! *because* geometry creates geometrical figures, which allow

making things; for example, let's say a cylinder, like..., let's say we're building a house and I need a rod for a column, thanks to the cylinder I have that rod to build the house.

- 45 Maria: Can you draw some geometrical figures, on your notebook, to show us all what you are referring to? [Figure 17 shows what student draw on his notebook]

Students use oral and written language, gestures and representations to draft and revise justifications for their claims, nonetheless Kazemi et al. (2021) findings, in regard to use of language by pupils, affirm that language development cannot be considered apart from how students and teachers shape this process.

DISCUSSION OF RESULTS

Argumentation is used by teachers to communicate with students during the mathematics class, includes different features that are used naturally by teachers as they discuss with peers or as they teach to schoolers. These elements have commonalities with the didactic-mathematical knowledge model proposed by Pino-Fan et al. (2015). The pre-service teachers use some features of teacher knowledge, such as the epistemic, interactional and mediational facets that are part to the didactical dimension for teaching, to design and taught their classes.

The didactic-mathematical knowledge (DMK) (Pino-Fan et al., 2015, 2016) considers three dimensions for the knowledge that a teacher needs to conduct his teaching efficiently: the mathematical dimension, didactical dimension and meta-didactic-mathematical dimension.

The epistemic facet refers to the knowledge that mathematics teacher must possess in order to teach mathematics (Pino-Fan et al., 2016). According to Pino-Fan et al. (2015) "it is clear that this category includes not only the proposed notions in the Schoenfeld & Kilpatrick's (2008, p. 32) model of mathematics education proficiency about 'knowing mathematics deeply and extensively but also the notions of Ball et al. (2008, p. 377-378)' about 'specialized content knowledge'. The teacher' explanations must be supported by mathematics knowledge, concepts, properties and theorems, which the teacher enunciates to justify processes for solving mathematics tasks. Enunciating and using such knowledge is related to the epistemic warrants (Nardi et al., 2011), which come close to the logical-deductive nature of mathematics.

The mediational facet of DMK refers to the means used to manage the learning, which include the knowledge of class material, as part of curriculum knowledge (Grossman, 1990; Shulman, 1986). During this research, the type of warrant' use of means was identified, as the use of software, calculators, graphs and

tangible material to draw conclusions about a mathematical feature or property, or to validate a conjecture. The use that pre-service teachers gave to tools refers to concluding a fact, property or characteristic and was used during both, class designing and teaching. A great deal of research has informed the relevance of mathematical objects (Drijvers et al., 2013) because their inadequate use could give way to the emergence of meanings not considered previously by teachers- that would show not enough prowess on the arguments to manage the mediational and epistemic facet in the DMK (Pino-Fan et al., 2015).

However, other features of the argumentative activity that the teachers use can be related to the interactional and mediational features. There is a relationship with the interactional aspect as the research data show that pre-service teachers taught their classes through a dialogue with students, given that the teachers think it is essential to 'interact' with students and "to build upon the questions and answers given by the students" (Franke et al., 2009). Teachers use not only the epistemic warrants but also other resources to keep students interested and participating in the class.

Example, model and metaphor have the intention of explaining, justifying and rebutting and are related to the mediational features as they are resources that teachers use to achieve their instructional objectives. Even though the criteria to use the example, model or metaphor following the specific teaching topic was not studied in this research.

Concerning research validity, it can be stated that content validity is not included, as an extended study process could not be studied due to institutional restrictions. However, there was a commitment to internal validity as the particular events and the explanations provided in the document can actually be sustained by the data. The findings describe the phenomena being researched, and we use peer examination of the data, use mechanical means to record, store and retrieve data, and we use multiple researchers (Lecompte & Preissle, 1993, p. 338). Regarding research reliability, the analysis was sent to both professors and researchers active in argumentation, and they were discussed with them. The study's limitations refer to the number of classes that could be recorded in audio, this due to institutional restrictions that limited access to classrooms. Likewise, after the research ended, the pre-service teachers could not be contacted to formulate subsequent questions that could have been included in the final research report.

CONCLUSION

Different teacher knowledge models present knowledge "domains" (Hill et al., 2008; Pino-Fan & Godino, 2015; Rowland et al., 2005; Schoenfeld & Kilpatric, 2008) that the teachers must know in order to

develop their teaching. Even though the identification of such domains is essential, some tools are required to integrate those domains into teaching practices of pre-service and in-service teachers. The didactic-mathematical knowledge model of Pino-Fan et al. (2015) proposes tools that teachers can use to develop their teaching efficiently (Castro et al., 2018). On this paper, we report the results of a research about the argumentation of pre-service teachers in two scenarios: Seminar of instructional design and classroom. In these two scenarios, the argumentation of teachers was studied assuming argumentation as a social and collaborative process, both rational and verbal, warrants are studied, as well as rhetorical resources and arguments' intentions in the context of mathematics class.

Teachers bring into play didactic-mathematical objects in accordance with didactic-mathematical practices they develop in order to manage class properly; the argumentation could be seen as a process that allows relating and organizing dimensions of the didactic-mathematical knowledge considering argumentation as a social process, where a teacher organizes mathematical knowledge that emerges in his mathematical practice while solving or discussing mathematical tasks. Ayalon and Hershkowitz (2018) report that in examining the 17 teachers' choices of tasks, they found that most of the teachers' chosen tasks were of the "proof-type argument" of demonstration, which is "at the top of the hierarchy" (Stylianides, 2009, p. 280); our findings are different, may be because the teachers in our study are preservice teachers.

The findings support seeing 'teaching' as the mean used by the teacher to articulate different features of the didactic-mathematical knowledge with features of argumentation: warrants, rhetorical resources, and intentions. The teacher should be aware that during argumentative practice, features of the didactic-mathematical knowledge are used, although based on complex personal decisions and classroom' variables, can also be associated, controlled and regulated during the class. Developing the argumentative competence as a mean through which the different aspects of the didactic dimension of the didactic-mathematical knowledge are put into action could be taken into consideration.

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REFERENCES

- Arnaus, R. (1999). La formación del profesorado: Un encuentro comprometido con la complejidad educativa [Teacher training: A meeting committed to educational complexity]. In J. Angulo, J. Barquín, & A. Pérez (Eds.), *Desarrollo profesional del docente: Política, investigación y práctica* (pp. 599-635). Ediciones Akal.
- Arzarello, F., & Sabena, C. (2011). Semiotic and theoretic control in argumentation and proof activities. *Educational Studies in Mathematics*, 77(2-3), 189-206. <https://doi.org/10.1007/s10649-010-9280-3>
- Ayalon, M., & Hershkowitz, R. (2018). Mathematics teachers' attention to potential classroom situations of argumentation. *The Journal of Mathematical Behavior*, 49(March 2018), 163-173. <https://doi.org/10.1016/j.jmathb.2017.11.010>
- Baccaglioni-Frank, A., & Mariotti, M. (2010). Generating conjectures in dynamic geometry: The maintaining dragging model. *International Journal of Computers for Mathematical Learning*, 15(3), 225-253. <https://doi.org/10.1007/s10758-010-9169-3>
- Balacheff, N. (1999). Is argumentation an obstacle? Invitation to a debate. *International Newsletter on the Teaching and Learning of Mathematical Proof*, May/June. <http://www.lettredelapreuve.org/OldPreuve/Newsletter/990506Theme/990506ThemeUK.html>
- Balacheff, N. (2000). *Procesos de prueba en los alumnos de matemáticas* [Testing processes in mathematics students]. Una Empresa Docente, Universidad de los Andes.
- Ball, D., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of Research on Teaching* (4th ed., pp. 433-456). American Educational Research Association.
- Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407. <https://doi.org/10.1177/0022487108324554>
- Bartolini Bussi, M., & Boni, M. (2003). Instruments for semiotic mediation in primary school classrooms. *For the Learning of Mathematics*, 23(2), 15-22. <http://www.jstor.org/stable/40248416>
- Bartolini Bussi, M., & Mariotti, M. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of International Research in Mathematics Education* (2nd revised edition, pp. 746-783). Lawrence Erlbaum.
- Bartolini Bussi, M., Boero, P., Ferri, F., Garuti, R., & Mariotti, M. (2006). Approaching and developing the culture of geometry theorems in school. In P. Boero (Ed.), *Theorems in school: From history, epistemology and cognition to classroom practice* (pp. 211-217). Rotterdam, Sense Publishers. https://doi.org/10.1163/9789087901691_013
- Boero, P., Douek, N., Morselli, F., & Pedemonte, B. (2010). Argumentation and proof: A contribution to theoretical perspectives and their classroom implementation. *Proceeding of the 34th Conference of the International Group for Psychology of Mathematics Education* (Vol. 1, pp. 179-205). PME.
- Brown, R. (2017). Using collective argumentation to engage students in a primary mathematics classroom. *Mathematics Education Research Journal*, 29, 183-199. <https://doi.org/10.1007/s13394-017-0198-2>
- Campbell, T. G., Boyle, J. D., & King, S. (2020). Proof and argumentation in K-12 mathematics: A review of conceptions, content, and support. *International Journal of Mathematical Education in Science and Technology*, 51(5), 754-774. <https://doi.org/10.1080/0020739X.2019.1626503>
- Candela, A. (1991). Argumentación y conocimiento científico escolar [Argumentación y conocimiento científico escolar]. *Infancia y aprendizaje*, 14(55), 13-28. <https://doi.org/10.1080/02103702.1991.10822302>
- Castro, W. F., Pino-Fan, L., & Velásquez-Echavarría, H. (2018). A proposal to enhance pre-service teachers noticing. *Eurasia Journal of Mathematics, Science & Technology Education*, 14(11), em1569. <https://doi.org/10.29333/ejmste/92017>
- Cervantes-Barraza, J., Cabañas-Sánchez, G., & Reid, D. (2019). Complex Argumentation in elementary school. *PNA*, 13(4), 221-246. <https://doi.org/10.30827/pna.v13i4.8279>
- Conner A, Singletary L., Smith R., Wagner, P., & Francisco R. (2014). Identifying kinds of reasoning in collective argumentation. *Mathematical Thinking and Learning*, 16(3), 181-200. <https://doi.org/10.1080/10986065.2014.921131>
- Conner, A, Singletary, L., Smith, R., Wagner, P., & Francisco, R. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics*, 86, 401-429. <https://doi.org/10.1007/s10649-014-9532-8>
- Crespo, C., Farfán, R., & Lezama, M. (2010). Argumentaciones y demostraciones: Una visión de la influencia de los escenarios socioculturales [Arguments and demonstrations: A vision of the influence of sociocultural settings]. *Revista*

- Latinoamericana de Investigación en Matemática Educativa*, 13(3), 283-306.
- De Villiers, M. (1993). El papel y la función de la demostración en matemáticas [The role and function of proof in mathematics]. *Epsilon*, 26, 15-29.
- Douek, N., & Scali, E. (2000). About argumentation and conceptualization. In T. Nakahara & M. Koyama (Eds.), *Proceeding of the 24th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 249-256. Hiroshima, Japan: PME.
- Drijvers, P.; Godino, J.; Font, V & Trouche, L. (2013). One episode, two lenses A reflective analysis of student learning with computer algebra from instrumental and onto-semiotic perspectives. *Educational Studies in Mathematics*, 82(1), 23-49. <https://doi.org/10.1007/s10649-012-9416-8>
- Durand-Guerrier, V., Boero, P., Douek, N., Epp, S. S., & Tanguay, D. (2011). Argumentation and proof in the mathematics classroom. In G. Hanna, & M. de Villiers (Eds.), *Proof and proving in mathematics education* (New ICMI Study Series, Vol. 15, pp. 349-367). Springer, Dordrecht. https://doi.org/10.1007/978-94-007-2129-6_15
- Duschl, R., & Osborne, J. (2002). Supporting and promoting argumentation discourse in science education. *Studies in Science Education*, 38, 39-72. <https://doi.org/10.1080/03057260208560187>
- Duschl, R., Schweingruber, H., & Shouse, A. (2007). *Taking science to school: Learning and teaching science in grades K-8*. National Academies Press.
- Ebby, C. (2000). Learning to teach mathematics differently: The interaction between coursework and fieldwork for pre-service teachers. *Journal of Mathematics Teacher Education*, 3, 69-97. <https://doi.org/10.1023/A:1009969527157>
- Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education*, 24, 8-40. <https://doi.org/10.2307/749384>
- English, L. (1997). Promoting a problem posing classroom. *Teaching Children Mathematics*, 4, 172-180. <https://doi.org/10.5951/TCM.4.3.0172>
- Erduran, S. (2007). Methodological foundations in the study of argumentation in science classrooms. In S. Erduran & M. P. Jimenez-Aleixandre (Eds.), *Argumentation in Science Education* (pp. 47-70). Springer. https://doi.org/10.1007/978-1-4020-6670-2_3
- Erkek, Ö., & Bostan, M. (2019). Prospective middle school mathematics teachers' global argumentation structures. *International Journal of Science and Mathematics Education*, 17, 613-633. <https://doi.org/10.1007/s10763-018-9884-0>
- Fielding-Wells, J. (2014). *Developing argumentation in mathematics: The role of evidence and context* (Unpublished doctoral thesis), Queensland University, Melbourne, Australia.
- Font, V., Nanclares, A., & Iván, J. (2003). Fenómenos relacionados con el uso de metáforas en el discurso del profesor: el caso de las gráficas de funciones [Phenomena related to the use of metaphors in the teacher's discourse: the case of function graphs]. *Revista Enseñanza de las Ciencias*, 21(3), 405-418. <https://doi.org/10.5565/rev/ensciencias.3917>
- Franke, M, Webb, N., Chan, A., Ing, M., Freund, D., & Battey, D. (2009). Teacher questioning to elicit students' mathematical thinking in elementary school classrooms. *Journal of Teacher Education*, 60(4), 380-392. <https://doi.org/10.1177/0022487109339906>
- Gabel, M., & Dreyfus, T. (2013). The flow of proof: The example of Euclidean algorithm. In A. Lindmeier, A. Heinze (Ed.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 321-328). PME.
- Garcia-Mila, M., Gilabert, S., Erduran, S., & Felton, M. (2013). The effect of argumentative task goal on the quality of argumentative discourse. *Science Education*, 97(4), 497-523. <https://doi.org/10.1002/sce.21057>
- Godino, J., & Batanero, C. (1994). Significado institucional y personal de los objetos matemáticos [Institutional and personal meaning of mathematical objects]. *Recherches en Didactique des Mathématiques*, 14(3), 325-355.
- Godino, J., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM. The International Journal on Mathematics Education*, 39(1), 127-135. <https://doi.org/10.1007/s11858-006-0004-1>
- Goizueta, M., & Planas, N. (2013a). El papel del contexto en la identificación de argumentaciones matemáticas por un grupo de profesores [The role of context in the identification of mathematical arguments by a group of teachers]. *PNA*, 7(4), 155-170.
- Goizueta, M., & Planas, N. (2013b). Temas emergentes del análisis de interpretaciones del profesorado sobre la argumentación en clase de matemáticas [Emerging themes from the analysis of teachers' interpretations of argumentation in mathematics class]. *Enseñanza de las Ciencias: Revista de Investigación y Experiencias Didácticas*, 31(1), 61-78. <https://doi.org/10.5565/rev/ec/v31n1.835>

- Grossman, P. (1990). *The making of a teacher: Teacher knowledge and teacher education*. Teachers College Press.
- Habermas, J. (1999). *Teoría de la acción comunicativa I. Racionalidad de la acción y racionalización social* [Theory of communicative action I. Rationality of action and social rationalization]. Taurus Ediciones.
- Harada, E. (2009). Algunas aclaraciones sobre el "Modelo" Argumentativo de Toulmin [Some clarifications about Toulmin's argumentative "Model"]. *ContactoS*, 73, 45-56.
- Harel, G., & Sowder, L. (1998). *Students' proof schemes: Results from exploratory studies*. In E. Dubinsky, A. Schoenfeld, J. Kaput, F. Hitt, (Eds.), *Research in Collegiate Mathematics Education VII* (pp. 234-283). American Mathematical Society. <https://doi.org/10.1090/cbmath/007/07>
- Hill, H., Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge of students. *Journal for Research in Mathematics Education*, 39, 372-400.
- Hoyle, C., & Küchermann, D. (2002). Students' understandings of logical implication. *Educational Studies in Mathematics*, 51, 193-223. <https://doi.org/10.1023/A:1023629608614>
- Jiménez-Aleixandre, M., & Pereiro Muñoz, C. (2002). Knowledge producers or knowledge consumers? Argumentation and decision making about environmental management. *International Journal of Science Education*, 24(11), 1171-1190. <https://doi.org/10.1080/09500690210134857>
- Jimenez-Aleixandre, M., & Erduran, S. (2007). Argumentation in science education: An overview. In S. Erduran & M. Jimenez-Aleixandre (Eds.), *Argumentation in Science Education* (Vol. 35, pp. 3-27). Springer. https://doi.org/10.1007/978-1-4020-6670-2_1
- Johnson, R. (2000). *Manifest rationality. A pragmatic theory of argument*. Lawrence Erlbaum Associates, Inc
- Jørgensen, C. (2007). The relevance of intention in argument evaluation. *Argumentation*, 21, 165-174. <https://doi.org/10.1007/s10503-007-9044-0>
- Kazemi, E., Ghouseini, H., Cordero-Siy, E., Prough, S., McVicar, E., & Resnick, A. F. (2021). Supporting teacher learning about argumentation through adaptive, school-based professional development. *ZDM Mathematics Education*, 53, 435-448. <https://doi.org/10.1007/s11858-021-01242-5>
- Knipping, C., & Reid, D. (2015). Reconstructing argumentation structures: A perspective on proving processes in secondary mathematics classroom interactions. In A. Bikner-Ahsbabs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education. Advances in mathematics education*. Springer, Dordrecht. https://doi.org/10.1007/978-94-017-9181-6_4
- Lakatos, I. (1978). *Pruebas y Refutaciones: La lógica del descubrimiento matemático* [Proofs and Refutations: The Logic of Mathematical Discovery]. Editorial Alianza.
- Lakoff, G., & Johnson, M. (1999). *Philosophy in the flesh: The embodied mind and its challenges to western thought*. Basic Books.
- Lakoff, G., & Nuñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.
- LeCompte, M., & Preissle, J. (1993). *Ethnography and qualitative design in educational research* (2nd ed.). Academic Press.
- Makar, K., Bakker, A., & Ben-Zvi, D. (2015). Scaffolding norms of argumentation-based inquiry in a primary mathematics classroom. *ZDM*, 47(7), 1107-1120. <https://doi.org/10.1007/s11858-015-0732-1>
- Manghi, D. (2010). Recursos semióticos del profesor de matemáticas: Funciones complementarias del habla y los gestos para la alfabetización científica escolar [Semiotic Resources of the Mathematics Teacher: Complementary Functions of Speech and Gestures for School Scientific Literacy]. *Estudios Pedagógicos*, 36(2), 99-115. <https://doi.org/10.4067/S0718-07052010000200006>
- McClain, K. (2009). When is an argument just an argument: The refinement of mathematical argumentation. In D. Stylianou, M. Blanton, & E. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 222-234). Routledge. <https://doi.org/10.4324/9780203882009-13>
- McNeill, K., & Pimentel, D. (2010). Scientific discourse in three urban classrooms: The role of the teacher in engaging high school students in argumentation. *Science Education*, 94(2), 203-229. <https://doi.org/10.1002/sce.20364>
- Mendonça, P., & Justi, R. (2013). The relationships between modelling and argumentation from the perspective of the model of modelling diagram. *International Journal of Science Education*, 35(14), 2007-2034. <https://doi.org/10.1080/09500693.2013.811615>
- Mercer, N. (2009). Developing argumentation: Lessons learned in the primary school. In N. Muller Mirza, & A.-N. Perret-Clermont (Eds.), *Argumentation and education*. Springer. https://doi.org/10.1007/978-0-387-98125-3_7
- Metaxas, N., Potari, D., & Zachariades, T. (2016). Analysis of a teacher's pedagogical arguments using Toulmin's model and argumentation schemes. *Educational Studies in Mathematics*, 93, 383-397. <https://doi.org/10.1007/s10649-016-9701-z>

- Ministerio de Educación Nacional de Colombia MEN (1998). *Lineamientos curriculares para matemáticas* [Curriculum guidelines for mathematics]. Editorial Magisterio.
- Miyakawa, T. (2017). Comparative analysis on the nature of proof to be taught in geometry: the cases of French and Japanese lower secondary schools. *Educational Studies in Mathematics*, 94, 37-54. <https://doi.org/10.1007/s10649-016-9711-x>
- Molina, O., Font, V., & Pino-Fan, L. (2019). Estructura dinámica de argumentos analógicos, abductivos y deductivos: un curso de geometría del espacio como contexto de reflexión [Dynamic structure of analogical, abductive and deductive arguments: a course in geometry of space as a context for reflection]. *Enseñanza de las Ciencias*, 37(1), 93-116. <https://doi.org/10.5565/rev/ensciencias.2484>
- Muller, N., Perret-Clermont, A., Tartas, V., & Iannaccone, A. (2009). Psychosocial processes in argumentation. In N. Muller, & A. Perret-Clermont (Eds.), *Argumentation and education: Theoretical foundations and practices* (pp. 67-90). Springer. https://doi.org/10.1007/978-0-387-98125-3_3
- Nardi, E., Biza, I., & Zachariades, T. (2012). 'Warrant' revisited: Integrating mathematics teachers' pedagogical and epistemological considerations into Toulmin's model for argumentation. *Educational Studies in Mathematics*, 79(2), 157-173. <https://doi.org/10.1007/s10649-011-9345-y>
- Nielsen, J. (2011). Dialectical features of students' argumentation: A critical review of argumentation studies in science education. *Research in Science Education*, 43(1), 371-393. <https://doi.org/10.1007/s11165-011-9266-x>
- Núñez, R. (2000). Mathematical idea analysis: What embodied cognitive science can say about the human nature of mathematics. In *Proceedings of the 24th International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 3-22). PME.
- Núñez, R., & Lakoff, G. (1998). What did Weierstrass really define? The cognitive structure of natural and epsilon-delta continuity. *Mathematical Cognition*, 4(2), 85-101. <https://doi.org/10.1080/135467998387343>
- Pedemonte, B. (2007). How can the relationship between argumentation and proof be analyzed? *Educational Studies in Mathematics*, 66, 23-41. <https://doi.org/10.1007/s10649-006-9057-x>
- Perelman, C. (1997). *El imperio retórico: retórica y argumentación* [The rhetorical empire: rhetoric and argumentation]. Grupo Editorial Norma.
- Perelman, C. (2007). Lógica formal y lógica informal [Formal logic and informal logic]. *Praxis Filosófica*, 25, 139-144.
- Perelman, C., & Olbrechts-Tyteca, L. (2006). *Tratado de la argumentación: La nueva retórica* [Treatise on argumentation: The new rhetoric]. Editorial Gredos.
- Pino-Fan, L., & Godino, J. (2015). Perspectiva ampliada del conocimiento didáctico-matemático del profesor [Extended perspective of the teacher's didactic-mathematical knowledge]. *PARADIGMA*, 36(1), 87-109.
- Pino-Fan, L., Assis, A., & Castro, W. (2015). Towards a methodology for the characterization of teachers' didactic-mathematical knowledge. *EURASIA Journal of Mathematics, Science & Technology Education*, 11(6), 1429-1456. <https://doi.org/10.12973/eurasia.2015.1403a>
- Pino-Fan, L., Assis, A., & Godino, J. (2015). Análisis del proceso de acoplamiento entre las facetas epistémica y cognitiva del conocimiento matemático en el contexto de una tarea exploratorio-investigativa sobre patrones [Analysis of the coupling process between the epistemic and cognitive facets of mathematical knowledge in the context of an exploratory-investigative task on patterns]. *Educación Matemática*, 27(1), 37-64. <https://doi.org/10.24844/EM>
- Pino-Fan, L., Godino, J., & Font, V. (2011). Faceta epistémica del conocimiento didáctico-matemático sobre la derivada [Epistemic facet of didactic-mathematical knowledge about the derivative]. *Educação Matemática Pesquisa*, 13(1), 141-178.
- Pino-Fan, L., Godino, J., & Font, V. (2016). Assessing key epistemic features of didactic-mathematical knowledge of prospective teachers: the case of the derivative. *Journal of Mathematics Teacher Education*, 21, 63-94. <https://doi.org/10.1007/s10857-016-9349-8>
- Pino-Fan, L., Godino, J., Font, V., & Castro, W. (2013). Prospective teacher's specialized content knowledge on derivative. In B. Ubuz, Ç. Haser, & M. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 3195-3205). CERME.
- Reed, C., & Long, D. (1997). Multiple subarguments in logic, argumentation, rhetoric and text generation. In D. M. Gabbay, et al. (Eds.), *Qualitative and Quantitative Reasoning* (pp. 496-510). Springer Verlag. <https://doi.org/10.1007/BFb0035644>
- Reid, D., & Zack, V. (2011). Aspects of teaching proving in upper elementary school. In D. Stylianou, M. Blanton, & E. Knuth (Eds.), *Teaching and Learning Proof across the Grades: a K-16 perspective* (pp. 133-145). Routledge; National Council of Teachers of Mathematics. <https://doi.org/10.4324/9780203882009-8>

- Reid, D., Knipping, C., & Crosby M. (2011). Refutations and the logic of practice. *PNA*, 6(1), 1-10.
- Reiss, K., & Renkl, A. (2002). Learning to prove: The idea of heuristic examples. *Zentralblatt für Didaktik der Mathematik, ZDM International Journal of Mathematics Education*, 34(1), 29-35. <https://doi.org/10.1007/BF02655690>
- Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, 8(3), 255-281. <https://doi.org/10.1007/s10857-005-0853-5>
- Rumsey, C., Guarino, J., Gildea, R., Cho, C., & Lockhart, B. (2019). Tools to support K-2 students in mathematical argumentation. *Teaching Children Mathematics*, 25(4), 208-217. <https://doi.org/10.5951/teacchilmath.25.4.0208>
- Sampson, V., Enderle, P., Grooms, J., & Witte, S. (2013). Writing to learning to write during the school science laboratory: Helping middle and high school student develop argumentative writing skills as they learn core ideas. *Science Education*, 97(5), 643-670. <https://doi.org/10.1002/sce.21069>
- Schoenfeld, A., & Kilpatrick, J. (2008). Towards a theory of proficiency in teaching mathematics. In D. Tirosh, & T. L. Wood (Eds.), *Tools and processes in mathematics teacher education* (pp. 321-354) Sense Publishers. https://doi.org/10.1163/9789087905460_016
- Shinno, Y., Miyakawa, T., Iwasaki, H., Kunimune, S., Mizoguchi, T., Ishii, T., & Abe, Y. (2018). Challenges in curriculum development for mathematical proof in secondary school: Cultural dimensions to be considered. *For the Learning of Mathematics*, 38(1), 26-30.
- Shulman, L. (1986). Those who understand: Knowledge growth in Teaching. *Educational Researcher*, 15(2), 4-14. <https://doi.org/10.3102/0013189X015002004>
- Simpson, A. (2015). The anatomy of a mathematical proof: Implications for analyses with Toulmin's scheme. *Educational Studies in Mathematics*, 90(1), 1-17. <https://doi.org/10.1007/s10649-015-9616-0>
- Soysal, Y. (2015). A critical review: Connecting nature of science and argumentation. *Science Education International*, 26(4), 501-521.
- Toulmin, S. (1958). *The uses of argument*. Cambridge University Press.
- Tristani, L., Sutawidjaja, A., As'ari, A., & Muksar, M. (2015). Modelling student mathematical argumentation with structural-intuitive and deductive warrant to solve mathematics problem. In *Proceeding of ICERD* (pp. 130-139).
- Turiano, F., & Boero, P. (2019). Beyond direct proof in the approach to the culture of theorems: a case study on 10-th grade students' difficulties and potential. In *Eleventh Congress of the European Society for Research in Mathematics Education* (No. 37). Freudenthal Group; Freudenthal Institute; ERM
- Van Dormolen, J. (1991). Metaphors mediating the teaching and understanding of mathematics. In J. Bishop, & S. Melling Olsen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 89-106). Kluwer. https://doi.org/10.1007/978-94-017-2195-0_5
- Van Eemeren, F. H., Houtlosser, P., & Henkemans, A. (2007). Argumentative indicators in discourse: A pragma-dialectical study (Vol. 12). Springer Science & Business Media. <https://doi.org/10.1007/978-1-4020-6244-5>
- Van Eemeren, F., Grootendorst, R., & Henkemans, A. (2006). *Argumentación: análisis, evaluación y presentación* [Argumentation: analysis, evaluation and presentation]. Biblos.
- Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 22-44). National Council of Teachers of Mathematics.

<http://www.ejmste.com>