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Problem Solving as a Professional Development Strategy for Teachers: A Case Study with Fractions

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ABSTRACT
In this paper we present a professional development course designed to impact on teachers' mathematical knowledge for teaching fractions. The main features of the course are the use of i) problem solving activities related with mathematical knowledge for teaching fractions ii) peer discussions and iii) monitor's interventions focused on answering with questions. The objective of this paper is to present some insights on how this type of course may contribute to the development of teachers' knowledge for teaching fractions. To do that we analyze in depth the case of one teacher, using the recording of his work during the course (video, audio, and written documents), his answers to two questionnaires and his responses to an interview six months after the course. Results show how the course's features contributed to the development of this teacher's specific fractions knowledge, knowledge of fractions and students, and of fractions and teaching.
Keywords: fractions, mathematical knowledge for teaching, peer discussion, problem solving, teacher professional development

INTRODUCTION

Curricular reforms being undertaken in many countries around the world place mathematical abilities as the core of mathematics learning, considering problem solving as a central ability that students should develop during their school years (e.g. Kaur & Yeap, 2009; NCTM, 2000). These reforms entail teachers to review their teaching models and to rethink the mathematics their students should learn. In some cases, this new perspective on mathematics learning also demands changes on teachers' conceptions about mathematics itself. These new curricula for mathematics pose a complex scene for in-service teachers who have mostly grown and have been trained in a completely different academic system than the one suggested by the reforms.

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State of the literature

- The influence of professional development programs on student learning depends on the way they affect teachers' knowledge and beliefs.
- Some features of professional development proposals that increase their effect on teachers' instruction are: content focus, active learning, coherence, duration and collective participation.
- Teachers' knowledge is a complex construct, composed of diverse and strongly interwoven parts

Contribution of this paper to the literature

- The type of activity in which teachers are involved is an important aspect to be considered in the analysis of the effect of professional development programs on teachers' knowledge.
- Problem solving offers the opportunity to design professional development strategies where creating scenes that model student centered learning classrooms. At the same time, it offers teachers the opportunity to go through their mathematical knowledge for teaching, in particular, their specialized content knowledge and their knowledge of content and students, and content and teaching.
- More research on the relationships between the characteristics of mathematics teachers' professional development programs and teachers' knowledge for teaching is needed.

In many countries, an additional difficulty to implement the reform is the weak mathematical knowledge of teachers, especially at the elementary level.

How do teachers may deal with this new situation? What type of actions are required for accompanying them into this process? Who should take charge of these actions and how? One way to address these questions is through professional development courses which should consider both, the implication of the curricular reforms on mathematics teaching and learning and the need for strengthening teachers' mathematical knowledge.

In this paper, we present a professional development course called *Fractions with Problem Solving*. The course, proposed for primary mathematics teachers, focuses on fractions with problem solving as a strategy. Its aim is to meet the new curriculum requirements having problem solving as a vehicle to review mathematical concepts and processes, as an ability to be developed by teachers and as an opportunity to model student centered learning classrooms. Using a case study, we explore how this type of courses should contribute to the development of teachers' mathematical knowledge for teaching (Ball, Thames & Phelps, 2008). The research question that guided the study is what features of the course do influence on the teachers' knowledge and how do they work to do that?

FRAMEWORK AND RESEARCH REVIEW

To conduct this study, we identify the potentially critical elements for the effectiveness of a professional development course from research literature (Desimone et al., 2002; Garet et al., 2001) and we use a case study to explore how these elements should work to moderate teachers' mathematical knowledge for teaching (Ball et al., 2008).

On the base of a broad study of different professional development courses in the United States, Garet et al. (2001) identified three core characteristics having significant positive effects on teachers' self-reported improvement in knowledge, skills, and changes in classroom practice: (a) the degree to which the activity had a content focus, that is, the degree to which it was focused on improving and deepening teachers' content knowledge; (b) the extent to which the activity offered teachers opportunities for active learning, such as opportunities to become actively engaged in a meaningful analysis of teaching and learning, for example, by reviewing students work or obtaining feedback on their teaching; and (c) the degree to which the activity promoted coherence in teachers' professional development, by incorporating experiences that were consistent with teachers' goals and aligned with state standards and assessments, and by encouraging continue professional communication among teachers. In the same line, Desimone et al. (2002) found that professional development courses focused on specific instructional practices increased teachers' use of those practices in the classroom. Furthermore, they found that specific features, such as active learning opportunities increased the effect of the professional development on teachers' instruction. All these research results were taken into account for the design of the course *Fractions with Problem Solving* whose main features will be given in the next section.

In relation with the mathematical knowledge that teachers need for teaching, we considered the characterization proposed by Ball et al. (2008), based on Shulman's works (Shulman, 1986; 1987). It considers two main domains of knowledge for teaching: *subject matter knowledge* and *pedagogical content knowledge.* Each of these domains is subdivided into three interrelated subdomains (**Figure 1**). The course we present in this study focuses on four of these six subdomains: *common content knowledge, specialized content knowledge, knowledge of content and students* and *knowledge of content and teaching*.

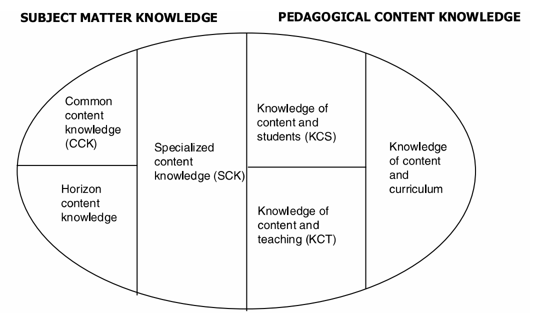


Figure 1. Domains of Content Knowledge for Teaching

As defined by Ball et al. (2008), *common content knowledge* (CCK) is the mathematical knowledge and skills that allow a person to give correct answers to a mathematical problem or to discover an error in a mathematical development or answer. It could be interpreted as what a person should know about mathematics when passing through school. In this sense CCK is strongly related with *mathematical proficiency*, defined by Kilpatrick et al. (2009) with five interwoven strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. On the other hand, *specialized content knowledge* (SCK) is a knowledge not typically needed for other purposes than teaching. It includes precise mathematical language, useful representations and justifications of algorithms (e.g. to differentiate between squares and rhombus, to use and interpret the graphic representation for adding fractions or to know why we multiply by the inverse of the divisor when dividing). CCK is not a mathematical knowledge unique to teachers, which is the case of *specialized content knowledge.*

On pedagogical subdomains, *knowledge of content and students* (KCS) combines knowledge about mathematics and about students when learning mathematics; common errors and typical difficulties students face when learning a new concept belong to this category, as well as to anticipate what students will consider interesting or confusing. Finally, *knowledge of content and teaching* (KCT) combines knowledge about mathematics and about teaching mathematics. It involves sequencing content for instruction, planning when to show an important example or deciding on a method or procedure to teach a given idea. During classroom instruction, KCT considers various types of decisions like when to stop for an explanation, for giving an example or pushing for deeper understanding.

The four components of the mathematical knowledge for teaching (MKT) discussed above were considered in selecting the mathematical activities proposed to teachers in the *Fractions with Problem Solving* course. They were also kept in mind when designing the questionnaires and the interview used as research instruments, and when analyzing the teachers' improvement in content knowledge for teaching.

METHODOLOGY

The course Fractions with Problem Solving

Fractions with Problem Solving was designed upon the request of an educational foundation concerned by their students' achievements on fractions and their teachers' knowledge about this mathematical topic. Thus, the overall purpose of this course was to enhance teachers' knowledge of fractions with emphasis on CCK and SCK, but also considering KCS and KCT. The main effort was put on the development of teachers' SCK, as a way to also access to teachers' CCK, KCS and KCT. In terms of the mathematical proficiency, emphasis was in conceptual understanding, operations and relations, problem solving, explanations and reasoning (Kilpatrick et al., 2009). Problem solving was chosen as a vehicle for getting into the mathematical content, considering coherence with the national curriculum, and teachers' needs.

The course ran for 25 hours, one week with 5 hours per day, after the end of the academic year and before summer vacations. In every 5-hours-session, teachers were organized in random groups of 2 or 3 teachers, they solved problems for about four hours and at the end of the session there was a plenary discussion. The workflow during the course is an essential feature of the professional development model we work on (Felmer & Perdomo-Díaz, to appear).

Mathematical problems used in the course

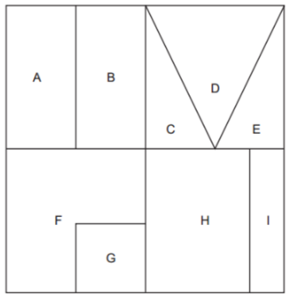
The selection of the mathematical activities or problems to be used in the course was based on the national curriculum and international research results about difficulties on fractions learning (e.g. Ball, 1990; Philipp, 2000; Ma, 2010; Wong and Evans, 2007). A set of 22 problems were selected for the course, distributed among the domains of MKT defined by Ball et al. (2008) as shown in **Table 1**.

Table 1. Problems distribution on MKT domains

CCK	SCK	KCS	KCT
13	5	3	1

In terms of content, first sessions were dedicated to the concept of fractions in their different facets, followed by addition and subtraction and the notion of least common multiple, and ending with the conceptual meaning of product and division of fractions. In **Table 2** we present some examples of the problems, with the mathematical content covered and the MKT subdomain related with each of them.

Table 2. Examples of problems used in the course

Problem 4: The Split Square [CCK, graphical representation of fractions] A square has been divided as shown. What is the fraction of the square represent by each letter?	 **Figure 2:** The split square
Problem 9. They are no the same [KCS, equality of fractions] Georg does not understand that fractions $\frac{2}{5}$ and $\frac{6}{15}$ represent the same number. He says if the denominator and the numerator are different then the numbers are different. He asks his uncle, who talked to him about equivalent fractions getting him even more confused. How could you solve Georg conflict?	
Problem 17. Addition of fractions [KCS, addition of fractions] Given the operation $\frac{1}{2}+\frac{3}{5}$, Patricia has answered that the solution is $\frac{1+3}{2+5}=\frac{4}{7}$. How would you explain to Patricia her error?	
Problem 19: The Subtraction [SCK, graphical representation of operations with fractions] How would you graphically represent $\frac{1}{5}-\frac{2}{3}$?	
Problem 22. Antonia's table [CCK, meaning of fractions, division of fractions] The surface of the table of Antonia is rectangle measuring 6/35 u^2. Knowing that one side of the table measures 3/7 u, how long is the other side?	

Work dynamics during problem solving

Teachers solved the problems in groups. The monitor provided problems to each group, one at a time, sequentially, according to the work pace of each group. The role of the monitor was mainly to stimulate group discussion, mathematical explanations, reasoning and conceptual understanding, in such a way that the solution of the problem came from teachers and not from the monitor. When teachers requested for some explanation or for some idea for starting or continuing with the current problem, the monitor answered them with a question and quickly left the group. When a group felt they had solved the current problem, the monitor asked them to explain their solution and the strategy used to get it, requesting for details and looking for clarification when a fuzzy argument was given. If the monitor realized that the problem was not yet solved or that one member of the group had not yet reached the solution, he asked a question and left the group. The monitor provided the next problem to the group only when he was convinced that every member of the group had solved the current problem and could explain the way the solution was obtained.

Plenary discussions

A plenary discussion was arranged during the last hour of each daily session. Monitor led the discussion with questions and he encouraged teachers' participation, but never gave explanations, interpretations or solutions. The source of knowledge and the conviction of a correct answer was the work previously carried out by the groups. The focus of the plenary discussion changed every day: the analysis of the solution to one of the problems; problem solving strategies; work dynamics considering the role of the monitor and participants; and possible teaching proposals for their students. Last day there was a general discussion about the course, about learning achievements and other accomplishments.

The case

This is an empirical qualitative study designed to shade light on how should the features of the course work to enhance teachers' mathematical knowledge for teaching. In order to explore the phenomenon in depth we consider a case study of one teacher. There were seven participants in the course, all mathematics teachers from grades 5^{th} to $8^{th.}$ For this study, we chose the case of John (pseudonym) for the analysis, because of the richness of information he provided at the different instances of the study.

Data collection

Data used for this case study come from different sources: two questionnaires, John's audio and video recording from each session and his working sheets, plenary discussions audio and video recording, and an interview.

The questionnaires

All teachers participating in the course filled two short questionnaires. The first one was answered at the end of the first session and included four open questions regarding the methodology of the course and the work dynamics during the session: How did you feel while solving the proposed problems? What did work in group bring you? What did plenary discussion bring you? What did the monitor bring you? The second questionnaire, answered at the end of the course, had six questions regarding teachers' professional experience, for example, the way they use problem solving in classroom, the way students work on them and if the problems given in the course could be used with their students.

John's recording and working sheets

John's group work was audio and video-recorded during the five sessions and all John's working sheets written during the course were copied. The same was done with other teachers participating in the course.

Audio and video recording of plenary discussions

Plenary discussions were audio and video-recorded with one general camera. We used all this material for the analysis of John's case, even episodes when John was not participating, since all the plenary discussions could have an influence on John's knowledge.

John's interview

Seven months after the course was given, a semi-structured interview was recorded for John. The interview had three parts; the first one was introductory, looking for general information like how many lectures were used to teach fractions and how long ago. The second part had the purpose of inquiring if there were some aspects of the course *Fractions with Problem Solving* that helped him to teach fraction this year, but without explicit mention of the course. The interviewer asked him about the way he taught different topics this year, his role and the role of his students and where did he learn his method for teaching. The last part of the interview was dedicated to specific questions about the possible influence of course in his teaching, if he felt better while teaching some topic and if the students learned more.

Data analysis

As this study focuses on teacher's mathematical knowledge and its evolution, the principal data for the analysis were John's recording and working sheets. Looking for episodes involving John's knowledge, recordings were watched and listened several times, together with John's working sheets. For example, John's questions to their peers or to the monitor, John's explanations, arguments, strategies or John's reactions to his peers or monitor's interventions. Relevant episodes about John's MKT were transcribed and coded as CCK, SCK, KCS or KCT. In the case of CCK, episodes were also coded according to the mathematical proficiency strand involved.

Plenary discussion recordings and other teachers' documents were used just to complement the information obtained by John's recordings and working sheets in case it was needed. John's answers to the questionnaires were mainly used to obtain general information about Johns' profile as mathematics teacher; its analysis consisted on reading, extracting and summarizing the information. The same process was used with John's interview. In this case, information selected corresponds to evidences of relationships between the features of the *Fractions and Problem Solving* course and the development of John's MKT.

The analysis of the research data for John's case was performed taken into account the research questions and the main lines defined in the framework, namely the type of mathematics knowledge and mathematical proficiency teachers need to teach and the characteristics that an effective professional development course needs to have.

RESULTS: THE CASE OF JOHN

At the time John participated in the course he had seven years of teaching experience, mainly in 3rd and 4th grade, with some little experience in 5th and 6th grade. From the interview we draw that he is devoted to his students, with good self-esteem, good self-assessment regarding content and teaching practices, curious and with good disposition to improve constantly. He had participated in various professional development instances, including a long program directed to primary teachers willing to teach from 5th to 8th grade, lasting for one year and a half, with more than 600 hours at a local university.

In the first questionnaire John stated that he uses "application problems that require more than one operation to be solved" in his lessons, with students working in couples. During the interview, John said that he uses problems "Because I feel the need for the child to find ways to solve problems, because then, I think, it ends up serving it for all situations".

In relation with the Fraction with Problem Solving course, we can say that John got really involved on it. John started working individually on the given problem; after a while he shared the solution or partial approaches to the solution with his peers. He was receptive on his colleagues' ideas, striving to understand and questioning them. Videos of the sessions show numerous episodes where John was reflecting on their own work and the work of his peers. Nevertheless, he just participated on the plenary discussions when the monitor asked for it. In the interview, John self-evaluated his knowledge about fractions with the highest possible mark in Chilean absolute scale from 1 to 7, except for two mathematical processes where he marked 5: turning fractions to decimals and vice versa, and decomposing numbers on prime factors. During the course, John indeed showed procedural skills; he was able to perform calculations and he correctly used the algorithms for operating with fractions. However, John resorted to cellular phone calculator whenever he needed the decimal version of a fraction. These are exhibitions of his proficiency on a procedural point of view of the CCK.

In the next subsections we will describe some episodes that reflect how the main features of the Fraction with Problem Solving course worked to contribute to the development of John's mathematical knowledge for teaching fractions.

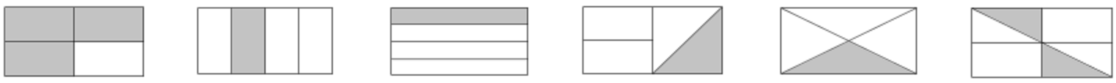
The type of problem and John's common content knowledge about fractions

Problem solving tasks used in the course contributed to make some John's conceptual difficulties emerge. It is the case of fractions' graphic representation. Solving problem 1 (**Table 3**), John and his group correctly identified what portion of the flag was shaded when divisions where just horizontal or vertical. However, they did not know what to do in (d). This episode reflects that *The School Banners* was really a problem for John and his group mates because they did not know a procedure that takes them directly to the answer of the activity (Schoenfeld, 1985).

Table 3. Problem related with CCK of fractions

Problem 1: The School Banners

Among sixth grade students, a contest was held to select the flag to represent them in the school anniversary week. The flag was to meet the following requirements: it must be rectangular, it must have only 2 colors and one of the colors should cover a quarter of the flag. Some of the designs submitted by the students are shown below (Figure 3). The teachers need some help, because they are not sure all flags meet the requirements. Help them and say which flag meets the requirements, justifying your selection.



a) b) c) d) e) f)

Figure 3: The School Banners

When the monitor was required by the group, he asked, "what if you divide the flag like the others?" From this question, John's group solved the problem. The idea of dividing a given figure in different ways was used later by John in Problem 4 (**Table 2**).

This episode shows that the graphical representations included in the problem played a key role for making John's weakness of CCK emerge and that the monitor's question contributed to John for overriding them.

Peer interactions and the development of John's common and specialized content knowledge and knowledge of content and teaching

John also displayed difficulties with some of the different meanings that fractions may have: part-whole, operator and number. The first episode about this was during the first plenary discussion, speaking about equivalent versus equal fractions. The video shows how John quietly asked: "how do we explain to a child that ¼ of 1kg is not the same than ¼ of 2 kg? Since it is the same amount, ¼". In another plenary discussion appeared that a fraction is a number and so it can be located on the real line. Facing this situation, John said that the place where a fraction is located on the real line "depends on the integer because if I have 100 in the real line and I want to locate where 1/3 is [...] it is different that if the real line has from 0 to 3. Then I can perfectly set 1/3, but if it is 100..."

During the interview, John extended this type of reflection toward the discrete and continuous model of fractions. Talking about the textbook he uses in his lessons, John said, "it does not make the distinction that the whole can be made of different parts. The lesson starts with a whole [...] at the middle of the chapter, the book shows them [the students] that the whole was not necessarily a unit, and give them examples with fruits." This reflection is related with Johns' KCT.

These are examples of the way peer interaction during the plenary discussions allows for arising John's misconceptions (CCK) at the same time that offers an opportunity to generate a discussion about mathematical knowledge that teachers need for teaching fractions (SCK) and reflect about methods or procedures for teaching that content (KCT).

Answering with questions and John's specialized content knowledge and knowledge of content and students

Another main feature of the course was the way the monitor reacted to requirements from group of teachers, he never gave an answer, he always made a question, looking for teachers' justifications of their answers, processes explanations or a new cognitive challenge. For example, some of the questions asked teachers to make sense of some of the procedures commonly used like to compare fractions, simplify or amplify fractions. This is directly related with SCK.

This particular aspect of monitor's role provoked that teachers, in particular John, started to question themselves. One example of this could be observed when John and his partner solved the problem *Antonia's table* (**Table 1**) using the fraction division algorithm. Once they solved the problem, John said, "for sure we will be asked about the reason justifying the algorithm that we used", and actually this happened shortly after.

John still recalled this situation during the interview and, even though he was not able to find the why of the algorithm for dividing fractions, he regarded the course saying "I went to the course to find the mathematical answer to something that is usually done mechanically: why the sum of fractions is done so, why the division of fractions is done so, and in finding the mathematical reason, horizons were opened for me, of how incorporating this strategy with the small ones [the students] which I work with at school, find the answer, or that they build it and try to find it". Upon the question: *In what sense was the course useful?* He answered, "to find the mathematical reason, to know that there is a reason. Because I may not always find it, but I know the reason exists, and I challenge myself to keep looking for it, keep looking, for example, why to divide a fraction one flips the numerator and denominator of the divisor and then multiply, and that I have not found yet."

Those are examples of how John started to question himself about the mathematical processes he uses and teach. From the "answering with questions" strategy of the monitor, John reflected about the explanation behind the used algorithms (SCK) and the needs and difficulties students might have to understand them (KCS).

FINAL DISCUSSION

Teaching is a profession constantly exposed to changes resulting from political decisions, social requirements, technological advances, didactical necessities, etc. Professional development programs play a central role in the way those changes are incorporated in teaching practices. There exists a wide diversity of mathematics teachers' professional development proposals, with different objectives, methodologies, etc. During the last decade, the amount of research on characterizing effective programs and evaluating their impact has increased. But still more studies about the way the features of the professional development influence on teachers and students are required (Desimone, 2009). It is the expectation of the authors that this paper contributes to the understanding of the form that PD can mobilize teachers' knowledge and practices.

The design of the course *Fraction with Problem Solving* includes the main characteristics that the international research literature associates with effective PD in terms of content, process and structure: focus on specific contents, active and collaborative participation, and working in professional communities (Garet et al., 2001; Koellner & Jacobs, 2015). It could be considered as a specified model since goals, resources and materials where designed and provided to reach a particular experience, although it has an adaptive component too in the way the monitor interacts with teachers (Koellner & Jacobs, 2015).

The episodes used for presenting John's case show how the course gave him the opportunity to discover some misconceptions about fractions (CCK), to realize the need of making sense of mathematical procedures and to know different problem solving strategies (SCK). During this process, John also reflected about the importance of having a deeper understanding of the mathematical concepts and processes to planning better lessons (KCT) and to attend students' questions and reasoning (KCS). This knowledge covers different facets of the mathematical knowledge for teaching as defined by Ball et al. (2008) and that are also present in other frameworks (e.g. Conner, Wilson & Kim, 2011; Pino-Fan, Assis & Castro, 2015).

Besides the contributions listed above, the workflow along the course also made John feel the same way his students feel when he poses problems, something well evaluated by the teacher. During the interview, John stated that "one tends to go to professional developments courses to find solutions" and in this course, he found "a course where I had to give the solution, which is a bit what I do with the kids, which is that: get the pencil, seek for the solution and tell me the solution". John added about the course: "it was the first time in which

I participated as a student under this system" and that, being in the place of his students he felt "comfortable, yet challenging, difficult."

Aside from the purpose of this research, the case of John reveals a reality taking place in many countries, including Chile. This reality is expressed usually in indirect ways, but here it appears in a crude and direct way, and it can be summarized as a lack of opportunities for pre-service and in-service teachers to develop their mathematical knowledge and their mathematics proficiency in a way that is sufficient to guide children in learning mathematics at school. John represents many teachers, potentially great teachers, with enthusiasm for their student learning, willing to give them the best, but lacking of the basic knowledge. This is especially dramatic if we take into account his youth as teacher, he has only seven years of teaching experience, and the various courses taken while teaching, including one with about 600 hours.

At this point one may ask, what is failing in initial formation of teachers? What is failing in professional development courses? The case of John suggests the need for research in order to understand the nature of the difficulties faced by teachers when addressing professional development courses and of carrying out research at the national level to design, implement and evaluate professional development strategies, something that it is already happening in other countries (e.g. Feuer, Floden, Chudowsky & Ahn, 2013).

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