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Profiles of young students' understanding of fractions on number lines

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Abstract

Number lines are acknowledged as an effective model for understanding fractions, yet students often face challenges in interpreting fractions on number lines. This study contributes to the field by investigating students' performance on fraction number line tasks that require the coordination of their fraction and number line knowledge. To explore this, test items were developed, and 122 fourth-grade students participated in the assessment. Students' written responses underwent analysis in three phases: a descriptive overview of overall student performance, latent profile analysis to identify subgroups with different competencies, and a qualitative analysis of each latent profile. The findings indicated lower performance among students across the tasks, revealing three distinct latent profiles with different competency characteristics: *intuitive*, *emergent*, and *advanced* understanding of fractions on the number line. From these findings, instructional implications were extrapolated for using number lines as a model for fractions.

Keywords: number line, fractions, latent profiles, young students

INTRODUCTION

It is well documented that fractions are a crucial concept for developing children's deep understanding and are considered as a predictor of success in later mathematics achievement (Sidney et al., 2019; Siegler et al., 2012). For instance, previous studies have shown that the ability to understand fraction concepts and manipulate fraction operations is strongly related to progression in algebraic thinking (Eriksson & Sumpter, 2021; Gunderson et al., 2019; Lee & Hackenberg, 2013; Reeder, 2017). However, it has also been argued that fractions are one of the most challenging concepts for children (Behr et al., 1983; Chiu & Hsieh, 2017; Karika & Csíkos, 2022; Lortie-Forgues et al., 2015; Siegler & Pyke, 2013) and that they comprise multifaceted subconstructs (Charalambous & Pitta-Pantazi, 2007; Kieren, 1976).

Using visual representations during instructions and problem-solving activities is an effective way to support students in making sense of fraction concepts (Sidney et al., 2019). Especially, the number line serves as a conceptual tool for developing awareness of units, understanding fraction magnitudes, recognizing various fraction types, and establishing fraction equivalence (Bruce et al., 2023; Cramer et al., 2017; Sidney et al., 2019; Siegler et al., 2011). Engaging in number line activities such as labeling various fractions or units enables students to apply their prior understanding of fundamental fraction ideas and fosters the development of more advanced fraction schemes (Charalambous & Pitta-Pantazi, 2007; Cramer et al., 2019; Hackenberg et al., 2016). In addition, Saxe et al. (2013) assert that number lines play a crucial role in bridging students' initial knowledge of integers with their subsequent knowledge of fractions. Despite the significance and importance of the number line, however, several studies have documented students' difficulties in comprehending and representing fractions on a number line (Izsák et al., 2008; Olive & Vomvoridi, 2006). For example, students often struggled with determining the unit on the number line (Behr et al., 1983; Saxe et al., 2013), and focused more on counting tick marks rather than on counting the intervals when naming a marked point (Shaughnessy, 2011), and applied inappropriate whole number reasoning to locate fractions on the number line (Petit et al., 2015). This suggests that a number line representation may also impose an additional cognitive load on students' thinking (Tunc-Pekkan, 2015) and students may use number lines in ways that differ from their teachers' expectations, resulting in number lines being perceived not as an affordance but as a constraint (Patahuddin et al., 2017).

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Contribution to the literature

- This study lays the foundation for development through a systematic review of previous research.
- For an in-depth understanding of students' performance on fraction number line tasks, it employed latent profile analysis (LPA) to identify distinct categories of students with varying characteristics.
- This study provides characteristics of three distinct latent profiles both quantitatively and qualitatively.

Table 1. Summary of fraction schemes (Hackenberg, 2013)

Fraction scheme	Description	Operation involved
Parts-within-wholes fraction scheme	Fractions as parts within wholes	Partitioning
Part-whole fraction scheme	Taking a part out of a whole without mentally destroying whole	Partitioning & disembedding
Partitive unit fraction scheme	Partitioning a shape into equal parts, taking one of partitions, & iterating it to make a whole	Partitioning, disembedding, & iterating
Partitive fraction scheme	Producing proper fractions as results & fractions as measurable quantities	Partitioning, disembedding, & iterating
Iterative fraction scheme	Producing both proper & improper fractions	Splitting & disembedding

Given the dissonance between the significance and the challenges of the number line, this study posits that the number line is a beneficial and advantageous model for enhancing students' understanding of fundamental concepts related to fractions. Several studies have documented students' strategies or misunderstandings when tackling fraction number line tasks and provided practical implications for improving students' performance (e.g., Behr et al., 1983; Cramer et al., 2017; Tunc-Pekkan, 2015; Yanik et al., 2008). Aligning with young students' understanding associated with number lines documented in previous studies, this study further delves into the possibility of identifying subgroups of students with distinct competence profiles through LPA and explores the characteristics of each latent profile both qualitatively and quantitatively. Specifically, the aims of this article were, as follows:

(a) to comprehend students' overall performance in fraction number line tasks,

(b) to identify profiles of students that reflect distinct performances in solving fraction number line tasks, and

(c) to scrutinize the characteristics of each profile that reveal varying understandings of fractions on the number line.

BACKGROUND LITERATURE

Fraction Understanding & Number Line Representation

Previous studies have delineated diverse interpretations of fractions, categorizing them into five subconstructs such as part-whole, quotient, operator, ratio, and measure (Behr et al., 1983; Kieren, 1976). The comprehension of fractions is grounded in knowledge of these subconstructs both individually and in their interconnections. According to Charalambous and Pitta-Pantazi (2007), fifth and sixth graders exhibited the highest performance on tasks related to the part-whole subconstruct, while their performance on tasks associated with the measure subconstruct was notably lower compared to other subconstructs.

Hackenberg (2013) suggested the fraction scheme theory as a framework for characterizing the developmental progression of students' fractional knowledge, as outlined in **Table 1**. The construction of fraction schemes is based on key mental operations, including,

(a) the partitioning operation, making a quantity into equal parts,

(b) the iterating operation, repeatedly instantiating a fractional part to create a larger fraction,

(c) the disembedding operation, extracting apart from a whole without mentally destroying the whole, and

(d) the splitting operation, a composition of partitioning and iterating.

It is noteworthy that students begin to mark fractions as measurable quantities upon being to be able to construct the partitive fraction scheme.

Among the various representations designed to aid students in understanding fractions, the number line is one of the most important models for providing a key link between conceptual and procedural knowledge of fractions (Cramer & Wyberg, 2009). Although the number line may visually appear as a simple line segment with arrows at each end and points placed at equal intervals (Teppo & van den Heuvel-Panhuizen, 2014, p. 46), its mathematical significance often varies considerably depending on an individual's emphasis on specific aspects of the representation. According to Gunderson et al. (2019), "the number line's uni-dimensionality is beneficial for learning because it aligns with a conceptual feature of real number magnitudes" (p. 15). Furthermore, Saxe et al. (2013) characterize the number line as a hybrid representation that involves the coordination of numeric and linear units.

By employing this linear measure model during fraction instructions, students can conceptualize fractions as numbers and diminish misconceptions that may arise from viewing fractions as representing two separate whole numbers. This is reinforced by the fact that every unique point of the number line corresponds to rational numbers (Witherspoon, 2019). Furthermore, the number line approach supports students' understanding of the relations between whole numbers and fractions, as well as the equivalence and order of fractions. This is due to the inherent structure of the number line, where numbers increase in magnitude from left to right, and two numbers assigned to the same point represent the same value (Saxe et al., 2013).

Given the significant role of the number line in developing and assessing the measure subconstruct of fractions (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007), this study asserts that the inherent nature and characteristics associated with the number line hold the potential to support the development of students' fraction schemes and concepts.

Students' Performance on Number Line Tasks

Previous studies (Behr et al., 1983; Cramer et al., 2017; Diezmann & Lowrie, 2006; Hannula, 2003; Pearn & Stephens, 2007; Tunc-Pekkan, 2015; Yanik et al., 2008) have documented the students' difficulties when working with number lines. Students have revealed lower competencies in locating or interpreting fractions on number lines compared with other alternative fraction representations such as circles or rectangles. Charalambous and Pitta-Pantazi (2007) analyzed students' understanding of the five subconstructs of fractions and found that the measure subconstruct was the most challenging, while the part-whole subconstruct was the easiest for students. In their study, discrete items or rectangular shapes were used for part-whole tasks, and number lines were used for measure tasks. Tunc-Pekkan (2015) also observed that students faced greater challenges when working with number lines as compared to other representations across various problem types. Surprisingly, regardless of the types of representations, students consistently performed lower on items that required creating the unit from given improper fraction quantities. The study suggested that the effectiveness of number lines for students might vary according to problem types.

Additionally, determining the unit on the number lines often posed difficulties for students (Cramer et al., 2017; Pearn & Stephens, 2007; Witherspoon, 2019; Yanik et al., 2008). They often tended to perceive the entire number line as the unit, rather than recognizing the intervals equal to the distance between zero and one, as the unit (Behr et al., 1983; Cramer et al., 2017; Ni, 2000). Cramer et al. (2019) highlighted that "students often draw on prior experiences with paper strips or fraction circles and misinterpret the entire segment of a given number line as the unit" (p. 181). According to Witherspoon (2019), for instance, only 31.0% of the students could locate 1/2 on a zero-to-three number line, while 96.0% could successfully locate 1/2 on a zero-to-one number line.

Some research has reported students' confusion with tick marks and partition errors on number lines (Cramer et al., 2017). Shaughnessy (2011) pointed out that students often focused on counting tick marks rather than considering the distances between intervals, and they employed strategies such as a two-count strategy related to the part-whole subconstruct. Teppo and van den Heuval-Panhuizen (2014) described in their work, stating that numbers presented in number lines are associated either with points or with directed lengths, and observing that "a point representation indicates a counting-based conception, … a directed length representation reflects a measurement-based conception" (p. 48). This aligns with Earnest (2007), who asserted students' incorrect use of tick marks as a counting function without considering the distance between tick marks.

Considering the reorganization hypothesis proposed by Steffe (2002) that "children's fractional schemes can emerge as accommodations in their numerical counting schemes" (p. 267), students' difficulties with fractions on number lines may be attributed to their whole-number reasoning. For instance, when arranging unit fractions on the number line, students tended to focus solely on the magnitudes of their dominators, leading them to position fractions such as one-fifth to the right of one-half (Pearn & Stephens, 2007; Petit et al., 2015). Pearn and Stephens (2007) termed this approach as "larger-is-bigger-thinking" (p. 604), illustrating students' weakness in establishing connections between whole numbers and fractional parts of the number line.

A broad conclusion drawn from the research is that, in comparison to other fraction representations, students find number lines more challenging, and the difficulties encountered in this context appear unique to this representation. While previous research extensively documents students' performances using number line representation, there seems to be a relative lack of attention devoted to investigating variations in their performances. This study aims to extend prior research by closely scrutinizing the strategies employed by young students in number line tasks and identifying distinct classes of students representing diverse performances. Our specific objective is to elucidate the approaches students use in solving number line tasks to better understand the variations in their performances.

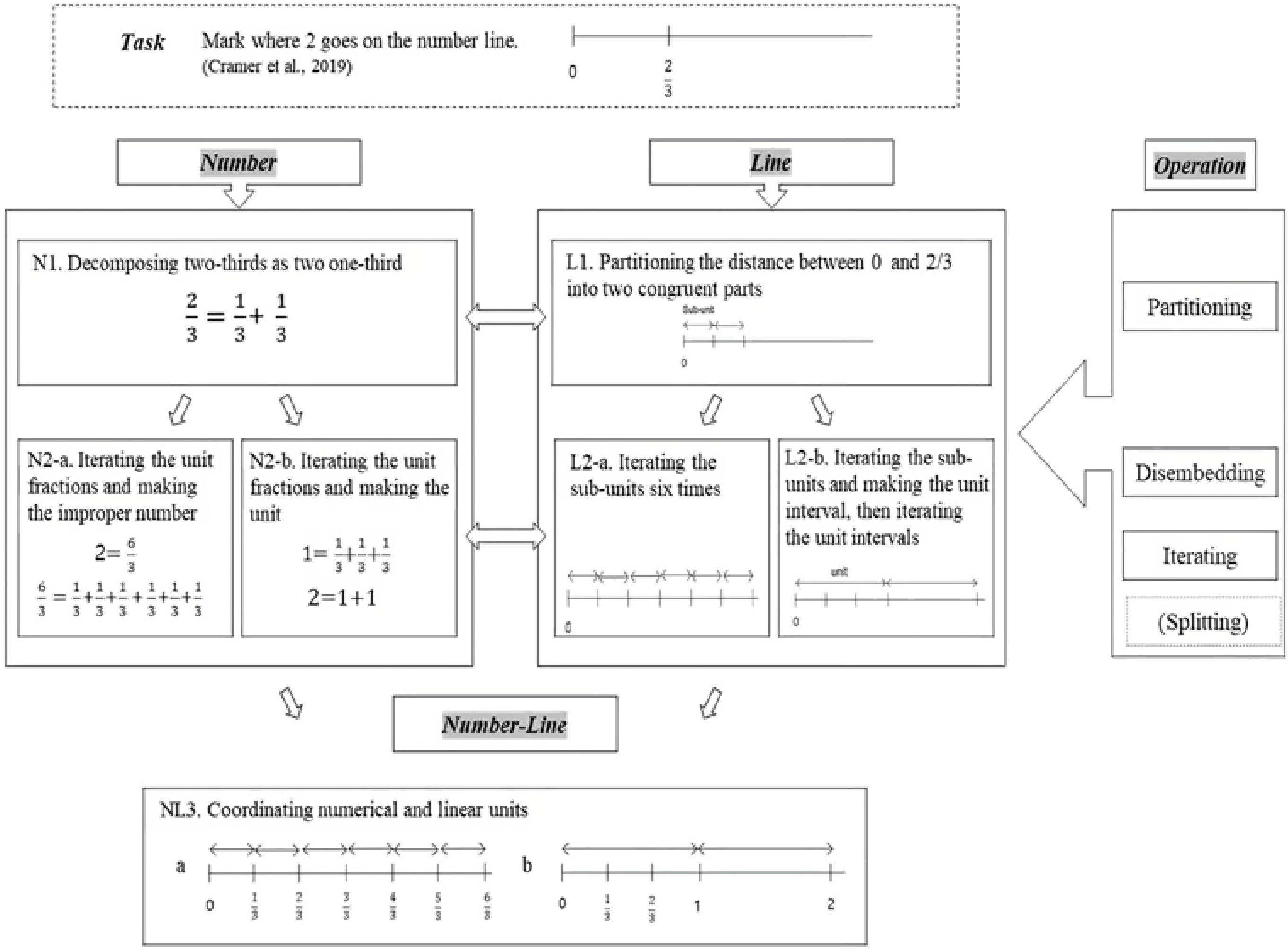


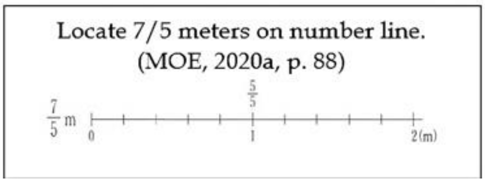
Figure 1. Framework for analyzing conceptual activities for task (adapted & revised from Saxe et al., 2010)

Cognitive Framework for Conceptualizing Students' Fraction Understanding of Number Line

To denote a fraction on a number line, students need to draw upon their existing knowledge about both numbers and lines (Saxe et al., 2013). According to Cramer et al. (2017), making sense of the number line as a model for fractions requires students to coordinate visual and symbolic information, including the units, partitioning, and relations with other numbers. In this investigation, we have modified a cognitive framework derived from prior research that focused on the operations inherent in fraction schemes (Hackenberg et al., 2016; Steffe & Olive, 2010) and conceptual activities that involve coordinating numerical and linear units on the number lines (Saxe et al., 2010; Tunc-Pekkan, 2015). By applying this framework, the current study conceptualizes students' fractional understanding of number lines. **Figure 1** illustrates the interwoven conceptual activities and the involved operations for the task of placing 2 on the number line, with only zero and 2/3 marked with tick marks.

Labeling two on the number line involves three interconnected conceptual activities: the treatment of numerical units, the treatment of the linear units, and the coordination of numerical and linear units (Saxe et al., 2010, p. 438). Initially, in terms of treating numerical units, 2/3 can be decomposed into two one-thirds (N1 in **Figure 1**), and this unit fraction can compose the targeted number two by either adding or multiplying it six times (N2-a in **Figure 1**). Alternatively, the unit fraction can compose unit 1 and the unit is added twice to achieve two (N2-b in **Figure 1**). Next, concerning the treatment of linear units, the distance between zero and 2/3 can be divided into two congruent segments (L1 in **Figure 1**), and this segment can be repeatedly added to form either the targeted segment (2) (L2-a in **Figure 1**) or the unit segment (L2-b in **Figure 1**). Lastly, the coordination of numerical units and linear units can result in a length of two units on the number line (NL3 in **Figure 1**).

The operations utilized in solving this task involve splitting and disembedding, both of which are associated with the construction of the iterative fraction scheme (Tunc-Pekkan, 2015). By constructing this scheme, students can label 2 on the number line by partitioning the given quantity into halves and simultaneously iterating 1/3 to reach 2. Throughout this process, students effectively employ 1/3 as a measurement unit, constructing three-level-of units, 1/3,



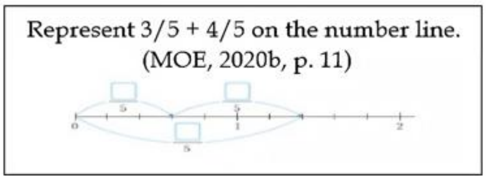


Figure 2. Examples of number lines presented in Korean mathematics textbooks (Ministry of Education [MOE] mathematics textbooks)

3/3, and 2 (or 6/3). It is crucial to note that when partitioning 2/3 into two equal parts, the result is two one-thirds, not two halves.

In summary, conceptual activities inherent in solving fraction number line tasks involve synthesizing the regulation of numerical and linear units independently, while also coordinating and executing operations simultaneously. The current study applied this framework to examine students' proficiency in fraction number line tasks and to delineate features of their comprehension of fractions when represented on number lines.

METHOD

Participants

This study involved three randomly selected elementary schools from three different school districts. The participants included 122 students in the second semester of 4th grade (65 males and 57 females) with moderate achievement in mathematics. None of them were identified as having major learning disabilities or other cognitive disadvantages, ensuring no one was excluded. The selection of this grade was done intentionally, as fraction concepts and number line representation associated with the concept of fractions are covered during the third and fourth grades of the Korean elementary mathematics curriculum.

The standards in the Korean mathematics curriculum for grade 3 and grade 4 include topics such as comparing fractions, understanding various types of fractions and their relationship (e.g., unit fraction, proper fraction, improper fraction, and mixed number), and performing addition and subtraction of fractions. Notably, the use of number lines is not explicitly mentioned in this written document. On the other hand, mathematics textbooks used in grade 3 and grade 4 incorporate diverse representations, including circles, rectangles, fraction bars, and number lines, to facilitate the learning of fractions. **Figure 2** provides examples of typical number lines presented in the fraction-related units of the textbooks, featuring equidistant intervals and consecutively sequenced numbers.

Instrument

An instrument was developed to investigate students' understanding of fractions on number lines, incorporating three types of tasks derived from previous studies:

(a) locating a fraction on the number line (Peran & Stephen, 2007; Tunc-Pekkan, 2015),

(b) naming the fraction on the number line (Larson, 1980; Saxe et al., 2007), and

(c) reconstructing the unit on the number line (Cramer et al., 2019).

Most of the tasks were adopted as they were in previous studies, while some tasks were modified by altering the fractions originally presented to avoid repetition of fractions across the tasks and to consider the participants' experience with the number line. For instance, the task intended for representing 3/4 on a zero-to-four number line was modified due to the repeated occurrence of 3/4 in other tasks from prior studies. Instead, it was adjusted to represent 2/3 on a zero-to-three number line. Furthermore, considering that third-grade students have limited experience in dividing number lines, the task originally intended for representing 2/3 on a zero-to-one number line was revised to represent 3/4 on a number line, which was deemed more accessible for the students.

The first type of task aimed to investigate students' ability to accurately place a given fraction on a number line. As the number lines in the tasks were either incomplete or empty, students needed to construct operations of partitioning, disembedding, and iterating. This process involved considering the unit of the number line as well as the denominators and numerators of fractions. As the length of number lines can impact students' performance (Larson, 1980), we incorporated tasks that required students to mark the proper fraction on zero-to-two, or zero-to-three number lines. In the second type of task, students were required to identify the names of the fractions on the number lines. Some of the number lines were not equidistant intervals and required partitioning into intervals of unequal lengths. In some tasks, a point on the number line was labeled as two equivalent fractions, depending on how students partition the intervals of the unit. The third task type was designed to assess whether students could determine the unit or whole numbers when given two labeled points on the number line, neither of which was zero or one. Students need to construct a splitting operation for constructing the unit on the number line.

Table 2. Specification of test in study

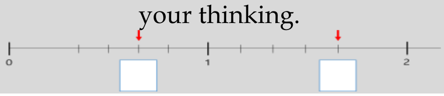
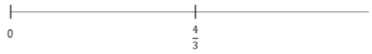
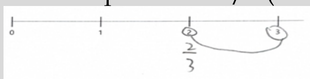
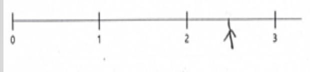
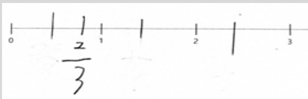
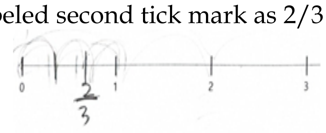
Type	Task	Fraction(s) involved	Type of number line	Example
Locating fraction on number line	1	3/4	Incomplete	#2. Show where 2/3 is on number line. 
	2	2/3	Incomplete	
	3	1/5, 2/5, & 3/5	Empty	
	4	1 2/8	Empty	
Naming fraction on number line	5	1/3, 2/3, 1 1/3, & 1 2/3	Filled	#6. Name fraction on number line. Explain your thinking. 
	6	4/6 & 1 4/6	Incomplete	
	7	2/8 (1/4)	Incomplete	
	8	5/6	Incomplete	
Reconstructing unit on number line	9	2	Incomplete	#11. Show where 1/2, 1 are on number line. 
	10	1 1/5	Incomplete	
	11	1/2 & 1	Incomplete	
	12	0 & 1	Incomplete	

Table 3. Assessment scoring framework

Level	Level description	Example (task 2: locate 2/3 on a 0-to-3 number line)
0	Not completed task	
1	Not considered magnitude of fractions & randomly placed fractions on number line	Labeled a point 2 as 2/3 (S46).  Marked an arrow between 2 & 3 (S77). 
2	Estimated position of a fraction on number line without conducting operations like partitioning & iterating	Labeled a tick mark between 0 & 1 (S3). 
3	Labeled a fraction on number line using operations like splitting or disembedding. Included insufficient aspects in students' performance (e.g., counting number of tick marks instead of number of intervals, partitioning a unit unequally).	Unevenly partitioned distance between 0 & 1 into thirds & labeled second tick mark as 2/3 (S40). 
4	Correctly labeled a fraction on number line by performing operations like splitting & disembedding.	Equally partitioned distance between 0 & 1 into thirds & labeled second tick mark as 2/3 (S50). 

The preliminary test was administered to five fourth-grade students to assess their comprehension of the task wording and the appropriateness of the task levels. Additionally, some feedback on the tasks from mathematics educators was received, resulting in minor revisions to certain terms and the task sequence.

Table 2 represents the finalized version of the test. Regarding the types of number line, filled number lines refer to ones that include appropriately partitioned equidistant points to represent given numbers. Incomplete number lines refer to ones that have uneven intervals between tick marks that require additional partitioning. Empty number lines lack both points and numbers, necessitating additional work such as labeling the units and subunits.

Procedure

In the fall of 2021, the assessment was administered by the teachers in their classrooms for approximately 40 minutes. The test was not a timed test so the teachers could allow more time than 40 minutes if necessary. The directions of assessment were provided for teachers to read to their students during administration.

The student's written records from the number line tasks were independently coded by two raters following a rubric created for the assessment scoring framework. As the aim of this study was to investigate students' understanding of fractions, we did not score students' responses either right or wrong. Instead, a five-point ordinal scale, as shown in Table 3, was used for scoring students' performance.

Two independent raters with doctor's degrees in mathematics education double-coded half of the data (n=61), and the inter-rater reliability was satisfactory to very good (.927). Any disagreements between the raters were discussed, and then the final coding for each item was adjusted accordingly. Table 3 presents examples of students' responses at different levels for task 2.

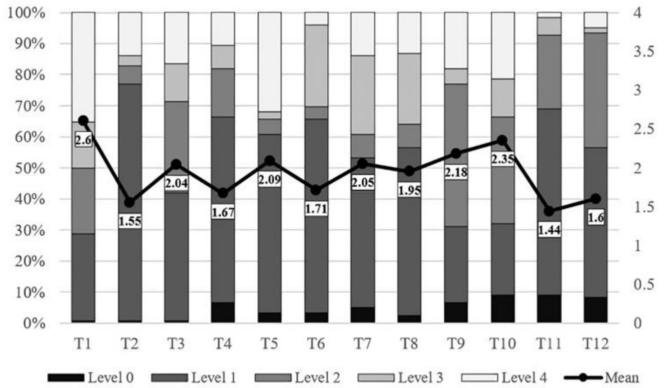


Figure 3. Distribution of performance levels & means (Source: Authors' own elaboration)

The item proved to be challenging for students because the given number was a proper fraction, yet the length of the number line exceeded one. Specifically, the students in level 1 appeared to apply the part-whole subconstruct or considered the given fraction as two separate numbers, indicating a lack of understanding of the measure subconstruct. The students in levels 2, 3, and 4 appeared to recognize the magnitude of 2/3 and made attempts to locate the fraction between zero and one on the number line, although the operations involved in their performances differed.

Data Analysis

The data analysis was undertaken in three phases. The first phase involved a descriptive analysis of overall students' performance. The mean values and standard deviations of each item, and the correlation coefficients between the items were calculated using SPSS 20.

In the second phase, an LPA was carried out using the software Mplus 8.10 (Muthén & Muthén, 2017) to identify categories of students with distinct characteristics. LPA is a statistical method used to find subtypes of related cases (latent classes) using a set of observed variables (Henry & Muthén, 2010, p. 193). The number of latent classes was determined based on different information criteria:

(a) the Akaike information criterion (AIC) (Akaike, 1987) and the adjusted Bayesian information criterion (BIC_{adj}) (Schwarz, 1978) in which lower absolute values indicate a better fitting model,

(b) the entropy values that approach one mean more certainty in the resulting classification, and

(c) two likelihood-ratio tests, Lo-Mendell-Rubin adjusted LRT test and bootstrapped parametric likelihood ratio test (BT LRT), which a low significant p-value indicates that current k-class model is more appropriate than k-1 class model.

This study relied more on BT LRT, as it is considered as most important (Nylund et al., 2007). Moreover, multiple analysis of variance (MANOVA) and post-hoc analysis were conducted to explore the differences between the profiles of students.

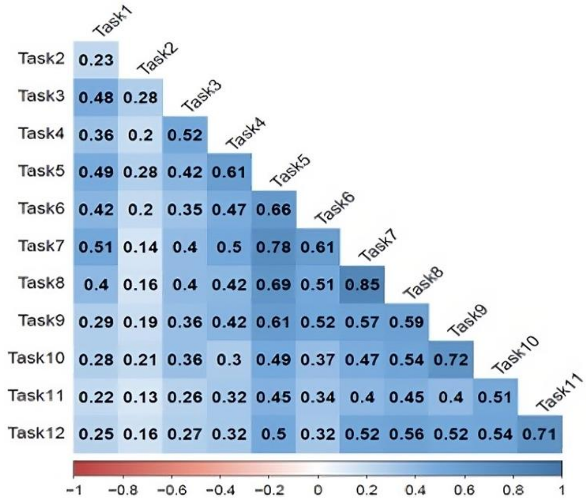


Figure 4. Correlation coefficients of items (Source: Authors' own elaboration)

In the final phase of the analysis, the characteristics of each latent profile were qualitatively examined using students' written responses. We described the strengths and difficulties observed in students' performance, providing illustrative examples of their responses. By comparing the similarities and differences in students' performance across the latent profiles, pedagogical insight into the use of number lines for teaching and learning the concept of fractions was provided.

RESULTS

Descriptive Results

Figure 3 reports the distribution of students' performance levels across twelve tasks with mean values for each task. The first thing to note is that performance level 1 and level 2 were predominant across almost all tasks, with lower proportions observed for performance level 3 and level 4. Especially, task 11 (level 4=1.6%) and task 12 (level 4=4.9%) had a limited number of students who accurately performed the number line tasks. The second important result is that the mean values of the tasks varied from 1.44 (task 11) to 2.60 (task 1), indicating distinct competencies among students in treating number lines. Specifically, fourth graders in this study demonstrated higher means on task 1 (mean [M]=2.599, standard deviation [SD]=1.517), task 10 (M=2.351, SD=1.201), task 9 (M=2.175, SD=1.057), and task 5 (M=2.093, SD=1.932) compared to other tasks. Conversely, they exhibited lower means on task 11 (M=1.44, SD=0.481), task 2 (M=1.550, SD=1.157), and task 12 (M=1.56, SD=0.597). In general, tasks with higher mean values featured a zero-to-one number line or involved proper fractions, while tasks with lower means incorporated number lines exceeding a length of one or included improper fractions.

Table 4. Fit indices for different numbers of classes

K	AIC	BIC	SABIC	Entropy	Class probabilities	Class size (%)	BLRT p
2	3,448.686	3,552.434	3,435.448	0.990	0.996/1.000	66.0/34.0	0
3	3,274.978	3,415.179	3,257.089	0.989	0.993/1.000/1.000	61.0/ 7.0/32.0	0
4	3,203.271	3,379.924	3,180.731	0.982	0.986/1.000/0.999/1.000	55.0/7.0/6.0/32.0	0
5	3,120.586	3,333.692	3,093.395	0.979	0.983/1.000/1.000/0.998/0.994	55.0/7.0/6.0/20.0/11.0	0

Note. K: Number of classes; AIC: Akaike information criterion; BIC: Bayesian information criterion; SABIC: Sample-size adjusted BIC; & BLRT: Bootstrapped likelihood ratio test

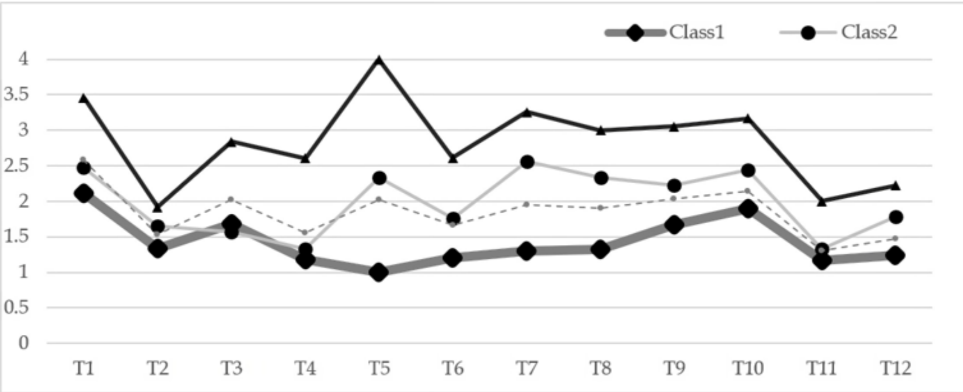


Figure 5. Distribution of means for three classes (Source: Authors' own elaboration)

Figure 4 displays the correlations across all items, where positive values span from 0.144 to 0.847. Here, Kendall's rank correlation coefficient, suitable for ordinal variables, was employed. In terms of task types, strong and consistent relationships are evident among tasks that are related to naming a fraction on the number line and reconstructing the unit on the number line.

Exceptionally, task 2 exhibited no significant correlations with other tasks. The lack of moderate or high correlations between tasks in different types indicates that they can be distinguished.

Number of Classes

LPA was conducted to determine the number of classes based on the student's performance in fraction number line tasks. We steadily increased the number of classes until the fit indices signaled that the number of classes was sufficient, rendering an additional class unnecessary (**Table 4**).

The values of information criteria, such as AIC, BIC, and SABIC decreased with an increasing number of classes. Given that small sample sizes may not support reliable data collection from classes (Nylund-Gibson & Choi, 2018), a three-class model was decided as the most appropriate one along with consideration taken for BLRT and class sizes. The entropy for this model was 0.989, and the posterior probabilities for the classes showed that the model had a high agreement in definitively placing most individuals into a particular class (all latent class possibilities >0.993).

Furthermore, the three-class model also revealed the most interesting profiles with qualitative differences. Since the inclusion of more classes only resulted in quantitative distinctions, the decision was made to use the three-class model for this analysis.

Profiles of Students' Performance on Fraction Number Line Tasks

Figure 5 shows distribution of means for three classes of students in the fraction number line tasks (see **Table 5** for detailed values of mean and standard deviation).

The line thickness in the graph is proportional to the number of cases within each latent class and indicates that class 1 students are the most abundant, followed by class 3, and then class 2. In most tasks, the means of class 2 and class 3 were higher than those of class 1. A significant difference in means by class was observed in task 5, indicating that the task is appropriate for diagnosing student's understanding of fractions on the number line. In contrast, task 2 and task 11 showed low means across all classes. Considering contents of tasks, this result suggests that students, even those in higher distinction classes, have insufficient understanding of recognizing fractions as a single number.

Class 1 students exhibited a lower level of performance compared to the other two classes on most of the tasks. Except for task 1 (M=2.120, SD=1.150), mean values for this class ranged from 0.96 (task 5) to 1.89 (task 10), revealing their challenges in working with fraction number line tasks. Class 2 students exhibited relatively satisfactory performance in specific items, including task 7 (M=2.560, SD=1.130), task 1 (M=2.450, SD=1.330), and task 5 (M=2.330, SD=0.500), indicating their ability to compare the magnitude of fractions on number lines. However, the mean values for all items were below three, suggesting that class 2 students often faced

Table 5. Means (Ms) & standard deviations (SDs) of three classes & results of MANOVA

Task	Class 1 (n=74)		Class 2 (n=9)		Class 3 (n=39)		*F*	*p*	η_p^2	Scheffe
	M	SD	M	SD	M	SD				
1	2.12	1.150	2.45	1.330	3.46	0.850	19.566	0.000	0.249	a & b<c
2	1.33	0.896	1.65	1.000	1.92	1.326	4.374	0.015	0.067	a<c
3	1.69	0.880	1.57	1.014	2.84	1.180	16.238	0.000	0.215	a<c
4	1.18	0.547	1.33	0.500	2.60	1.274	38.409	0.000	0.390	a & b<c
5	0.96	0.200	2.33	0.500	4.00	0.000	2,460.150	0.000	0.980	a<b<c
6	1.20	0.616	1.76	1.093	2.61	0.963	45.459	0.000	0.431	a & b<c
7	1.29	0.706	2.56	1.130	3.26	0.751	97.166	0.000	0.62	a<b<c
8	1.32	0.736	2.33	1.225	3.00	1.000	51.376	0.000	0.461	a<b & a<c
9	1.67	0.801	2.22	0.667	3.05	1.135	32.309	0.000	0.349	a<c
10	1.89	1.007	2.44	1.014	3.17	1.306	15.905	0.000	0.207	a<c
11	1.16	0.481	1.33	0.707	2.00	1.005	13.871	0.000	0.185	a<c
12	1.24	0.535	1.78	0.972	2.22	0.999	20.326	0.000	0.251	a<c

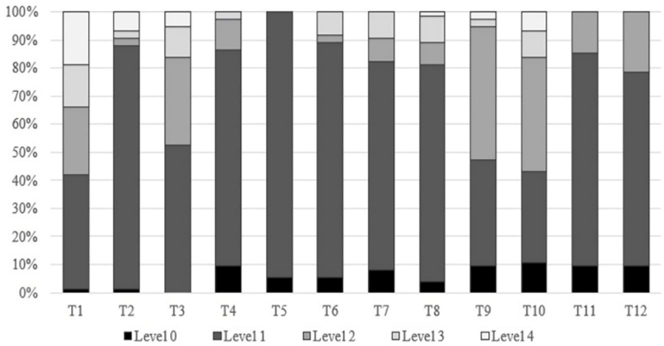


Figure 6. Distribution of performance levels in class 1 (Source: Authors' own elaboration)

difficulties working with number lines such as partitioning and iterating.

Class 3 students demonstrated higher competence across all items compared to the other two classes. They displayed a notably strong performance in certain items, such as task 5 (M=4.000, SD=0.000), task 1 (M=3.460, SD=0.850), task 7 (M=3.260, SD=0.751), and task 10 (M=3.170, SD=1.306), surpassing the threshold of the mean values above three.

Overall, labels were assigned to the latent classes based on these quantitative results. First, as class 1 students exhibited below-average performance and a lack of understanding in both fractions and number line representations, this class represents an *Intuitive understanding of fractions on number line* profile. Second, class 2 students had above-average performance in some tasks and below-average performance in others, making this class represent an *Emergent understanding of fractions on number line* profile. Finally, class 3 students displayed outperformed performance on all types of tasks, indicating an *Advanced understanding of fractions on number line* profile.

MANOVA was carried out to examine differences among the three profiles of the students. **Table 5** shows the means and standard deviations of the three classes along with MANOVA results. The analysis results revealed statistically significant differences among the three profiles of students (Pillai's trace=1.062, F=10.198, p<.001, η_p^2=0.531). Most substantial differences among three classes were observed in task 5 (F=2,460.150, p=0.000, η_p^2=0.980), task 6 (F=45.459, p=0.000, η_p^2= 0.431), task 7 (F=97.166, p=0.000, η_p^2=0.620), and task 8 (F=51.376, p=0.000, η_p^2=0.461), all of which are associated with the task type of naming a fraction on the number line. Furthermore, post hoc analysis (Scheffé post-hoc criterion) indicated that the statistically significant differences in the performance between class 1 and class 3 were more apparent compared to those in other classes, indicating that class 3 students outperformed class 1 in all tasks. Class 2 students outperformed class 1 students in three tasks (tasks 5, 7, and 8), while class 3 students exhibited higher performances than class 2 students in five tasks (tasks 1, 4, 5, 6, and 7). Most tasks that showed statistically significant differences among the classes were related to the task type of naming a fraction on the number line. Additionally, locating fractions or reconstructing the unit on the number line appeared to be challenging for students, even if they were able to recognize the names of fractions on the number lines.

Characteristics of Latent Profiles

Intuitive understanding of fractions on number line profile

Figure 6 illustrates the distribution of performance levels of class 1 students. Considering the criteria for scoring performance levels (see **Table 3**), the predominant level 1 performance observed across tasks suggests a deficiency in their ability to interpret fractions on the number lines. Specifically, the students exhibited extremely poor performance in tasks related to naming a fraction on the number line (task 5-task 8), with over 80.0% of them consistently in the level 1 performance range. In addition, more than 85.0% of class 1 students could not locate 2/3 between zero and one on a zero-to-three number line (task 2). On the other hand, over 50.0% of the students demonstrated proficiency in locating fractions or whole numbers on the number lines (tasks 1,

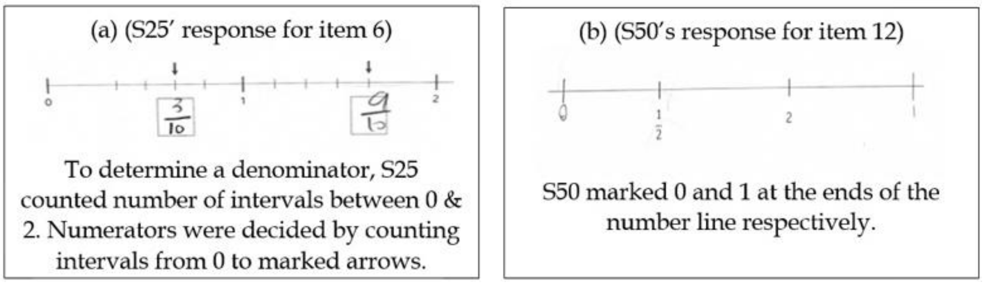


Figure 7. Examples of responses about unit error (Source: Field study)

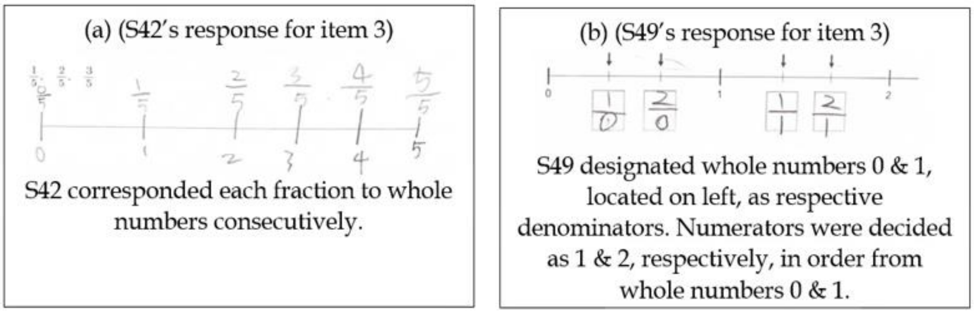


Figure 8. Examples of responses that apply incorrect reasoning (Source: Field study)

9, and 10), which require consideration of the ordinal aspects of numbers without necessarily adhering to equal spaces between intervals.

One of the most evident misunderstandings observed in the students' responses in class 1 was interpreting the entire line as the unit. **Figure 7** provides examples of responses from class 1 students related to this. One student (S25) responded, as shown in part a in **Figure 7** and stated, "there are ten intervals. So, it is $\frac{}{10} + \frac{3}{} = \frac{3}{10}$", indicating ten intervals for the denominator and the third arrow for the numerator. It seemed that this student ignored the number one and number two on the number line and could not recognize un-equidistant tick marks. Some students labeled zero and one at the ends of the number line, respectively, regardless of the numbers presented on the line, as shown in part b in **Figure 7**. This revealed students' deficiency in comprehending the nature of number line representation implying "left-smaller, right-larger" or fractions' relative magnitudes.

Another noticeable error among class 1 students came from confused reasoning, where they conflated whole numbers or decimal numbers when interpreting fractions on the number lines. For instance, some students drew tick marks and labeled them both as whole numbers and fractions, as shown in part a in **Figure 8**. It seemed that students' counts along the tick marks matched the ordered labeled values, but coordination between metric distance and numeric labels did not occur. Also, some students determined the denominator value by following the nearest and smaller whole number, explaining "Because it is less than (pointing to the number one on the right) this number, so the denominator has to be zero" (see part b in **Figure 8**). Students failed to perceive fractions as single entities, instead attributing distinct meanings to numerators and denominators independently, thus constructing fractions with separate interpretations for each.

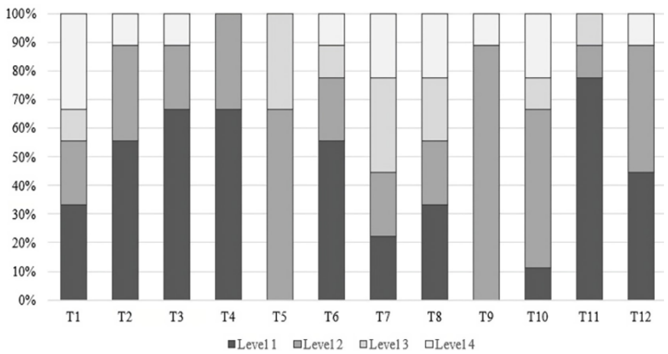


Figure 9. Distribution of performance levels in class 2 (Source: Authors' own elaboration)

Emergent understanding of fractions on number line profile

Figure 9 displays the distribution of performance levels of class 2 students. In this profile, students' performance levels increased in level 2 and level 3, while level 1 decreased for several tasks, indicating that students demonstrated the ability to compare the fractions' relative magnitudes on the number lines. In particular, the performance levels in some tasks showed remarkable improvement compared to class 1 students. However, it is important to note that there are variations in students' performance levels depending on the tasks, and performance level 2 and level 3 indicate an inaccurate and incomplete performance of fractions on the number lines.

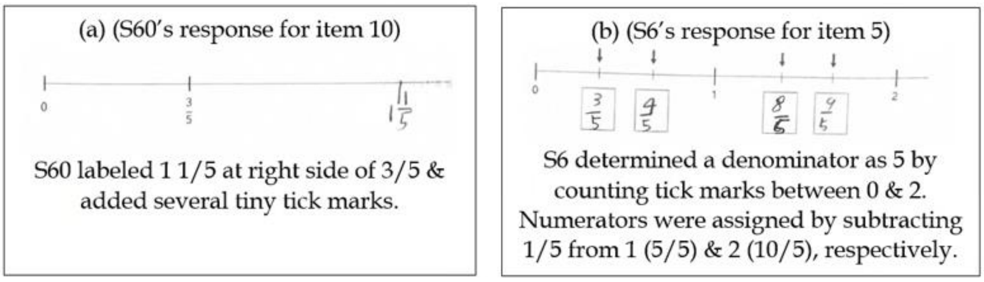


Figure 10. Examples of students' responses in emergent understanding of fraction profile (Source: Authors' own elaboration)

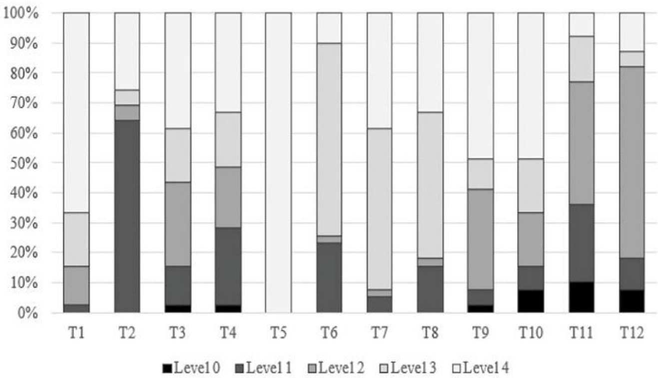


Figure 11. Distribution of performance levels in class 3 (Source: Authors' own elaboration)

Most students in this profile adjusted the magnitudes of fractions and whole numbers on the number lines. For instance, student S60 demonstrated an understanding of the relative magnitude of 1 1/5 by positioning it to the right of 3/5 on the number line (see part a in **Figure 10**). The small lines added to the right of 1 1/5 suggest his basic awareness of the structure of the number lines. However, the student appeared to disregard the coordination of numerical units and linear units when placing 1 1/5 on the number line. On the other hand, students still exhibited unit errors and counted tick marks instead of intervals, as shown in part b in **Figure 10**. Student S6 determined an interval as 1/5, and then subtracted it from one and two, respectively.

Advanced understanding of fractions on number line profile

Students in this profile displayed higher levels of performance compared to those in other profiles as depicted in **Figure 11**.

Overall, there were increases in performance at level 4 and decreases in performance at level 1 across tasks, indicating that students accurately labeled fractions on the number lines by employing operations such as splitting and disembedding. Especially, students distinctly showed advanced performance for the tasks related to naming fractions on the number line (task 5-task 8). On the other hand, it is noteworthy that the proportions of performance at level 1-level 3 are more remarkable than the ones at level 4, and the students' performance at level 1 and level 2 still exhibited a high proportion for specific tasks (e.g., tasks 2, 11, and 12).

As shown in **Figure 12**, students in this profile demonstrated their comprehension of the unit and equal partitioning on the number lines. Student S26 noticed the inconsistent intervals and reasoned that the longer intervals would be twice the length of the shorter ones by using loops and labeling longer intervals as ㉠ and ㉡ (see part a in **Figure 12**). Another student, S43, split the space between 1/2 and two into thirds, labeling the first one as one and the second one as 1 1/2. The student additionally marked 5/2 to the right of two, considering a distance of 1/2, however, placed zero at the far left of the number line without accounting for any distance (see part b in **Figure 12**). Consequently, the students' performance in this profile encompassed key operations, revealing an advanced understanding of fractions on number lines, despite incompletely executing some tasks.

DISCUSSION & CONCLUSIONS

The first aim of this study was to comprehend students' overall performance of fraction number line tasks. The results showed that, across all the tasks, students had difficulties in solving the fraction number

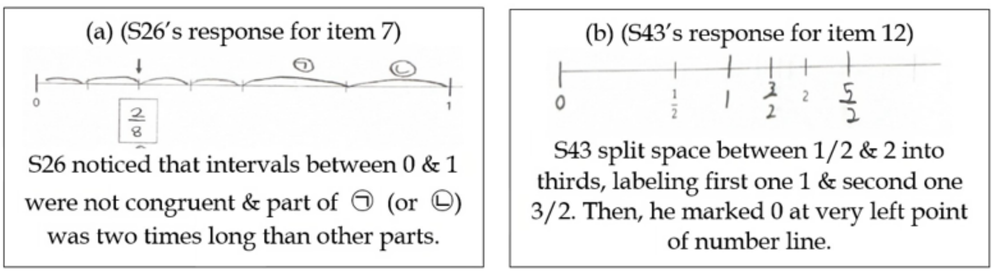


Figure 12. Examples of students' responses in advanced understanding of fraction profile (Source: Field study)

line task. Considering level 4 scoring as indicative of accurate task performance, only below 20.0% of the fourth graders achieved proficiency in the overall tasks, with exceptions for specific tasks (approximately 35.0% and 32.0% correct answers for task 1 and task 5, respectively). This finding is in line with research that highlights the challenges students face in number line tasks (Charalambous & Pitta-Pantazi, 2007; Tunc-Pekkan, 2015). As Patahuddin et al. (2017) remarked on the intricate nature of the number lines, asserting "behind its apparent simplicity, this mathematical object may involve layers of complexity when looked at from the perspectives of affordances" (p. 909), these findings imply that fractions on the number line appear to be challenging for students to grasp.

Indeed, it is understandable that a significant number of students in this study exhibited lower performance levels, given the unconventional nature of the tasks, ones rarely encountered in their textbooks. However, considering that the number lines are a representation introduced to students from very early grades, this strikingly low performance on the number line tasks is noteworthy. As Patahuddin et al. (2017) pointed out that "partition may be there on printed number line, but it is only when student can perceive it in relation to numbers that it becomes an affordance" (p. 910), the features of number lines might not be inherent to students, which is often contrary to teachers' expectations. According to Larson (1980), "a number line of length one is very similar to the part-whole (area) model that is usually the first and most constant model used in developing function concepts" (p. 426-427). Therefore, though the findings of this study are suggestive of the predominance of students' problematic use of fraction number lines, they also emphasize the importance of addressing the forms and functions of number lines in instructions for students to improve further conceptual understanding.

The second aim of this study was to identify profiles of students that reflected distinct performances in solving fraction tasks on the number lines. The analysis supported the existence of three categories of students and provided a framework that clarifies distinctions between students with different profiles. The first profile involved the largest proportion of students who randomly located fractions on number lines, ignoring not only the ordinal properties of the numbers but also the coordination of numerical and linear units. Any operations on fractions and coordination of units were not observed in students' performances in this profile, thus the group was characterized as having an intuitive understanding of fractions on number lines. The second class of students seemed to exhibit an awareness of fractions' relative magnitudes but used estimation strategies in placing fractions on number lines rather than considering partitioning or iterating operations, thus they were characterized as an emergent understanding of fractions on number lines, revealing partial coordination of numerical units, linear units, and order. Finally, the third class of students not only demonstrated a relatively in-depth awareness of fractions magnitudes but also managed the number lines by using the splitting and disembedding operations. This class was characterized by an advanced understanding of fractions on number lines. However, it is important to note that not all students in this profile exhibited such competencies across all tasks.

Especially, given that the largest group of students was categorized as having an intuitive understanding of fractions on the number line profile, encountering difficulties in reasonably interpreting the fractions on the number lines, it suggests that the measure subconstruct of the fraction is a prerequisite condition for successfully working with number lines (Charalambous & Pitta-Pantazi, 2007). Pearn and Stephens (2007) also highlighted that "students' use of number lines firstly to probe students' understanding of fractions as numbers capable of being represented on a number line" (p. 602). Although the three profiles of students showed different performances in fraction number line tasks, there is a cautionary note here: all three profile students commonly revealed a misunderstanding that considered the size of the whole number line as the unit and overapplied part-whole subconstructs regardless of the length of number lines. Thus, this study supports the perspective that understanding the unit needs to be a primary focus when utilizing number lines for fractions learning (Behr et al., 1983; Larson, 1980; Saxe et al., 2013), as highlighted by Steffe (2002) that constructions of the continuous units of students' number sequences are important as they entail the coordination of operations and order of positions.

The third objective of the study aimed to thoroughly examine the characteristics of each profile that manifested distinct understandings of fractions on number lines. Upon scrutinizing students' responses, the strategies employed by students in solving fraction number line tasks offered insights into their interpretations of both fractions and number lines. Specifically, it was possible to ascertain whether and how students executed operations such as partitioning, disembedding, iterating, and splitting in performing fraction number line tasks. Furthermore, the results revealed that certain students counted tick marks in canonical numeric progressions without coordinating metric distances with numeric values. They also demonstrated a tendency to designate denominators as 10 without questioning or added tick marks to divide the unit into 10 intervals, irrespective of uniformed spaces. This suggests that students' prior experiences with number lines in other numerical domains, such as whole numbers or decimal numbers, might influence their approach to working with fractions on number lines.

This observation was confirmed in another research finding by Cramer et al. (2019), which examined students' misunderstanding related to the number line as a model for fractions. This included students' misinterpretation of partitioning, tick marks, unit, and equivalence. Considering these challenges, even among students demonstrating an advanced understanding of fractions, there is a clear need for instructional support for comprehending number lines. Teppo and van den Heuvel-Panhuizen (2014) asserted "the number line served as a structured reference context that shifted the number line from being a model of a particular context to that of a model for reasoning about underlying mathematical relations and structure" (p. 56). Thus, students need to become aware of the explicit representational nuances and implicit meanings of number lines. The use of number lines has been regarded as one of the best practices for fraction instructions to facilitate students' sense-making (Siegler et al., 2012), as the number lines can function as a thinking tool about fraction ideas as well as an action tool for embodying those ideas (Bruce et al., 2023). Likewise, it is crucial for teachers to help students grasp the key ideas of fractions before rushing to introduce number lines (Charalambous & Pitta-Pantazi, 2007).

The research findings should be considered within the context of the limitations imposed on this study. First, the participants of the present study included only 122 fourth graders from three different elementary schools. As the sample of this study was small and limited to only one grade, there is a limitation in generalizing the identified trends of the profiles from this study to the entire elementary student population. Another critical limitation of this study is associated with relying solely on students' written responses rather than conducting interviews in collaboration. Despite these limitations, we could suggest future research to gain a more comprehensive understanding of students' challenges with number lines. Considering that the assessment items of this study exclusively focused on the number line representation, future research could incorporate additional fraction representations and compare the trends in students' latent profiles.

This study was conducted with 122 fourth graders to understand how children performed in a written assessment test that required them to coordinate their fraction and number line knowledge and identify some latent profiles based on students' performances. The findings of this study have some instructional implications. Mainly, the identified latent profiles of students may provide teachers with a better understanding of students' strengths and the limitations in dealing with fraction number line tasks. Furthermore, the descriptions of profiles of students may inform teachers about students' specific operations when working with fraction number line tasks. Another implication of this study is that, when introducing the number line as a model for fractions, careful consideration should be given to students' comprehension of both the number line representation itself and the fundamental concepts of fractions.

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Ethical statement: The author stated that the study is based on anonymized data from elementary school classes. Before conducting the study, the author obtained pertinent authorizations from the participants' schools and parents to develop and analyze the students' competencies. Additionally, the teachers and students were informed about the study and participated voluntarily.

Declaration of interest: No conflict of interest is declared by the author.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the author.

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