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Promoting the use of the Python programming language to analyze contextualized situations on derivatives and integrals considering the fundamental theorem of calculus

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Abstract

The aim of this article is twofold: first, to design a Python programming language proposal to analyze contextualized situations involving differential and integral calculus considering the case of the fundamental theorem of calculus (FTC); second, the proposal is applied to university students of differential calculus to promote the understanding of the derivative and the integral based on the Python programming language. This study is motivated because students have many difficulties to represent and applying calculus concepts in real situations. For this purpose, the theoretical model on the process of formation of mathematical concepts (PFC) is used, which consists of analyzing, abstracting, generalizing and synthesizing; the Python programming language and what concerns the FTC. The research was developed under a qualitative methodology in 4 stages: first, the proposal was designed based on the Python programming language; second, the participants were selected; third, the proposals were applied through task-based interviews; and finally, data analysis using the PFC theoretical model and the analytical framework of computational actions. The results show that the Python programming language is a tool that facilitates the analysis of contextualized situations involving differential and integral calculus, considering, in this case, the FTC; the application of mathematical tasks to students generated a deep conceptual development reflected in the interpretations they issued after viewing and contrasting the graphs of the rate of variation, the accumulated integral and the relationship between them when deriving the accumulated integral.

Keywords: data science, mathematics, Python, derivative, integral

INTRODUCTION

The teaching of mathematics at the university level has been characterized by teaching processes based on the development of the algebraic aspect, forging in students the memorization of processes and techniques that allow them to find a result, but with little meaning applied to the close context or associated with the undergraduate degree they study. Researchers such as Cervantes-Barraza et al. (2019b) argue that differential and integral calculus and differential equations courses are encouraged by developing problems or exercises that involve algorithmic procedures and even memorization of rules or formulas without achieving an understanding of the definition of the concepts. The authors in their research addressed the study of ordinary differential equations (ODEs) from a graphical-visual approach from the qualitative theory of differential equations, enhancing the construction of deductive arguments and promoting the conceptual understanding of the concept of ODE, its general equation, its graphical representation and geometric interpretation for the case of orthogonal curves.

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Contribution to the literature

- The Python programming language is a tool that facilitates the analysis of contextualized situations involving differential and integral calculus.
- The application of mathematical tasks to students generated a deep conceptual development with a dynamic and graphical view of the derivative and the integral.
- The FTC was interpreted and the graphs of the rate of change, the accumulated integral, and the relationship between them can be viewed and contrasted when deriving the accumulated integral.

A review of the literature regarding the teaching process of the fundamental theorem of calculus (FTC) allows to show the attempt of several researchers to glimpse from a technological perspective the conceptual elements of the theorem under study. For his part, Meza (2009) addressed in his study the problem of how to improve the teaching of the FTC by implementing digital tools. The main focus is the development of software that allows the dynamic visualization of the FTC, facilitating a deeper understanding. Through an epistemological analysis of the FTC combined with experimental tests in educational environments where the assistant software is used, the results showed that the use of dynamic visualizations significantly increases the understanding of abstract concepts in calculus, improving the learning experience. Oliveira and Bellemain (2020) addressed the difficulty of students to connect theoretical concepts with practical applications in learning calculus, focusing attention on the teaching of differential and integral calculus, particularly the FTC, from a historical and praxeological perspective. The results show that this approach allows students to acquire a more integrated understanding of mathematical concepts, which is reflected in greater effectiveness in problem solving.

Research has pointed out as a central problem the analysis of the disconnect between intuitive and formal concepts in the learning of calculus (Robles Arredondo & Tellechea Armenta, 2014). The study addresses a different epistemological and methodological approach to teach the FTC, through the use of didactic activities focused on exploration and guided discovery, combining the history of the development of the FTC with key concepts such as variation and accumulation. Achieving an alternative approach that facilitates the understanding of the FTC by integrating historical and epistemological concepts in teaching. Under this problem, Grande (2016) analyzed the teaching of the FTC from an epistemological perspective, emphasizing the use of intuitive and visual approaches. The added value of the study showed that the implementation of visual techniques and graphical representations to reinforce mathematical intuition; achieving that the combination of visualization and intuition improves the ability of students to connect ideas and solve problems related to the FTC.

Other authors have been concerned with researching the concepts of Calculus, especially the derivative and the integral, because mathematics and engineering students, future mathematics teachers and some in-service mathematics teachers have difficulties in solving intra- and extra-mathematical problems because they do not connect the multiple representations of these concepts and do not use the meanings appropriately (Galindo-Illanes & Breda, 2024; Pino-Fan et al., 2018; Rodríguez-Nieto et al., 2022, 2023a). Furthermore, understanding of the derivative is not fully achieved, because teachers do not propose challenging tasks where students interpret graphs and analyze the behavior of functions considering the derivative and the antiderivative together with modeling processes (Elmania et al., 2024; Ledezma et al., 2024; Rodríguez-Nieto et al., 2023b).

For their part, Borji et al. (2024) used a previously proposed model of the mental constructions that students can use to understand partial derivatives, considering a set of activities designed to facilitate such constructions. The model is based on the local linearity of differentiable functions of two variables, and the associated activities explore the relationship between partial derivatives and the tangent plane through various representations. Mastery of limits, continuity, and derivatives is crucial for the study of differential and integral calculus, which forms the basis of mathematics in several university courses. However, many students face difficulties in relating theoretical concepts to their practical application, which hinders the development of essential calculus skills. The abstract nature of this discipline can generate a negative perception towards mathematics and science, which impacts academic performance and decreases interest in advanced courses in the subject (de Vera et al., 2022; García-García & Dolores-Flores, 2021; Kunwar, 2021; Sofronas et al., 2015). In this context, the FTC plays a central role in establishing the connection between differentiation and integration, showing that they are inverse processes. However, the lack of a solid foundation in limits, derivatives, and continuity makes their understanding difficult, underlining the need for more effective pedagogical approaches (Munyaruhengeri et al., 2024).

The collected works of AlAli et al. (2024) offer a comprehensive exploration of current challenges and innovations in education, particularly in the Middle East. In their study on digital transformation in Jordanian basic education schools, the authors highlight how educational wastage–characterized by student dropouts and grade repetition–can be mitigated through strategic integration of digital technologies and teacher training (AlAli & Wardat, 2024a). Another article investigates the low performance of students in PISA assessments, identifying key contributing factors such as inadequate teaching strategies, socio-economic conditions, and a lack of student motivation (AlAli & Wardat, 2024b). In their analysis of generative AI in education, the authors present both opportunities (e.g., personalized learning, automation of assessments) and challenges (e.g., ethical concerns and teacher readiness), emphasizing the need for careful policy and pedagogical frameworks (AlAli & Wardat, 2024c).

In the context of mathematics education, AlAli et al. (2024) evaluate the effectiveness of STEM-aligned teaching practices among gifted math teachers, revealing strong correlations between innovative instruction and student engagement. Their work on gamification, specifically through the use of Kahoot, demonstrates significant improvements in both student motivation and academic achievement (Jarrah et al., 2025). Additionally, the study by Tashtoush et al. (2023) on cyberbullying in Abu Dhabi school's sheds light on its negative impact on students' willingness to engage in learning, stressing the importance of fostering safe digital environments. Collectively, these studies underscore the transformative potential of technology and innovation in addressing persistent educational issues while also calling for greater support systems to ensure equity and effectiveness in implementation.

Based on a synthesis of the literature review, this study integrates the programming language with the analysis of contextualized situations in order to address the existing problems in differential and integral calculus courses. An alternative that seeks, through the experimentation of physical phenomena and the construction of basic programming codes in Python, in order to enhance the understanding and construction of mathematical concepts, emphasizing the FTC. The research questions that guide this study are:

1. How does the design of a sequence of mathematical tasks focused on the use of the Python programming language allow the analysis of contextualized situations that involve differential and integral calculus considering the case of the FTC?
2. How does the application of the sequence of mathematical tasks designed based on the Python programming language facilitate understanding in university students regarding the concepts of the derivative and the integral?

In this sense, the activity of experimenting with physical phenomena is implemented as a didactic strategy that is included in the process of formation of mathematical concepts (PFC), which involves a series of cognitive actions that students go through when they perform mathematical tasks. These activities are encouraged by the teacher through questions, questioning combined with the didactic intention (Cervantes-Barraza, 2021; Díaz, 2007).

The proposal presented in this paper has a twofold objective:

(1) to design a Python programming language proposal to analyze contextualized situations involving differential and integral calculus considering the case of the FTC and

(2) to apply the proposal to university students of differential calculus to promote the understanding of the derivative and the integral based on the Python programming language.

In particular, this study focuses on the construction of the concept of derivative and integral of a real variable function by integrating the PFC and codes in the Python programming language. It should be noted that the concepts of derivative and integration are addressed with the purpose of concluding the results of the FTC.

THEORETICAL FRAMEWORK

Theoretical Bases of Pedagogical Order

It is focused on addressing the contemporary approach of constructivism. Knowledge under this approach is considered as a result of individual construction processes guided or supported by collective interaction processes (e.g., teacher and students) where each individual builds based on their own actions from the understanding of the concepts, meanings and sense attributed. In this order of ideas, the social construction of knowledge in mathematics is recognized as a process that involves educational actors linked to a didactic medium that enhances the learning process of students. The position and principle of the theory of symbolic interactionism states that the student learns a concept in mathematics when he or she is able to construct valid mathematical arguments (Schwarz, 2009) and argue in a context of interaction with peers with the aim of convincing an audience about the veracity of his or her claims (Cervantes-Barraza & Cabaña-Sánchez, 2022; Cervantes-Barraza et al., 2019a; Knipping & Reid, 2015; Krummheuer, 1995, 2015).

In a mathematics class, the learning process is mediated by dialogic and argumentative processes, guided by didactic instruments and tools, such as mathematical tasks, concrete materials, mobile applications, dynamic geometry and mathematics software, among others (Solar & Deulofeu, 2016). The implementations of didactic media seek to enhance the learning processes of students, which occur in an internal cognitive and social way in a well-situated mathematical context. The theory of socio-epistemology proposed by Ricardo Cantoral seeks to decentralize the mathematical object under study and recreate contextualized situations or tasks that allow the student to identify how mathematical concepts and their operations are linked to the latent reality of the students. For them, a deep understanding of the meanings associated with mathematical concepts and an appropriate implementation of didactic resources and tools is required.

The role of the teacher in relation to pedagogical-didactic principles involves following work routes with their students under a learning process guide approach. The teacher, from the design of contextualized mathematical tasks (Cervantes-Barraza & Aroca-Araujo, 2023; Pochulu et al., 2016), allows their students to interact with problems posed didactically in contexts familiar to the students in order to ensure that they build mathematical connections, these play a fundamental role in the argumentation process, since the connection is important for the establishment and identification of the argument and the justification that supports it (Font & Rodríguez-Nieto, 2024; Rodríguez-Nieto et al., 2023a), argue in collective contexts with peers in the classroom (Cervantes-Barraza et al., 2020) and apply what they have learned in situations with real problems.

Theoretical Bases of Didactic Order

Knowledge from a materialism perspective of the theory of knowledge is constituted in sensitive or material knowledge and theoretical knowledge. One of the premises of this theory is that sensitive knowledge is the product of direct reflections of objects (reality) on people's consciousness; in this construction process, sensation, notion and perception intervene (Díaz, 2007). This type of knowledge is ideal for carrying out the processes of interaction and exploration of conceptual notions with students during the teaching process. On the other hand, the construction of theoretical knowledge is the product of indirect reflections of subjective reality on man's consciousness; the process of construction of this type of knowledge requires cognitive activities that must be enhanced by the teacher and involve the concept, judgment and mathematical reasoning.

Within the framework of the construction of theoretical knowledge, the author points out that the concept formation process (CFP) requires four cognitive activities: analyze, abstract, generalize and synthesize (Díaz, 2007). It is then verified that a student constructed the mathematical concept when he/she goes through each of the activities promoted by the teacher, who from his/her role as guide of the teaching process forges in the students cognitive actions that imply the analysis of the invariant characteristics of the concept under study, after identifying them, carry out a debugging process to determine the essential characteristics, with this the student generalizes for other cases and finally, gathers the previous elements in order to build a synthesis that will consolidate the definition of the concept from a conceptual and operational point of view.

Díaz (2007) defines analysis as the process of interaction between a student and a teaching medium (i.e., mathematical task, exploration activity, among others) in order to identify and describe the invariant and common characteristics of mathematical objects. Abstraction consists of refining the identified characteristics that do not define the mathematical concept, generalization involves attributing the identified invariant characteristics to other mathematical objects that refer to the concept under study. And synthesis is the product of gathering the invariant properties and characteristics that define the mathematical concept and consolidating them into a single set.

In response to the methodological question, how to implement the CFP approach in the teaching process of university mathematics? the need to design a sequence of mathematical tasks that contain as a study center a mathematical concept, a learning objective, a slogan posed in terms of an investigative situation or exploration of a physical phenomenon, is highlighted. This must contain the necessary information to empower students and involve the activities of analysis, abstraction, generalization and synthesis.

Disciplinary Theoretical Bases of Differential and Integral Calculus

In the history of mathematics, it is recognized that the creation of mathematical objects, concepts and operations are the product of unsolvable problems that motivated more than one mathematician to propose possible solutions and with these concepts about objects and operations such as geometric figures, arithmetic operations, derivative, integration, differential equations among others. In contrast to epistemology and the process of development of mathematical thinking, it is recognized that mathematical concepts are the product of indirect reflections of human consciousness, receive exclusive treatments with cognitive activities through the path of mathematical reasoning and have a path that marks their discovery, development and consolidation.

Researchers such as Ramírez (2009) and Meza (2009) highlight that the creation of the derivative began with the Greeks as a mathematical community, who expressed four major problems that gave life to the derivative and integration function, and these were related to speed, the problem of the tangent line to a curve, the area under any curve and the problem of maximums and minimums. The problems described above motivated mathematicians such as Leibniz and Newton at the end of the 17th century with results that allowed them to develop concepts such as fluxion and source related to the speed problem. For his part, Leibniz developed the concepts of differential and integral framed by the problems of the construction of a tangent line to any curve. Under a historical review, Mateus Nieves (2011) points out that

> It was not until the end of the 17th century that calculus became algebraic, a phenomenon comparable to what Viète had done in the theory of equations, and what Descartes and Fermat had done in geometry. Perhaps the most important thing that was achieved was the reduction to antidifferentiation of area, volume and other problems that had been treated as summations. Thus, the four main problems of the time (relative change, tangents, maxima and minima and summation) were all reduced to differentiation and antidifferentiation (p. 116).

The attempt to solve problems led the above-mentioned mathematicians to find a relationship between their works, they discovered the important inverse relationship that is today known as the "FTC". This discovery favored the algorithmic development of calculus and provided a generic formulation of the relationship between the tangent problem and the area problem, or in our modern notation, between the derivative and the integral.

Under the principles of materialism of the theory of knowledge, the epistemological position of mathematics and the construction of mathematical concepts imply a set of pure knowledge that during its development was comprehensible only to people with high knowledge in mathematics. However, research proposals motivated by pure mathematicians and mathematics educators interested in teaching advanced mathematics, created the opportunity to teach contextualized mathematics and purified of formalistic bodies for people who did not have advanced knowledge in mathematics, for this purpose, the creation of educational mathematics or mathematics education as a discipline in Latin America, mathematics didactics for Spain, the French school for Europe or in North America mathematics education is recognized.

Researchers from all over the world have contributed to various lines of research (mathematics teaching, history of mathematics, socio-epistemology, technologies in mathematics education, among others) with the same objective of contributing to the improvement of the teaching process of school and university mathematics. Theories outside of education were used as a basis for the creation of new constructs, methods and methodologies specific to the discipline of mathematics education. One of the theoretical and methodological proposals that showed a change in the teaching of mathematics underlines the appearance of didactic transposition, seen as the opportunity to extract wise knowledge and adapt it to the educational context in order to access conceptual constructs through contextualization processes that allow the essence of mathematics to be rescued without addressing formal elements of axiomatic and theoretical mathematics.

The need to present students with other possible epistemologies of the concepts that triggered the creation of differential and integral calculus is highlighted, this favors the teaching and learning process, because the usual epistemology of the pure mathematical object does not necessarily respond to the needs and implications of the relationships between the object to be taught, the object taught, the culture, the contexts of use and other relationships that underlie its teaching (Ramírez, 2009). In this sense, the latent problem in differential and integral calculus courses is latent, where it has been evidenced that the historical development of calculus first emerged integration, then the derivation process, after these the limit process and finally in 1960 the function as a mathematical object and the rigor of mathematical language; while in the didactics of mathematics first the function as an object is taught from the rigor of mathematical language, then the limit, the derivative and finally the integral. In this sense, there are epistemological, psychological, didactic obstacles or semiotic conflicts (D'Amore et al., 2007; Radford, 1997).

Prior Knowledge

In order to address the study of the FTC, students must have prior knowledge that facilitates the development of the learning process under the construction of deductive reasoning and construction of mathematical arguments (Balacheff, 1999). Prior knowledge refers to the concepts of objects and operations that connect differential and integral calculus. Such concepts are consecutive in nature and students are required to have an adequate conceptual and operational appropriation. The prior concepts necessary for the development of the session are functions, limits of functions, continuity of functions, calculation of derivatives of functions, calculation of indefinite and definite integrals.

In the framework of the development of differential calculus courses, the curricular plans propose to focus attention on the teaching of a fundamental concept that will be a tool for understanding advanced concepts of calculus. The concept of a function of a variable is defined as "a rule that assigns to each element x of a domain set D exactly one element, called $f(x)$, of an arrival set E" (Stewart, 2010, p. 10). Addressing this concept allows students to identify the relationship between two numerical sets, which can be translated as two variables in the context of experimentation and the elements correspond to numerical data that have a relationship.

Functions allow the study of limiting situations in the construction of notions of existence of values that the function under study cannot take, but there is an approximation in both directions towards that point that allows convergence on a numerical value. In this sense, the limit of a function is defined as follows:

> Suppose that $f(x)$ is defined as x is close to the number a (this means that f is defined on some open interval containing a, except possibly at a itself). So, we write $\lim_{x\to a} f(x) = L$ and we say that "the limit of $f(x)$, as x approaches a, is equal to L" if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like), by taking values of x sufficiently close to a (on both sides of a), but not equal to a (Stewart, 2010, p. 87).

The analysis of the characteristics of functions opens an opportunity to determine whether the functions modeled or studied present interruptions or discontinuity, so the study of continuity and its conditions is required. Following the route of mathematical concepts of differential calculus, the concept of continuity of a function is related to the derivation and integration. In this sense, a function f is continuous on an interval if it is continuous at each number in the interval. If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint as continuous on the right or continuous on the left. Therefore, a function f is continuous at a number $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

The derivative of a function under the geometric notion is defined as follows: Given any number x for which this limit exists, we assign to x the number $f(x)$. So, we consider $f'(x)$ as a new function, called the derivative of f and defined by means of Eq. (1):

$$f'(x) = \lim_{h\to o} \frac{f(x+h)-f(x)}{h}. \quad (1)$$

We know that the value of $f'(x)$ at x, $f'(x)$ can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f\ (x))$. The function $f'(x)$ is known as the derivative of f because it has been "derived" from f by the operation of finding the limit in the equation. The domain of $f'(x)$ is the set $(x \,|\, f'(x)\ exists)$ and can be smaller than the domain of f (Stewart, 2010, p. 154).

As part of integral calculus, the study of techniques to calculate the area under curves is manifested through the definite integral, a mathematical concept that allows determining the area under a curve comprised by the upper and lower limits that delimit the integration region. The concept of definite integral according to Stewart (2010, p. 372) is defined as follows: If f is a continuous function defined for $a \ll x \ll b$, We divide the interval $[a, b]$ in n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Let $x_0(= a), x_1, x_2, \ldots, x_n(= b)$ the extreme points of these subintervals and let $x_1{}^*, x_2{}^*, \ldots, x_n{}^*$ the points show in these subintervals, so that $x_i{}^*$ are found in the i^{th} subinterval $[x_{i-1}, x_i]$. Then the definite integral of f, from a to b is given in Eq. (2):

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i{}^*)\,\Delta x. \quad (2)$$

Provided that this limit exists and gives the same value for all possible choices of the sample points. If it exists, we say that f is integrable on $[a, b]$.

The indefinite integral is considered to be the representative of a whole family of functions, that is, an antiderivative for each value of the constant C. A definite integral $\int_a^b f(x)dx$ is a number, while an indefinite integral $\int f(x)dx$ is a function or a family of functions. Whenever this limit exists and it has the same value for all possible choices of the sample points. If it exists, we say that f is integrable over $[a, b]$. Then, Eq. (3) is given:

$$\int_a^b f(x)dx = \int f(x)dx\ \big]_a^b. \quad (3)$$

The Fundamental Theorem of Calculus

Part I of the FTC

If f is continuous on $[a, b]$, then the function g defined by Eq. (4):

$$\boldsymbol{g(x) = \int_a^x f(t)dt\ a \ll x \ll b}. \quad (4)$$

It is continuous on $[a, b]$ and derivable on (a, b), and we get Eq. (5):

$$\boldsymbol{g'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x).} \quad (5)$$

A natural language translation of the first part of the FTC implies that the derivative of a definite integral with respect to its upper limit is the integrand evaluated at that limit. It is worth noting that the FTC not only establishes a relationship between integration and differentiation but also guarantees that any integrable function has an antiderivative. Specifically, it guarantees that any continuous function has an antiderivative. This theorem was prioritized over other calculus concepts because it encapsulates the central ideas of both differential and integral calculus–namely, derivatives and integrals. It is essential for students to understand how these two fundamental concepts are interconnected, as this relationship forms the foundation of much of the reasoning and problem-solving in calculus.

Demonstration: Graphically, the function $f(x)$ represents a curve on the Cartesian plane XY, the integration region is defined by indicating the interval with the integration limits a and b, the values x and $x + h$ contained in (a, b) are established (see **Figure 1**).

From the construction of the graph, it is seen that the images of the values x and $x + h$ correspond and form an integration region delimited from the lower limit a to $x + h$. Therefore, the geometric meaning of integral is applied as the area under the curve $f(x)$ and we obtain Eq. (6):

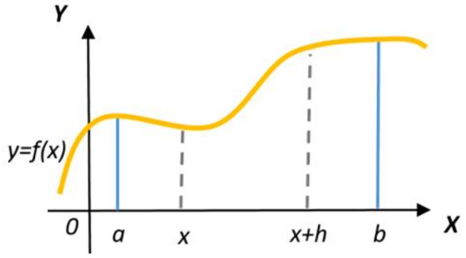


Figure 1. Graph of the function *f(x)* and the integration limits (Source: Authors' own elaboration)

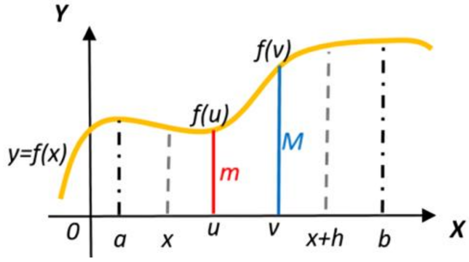


Figure 2. Graph of the function $f(x)$ and the values of u and v (Source: Authors' own elaboration)

$$g(x+h)-g(x)=\int_a^{x+h} f(t)dt-\int_a^x f(t)dt. \quad (6)$$

Operating and applying properties of integrals, we have Eq. (7):

$$g(x+h)-g(x)=\left(\int_a^x f(t)dt+\int_x^{x+h} f(t)dt\right)-\int_a^x f(t)dt. \quad (7)$$

Similar terms are simplified to get Eq. (8):

$$g(x+h)-g(x)=\int_a^{x+h} f(t)dt. \quad (8)$$

If we analyze non-zero values for h, we have Eq. (9):

$$\frac{\boldsymbol{g(x+h)-g(x)}}{\boldsymbol{h}}=\frac{\boldsymbol{\int_a^{x+h} f(t)dt}}{\boldsymbol{h}}. \quad (9)$$

Suppose that $h>0$, and since f is continuous on $[x,x+h]$ and implementing the mean value theorem, which allows us to identify that there exist two numbers u and v in $[x,x+h]$ such that $f(u)=m$ and $f(v)=M$, where, m and M are the minimum and maximum values of f on $[x,x+h]$ (see **Figure 2**).

Based on the comparison property of integrals, we have Eq. (10):

$$mh \ll \int_a^{x+h} f(t)dt \ll Mh. \quad (10)$$

Replacing the values of the function evaluated at points m and M, we have Eq. (11):

$$f(u)h \ll \int_a^{x+h} f(t)dt \ll f(v)h. \quad (11)$$

Under the condition that $h>0$, the entire expression can be multiplied by the multiplicative inverse of h, implying Eq. (12):

$$\boldsymbol{f(u) \ll \frac{1}{h}\int_a^{x+h} f(t)dt \ll f(v)}. \quad (12)$$

Replacing the value obtained in Eq. (9) in Eq. (12), we have Eq. (13):

$$f(u) \ll \frac{\boldsymbol{g(x+h)-g(x)}}{\boldsymbol{h}} \ll f(v). \quad (13)$$

Now with $h \to 0$, we have that $u \to x$ and $v \to x$ since u and $v \in [x,x+h]$. Therefore, by applying the definition of a limit of functions we have Eq. (14) and Eq. (15):

$$\lim_{u\to 0} f(u)=\lim_{x\to u} f(x)=f(x). \quad (14)$$

$$\lim_{v\to 0} f(v)=\lim_{x\to v} f(x)=f(x). \quad (15)$$

Based on Eq. (14) and Eq. (15) and by the comprehension theorem, it is concluded Eq. (16):

$$\boldsymbol{g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=f(x)}. \quad (16)$$

According to Leibniz's notation for derivatives, the first part of the FTC can be expressed, as Eq. (17):

$$\boldsymbol{\frac{d}{dx}\int_a^x f(t)dt=f(x)}. \quad (17)$$

Part II of the FTC

If f is continuous on $[a,b]$, then we have Eq. (18):

$$\boldsymbol{\int_a^b f(x)dx=F(b)-F(a)}, \quad (18)$$

where F is an antiderivative of f, that is, a function such that $\boldsymbol{F'(x)=f(x)}$.

Proof: Let $g(x)=\int_a^x f(t)dt$ According to the first part of FTC, it is known that $g'(x)=f(x)$, that is, g is an antiderivative of f. If F is any other antiderivative of f on $[a,b]$, then the difference between F and g is a constant c (Eq. [19]):

$$F(x)=g(x)+c. \quad (19)$$

For $a<x<b$, but both F and g continue on [a, b] and thus, when obtaining the limits of both sides of Eq. (19), that is, when $x\to a^+$ y $x\to b^-$, we see that it is also fulfilled when $x=a$ and $x=b$.

If we let $x=a$ in the formula for $g(x)$, we get Eq. (20):

$$g(a)=\int_a^a f(t)dt=0. \quad (20)$$

So, when evaluating Eq. (19), with $x=a$ and $x=b$, we have Eq. (21):

$$\begin{gathered} F(b)-F(a)=[g(b)+c]-[g(a)+c] \\ F(b)-F(a)=g(b)-g(a) \\ F(b)-F(a)=\int_a^b f(t)dt. \end{gathered} \quad (21)$$

Newton's Law of Cooling

This law describes the process where the temperature t of an object changes with time due to heat transfer between the object and its environment. In the theory of differential equations this cooling law is expressed, as follows in Eq. (22):

$$\frac{dT(t)}{dt}=-k(T(t)-T_{amb}), \quad (22)$$

where $\boldsymbol{T(t)}$ is the temperature of the object at time t, $\boldsymbol{T_{amb}}$ is the ambient temperature, which is assumed to be constant, $\boldsymbol{k}$ is a constant of positive proportionality that

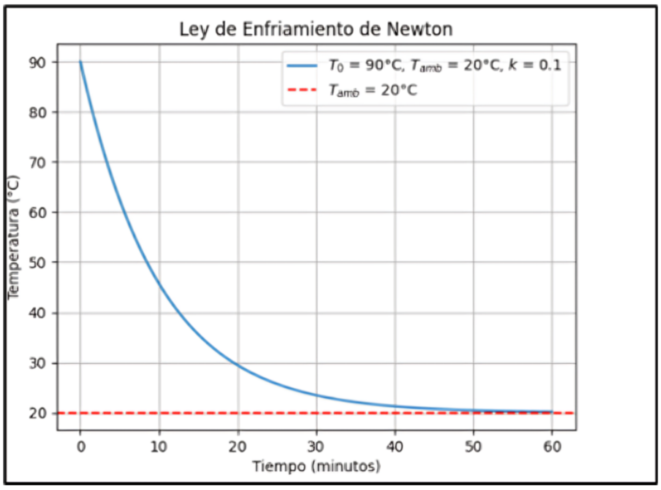


Figure 3. Graph of the function $T(t)$ when $T_0 = 90°$, $T_{amb} = 20°$ and $k = 0.1$ (Source: Authors' own elaboration)

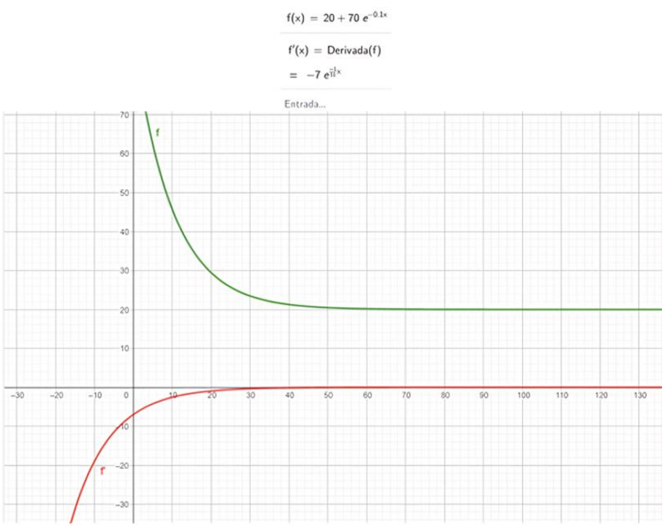


Figure 4. Graph of the function $T(t)$ and its respective rate of change (Source: Authors' own elaboration)

depends on the properties of the object and the medium, and $\frac{dT(t)}{dt}$ is the rate of change of the temperature of the object of the properties of the object and the medium.

By solving the differential equation that models the heat transfer phenomenon by applying the method of separation of variables and operating, the general solution of the ODE can be described in terms of a function $T(t)$ in terms of the temperature transferred in a given time t (Eq. [23]):

$$T(t) = T_{amb} + (T_0 - T_{amb})e^{-kt}, \tag{23}$$

where T_0 is the initial temperature of the object at $t = 0$ (see **Figure 3**).

For the particular case of calculating the temperature at 30 minutes, we proceed to evaluate $T(30)$, which generates Eq. (24):

$$T(30) = 20 + (90 - 20)e^{-(0.1).30}$$
$$T(30) = 20 + 70e^{-3} \tag{24}$$
$$T(30) = 20 + 3.48$$

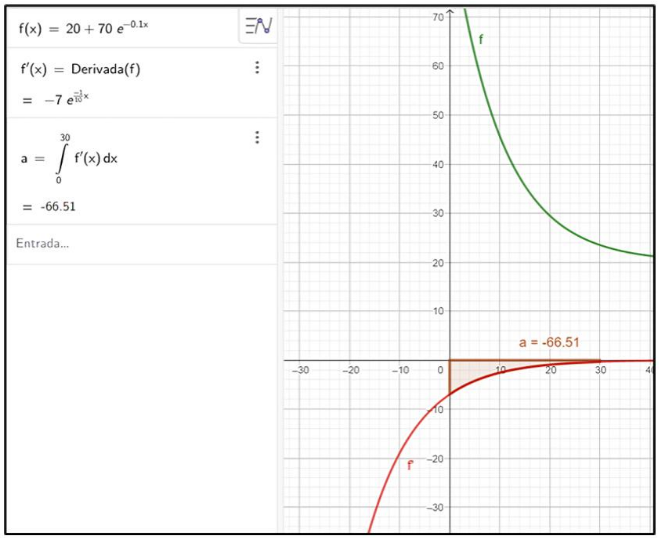


Figure 5. Graph of the function $T(t)$, its respective rate of change and the integral of the rate of change (Source: Authors' own elaboration)

$$T(30) = 23,48°.$$

The temperature for minute 30 corresponds to 23.48°. It is also noted that the function that models the cooling behavior of the coffee cup is given in Eq. (25):

$$T(t) = 20 + 70e^{-0.1t}. \tag{25}$$

The rate of temperature changes with respect to time would be in Eq. (26) (**Figure 4**):

$$\frac{dT(t)}{dt} = -7e^{-0.1t}. \tag{26}$$

Verification of the FTC applied to the coffee cup experiment and Newton's law. Calculation of the rate of change integral gives us the accumulated change in temperature in Eq. (27) (**Figure 5**):

$$\int \frac{dT(t)}{dt} = \int_a^x -7e^{-0.1t}\,dt = -7e^{-0.1x}. \tag{27}$$

PFC and Programming

In this section we made the map in Python using the coding steps and related with the PFC stages (from analysis, abstraction and generalization to→ synthesis).

STAGE 1: ANALYSIS OF THE REAL DATA

```
#Import math libraries
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import cumtrapz
#Upload the Excel file
#Replace ‘file.xlsx’ with the name of your Excel file
df = pd.read_excel(‘/content/datosgrupo1.xlsx’)
#Asumiendo que el archivo tiene columnas llamadas ‘Tiempo’ y ‘Temperatura’
tiempo = df[‘Tiempo (minutos)’]
temperatura = df[‘Temperatura (°C)’]
```

STAGE 2: ABSTRACTION PROCESS

```
#Calculate the derivative of temperature with respect to time (rate of change)
tasa_cambio = np.gradient(temperatura, tiempo)
#Calculate the accumulated integral of the rate of change to obtain the total change in temperature
integral_acumulada = cumtrapz(tasa_cambio, tiempo, initial=0)
#Calculate the derivative of the accumulated integral (it should approximate the original rate of change)
derivada_integral = np.gradient(integral_acumulada, tiempo)
```

STAGE 3: GENERALITATION OF FINDINGS

```
#Create new columns in the DataFrame for the results
df['Tasa de Cambio (dT/dt)'] = tasa_cambio
df['Integral Acumulada de la Tasa de Cambio'] = integral_acumulada
df['Derivada de la Integral Acumulada'] = derivada_integral
#Display the first rows of the DataFrame
print(df.head())
df['Tasa de Cambio (dT/dt)'] = tasa_cambio
df['Integral Acumulada de la Tasa de Cambio'] = integral_acumulada
df['Derivada de la Integral Acumulada'] = derivada_integral
print(df.head())
```

STAGE 4: SYNTESIS OF THE FINDINS

```
#Plot the temperature, the rate of change, the accumulated integral, and the derivative of the integral
plt.figure(figsize=(12, 8))
plt.subplot(3, 1, 1)
plt.plot(tiempo, temperatura, label='Temperatura (°C)', color='blue')
plt.ylabel('Temperatura (°C)')
plt.title('Análisis de Temperatura y Tasa de Cambio')
plt.subplot(3, 1, 2)
plt.plot(tiempo, tasa_cambio, label='Tasa de Cambio (dT/dt)', linestyle='--', color='red')
plt.plot(tiempo, derivada_integral, label='Derivada de la Integral', linestyle=':', color='green')
plt.ylabel('Tasa de Cambio (°C/min)')
plt.legend()
plt.subplot(3, 1, 3)
plt.plot(tiempo, integral_acumulada, label='Integral Acumulada de la Tasa de Cambio', linestyle='-.', color='purple')
plt.xlabel('Tiempo (minutos)')
plt.ylabel('Cambio Acumulado (°C)')
plt.legend()
plt.tight_layout()
plt.show()
```

METHODOLOGY

The methodology of this research follows a qualitative approach (Cohen et al., 2018), aiming to describe and interpret the results of educational phenomena involved in the mathematics teaching process. The descriptive qualitative approach serves as an appropriate means to construct a framework of descriptors that can consolidate and thoroughly explain the educational phenomenon under study. This research focuses on integrating an analytical framework for recognizing students' computational and statistical actions with the fundamental concepts of differential and integral calculus. This is realized through the design and implementation of interactive mathematical tasks based on the analysis of real data.

Participants

The participants were selected voluntarily, based on their own interest in taking part in the implementation of the didactic proposal described in this study. It is recommended that they be undergraduate students enrolled in a mathematics degree program, academically registered in the fifth semester or higher. Additionally, some minimum participation criteria are outlined, which require students to have basic knowledge of differential and integral calculus and familiarity with some open databases. Another selection criterion was the prior programming experience of students, and calculus proficiency. Students were selected and asked about the level of programming and its interest in participating in a research study. About the calculus proficiency, students took a basic mathematics course and have the basic calculus concepts in mind. Some demographic information about the participants are the sample size were 38 students, their academic level was higher education, gender balance: female 40% and male 60%. All the students are from different socioeconomic levels and they did not come from ethnic groups.

Didactic Instruments

The didactic instruments to be implemented during the session include:

(1) an OVA (virtual learning object) projected onto the board using a video beam,

(2) a cup of hot coffee and a mercury thermometer,

(3) a mobile phone or a laptop, and

(4) a table of values in Excel to be analyzed in Python.

Table 1. Students' computational and statistical actions in a data science course (adapted from Woodard & Lee, 2021)

Computational and statistical actions	The student is able to	Integration of differential and integral calculus concepts
Automation of computational procedures	• Uses technology to create graphs, summarize, and interpret datasets. • Uses technology to perform statistical calculations and applies the results to make appropriate decisions.	• **Derivative as a rate of change:** Calculates the population growth rate of a country over different periods using derivatives, analyses periods of higher or lower growth, and links them to historical events.
Computational thinking	• Develops a solution strategy and communicates it through software, programs, or code. • Demonstrates critical and abstract thinking in computing. • Reviews existing code to improve it and apply it to new tasks.	• In addition to loading and cleaning data, writes code for computing numerical derivatives and integrals to fit the data.
Application of new methods	• Develops various solutions to statistical questions. • Proposes original methods for analysis while using technology. • Identifies and understands organized information to use it in solving a new problem.	• Writes code to visualize solutions of derivatives and integrals in order to answer specific questions.
Pattern recognition and decision-making	• Identifies patterns in statistical analysis and uses the information to determine the next step. • Recognizes patterns in code structure to aid in problem-solving.	• Analyses and interprets the results of graphs generated from data to recognize patterns, variations in the phenomenon, or the situation under study.

Recognition of Students' Computational and Statistical Actions

The adaptation of the analytical framework for recognizing students' computational and statistical actions will be implemented as a fundamental part of the data science learning process, as proposed by Woodard and Lee (2021). To this end, students are expected to engage in the four categories of action as follows:

(1) automation of computational procedures,

(2) computational thinking,

(3) application of new methods, and

(4) pattern recognition and decision-making.

Table 1 presents the corresponding actions for each category. The integration of computational actions for learning data science is linked to the definitions and interpretations of the fundamental concepts of differential and integral calculus. To materialize this integration, interactive mathematical tasks are designed to provide students with contextualized experiences using real databases and problem scenarios where the definitions of differential and integral calculus concepts are applied.

Designing Contextualized Mathematical Tasks With Real Data

Recent studies in the field of mathematical task design research (Cervantes-Barraza & Aroca, 2023) highlight the need to develop interactive and argument-based mathematical tasks in both school and university contexts. These tasks should incorporate technological tools (e.g., geometry software and mobile mathematics applications), interactive environments (e.g., interactive presentations and OVAs), videos, and open databases, among others.

Mathematical tasks are designed following a methodological structure described by Cervantes-Barraza and Aroca (2023). Each task follows a basic structure, including the task name, learning objective, mathematical concept, instructions, research question, and Python code implementation. In this context, five mathematical tasks are presented with the objective of applying fundamental concepts of differential and integral calculus while introducing students to data analysis using Python and practical tools such as pandas, numpy, and matplotlib, which are essential for handling open data analysis (Lasso Cardona et al., 2022). The students did not create programming codes, they were given some wrote codes or interacted with pre-built scripts in order to understand their structure and could modify them into the experimental context.

Mathematical task: "The cup of coffee and differential and integral calculus"

For effective teaching and learning of differential calculus, it is essential that instruction is both structured and engaging. Educators should aim to harmonize the two key components of calculus teaching: the heuristic component and the specific mathematical thinking content, which involves the concepts to be taught (Mateus Nieves, 2011). The planned activities will take place in a virtual learning environment facilitated by an interactive OVA created by the session instructor.

The OVA can be accessed at: https://view.genially.com/67c8c05e3d96d5c9d32d6477/learning-experience-didactic-unit-ftc-experimentation

Table 2. Description of planned activities for the teaching session

Activity	Description	Duration
Initial questions	Conceptual questions will be posed to connect students' prior knowledge with the topic under study.	10 minutes
Initial task	Analyze how the temperature of the liquid changes over time using prior concepts.	10 minutes
Task 1	Experiment with how the derivative (rate of change) relates to the original function (temperature) and how the coffee's temperature decreases over time, illustrating Newton's law of cooling.	20 minutes
Task 2	Intuitively understand the FTC by demonstrating how the integral of the rate of change provides the accumulated change in temperature and how the derivative of this integral returns the original rate of change function.	20 minutes
Task 3	Conclude on the relationship between differentiation and integration, including a description of the FTC.	20 minutes

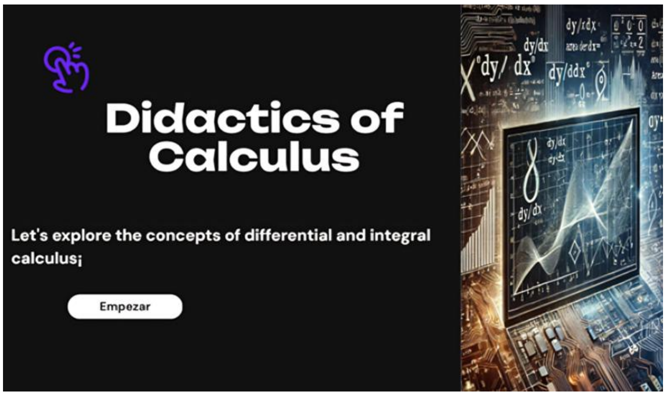


Figure 6. Start page of the interactive OVA format (Source: Authors' own elaboration)

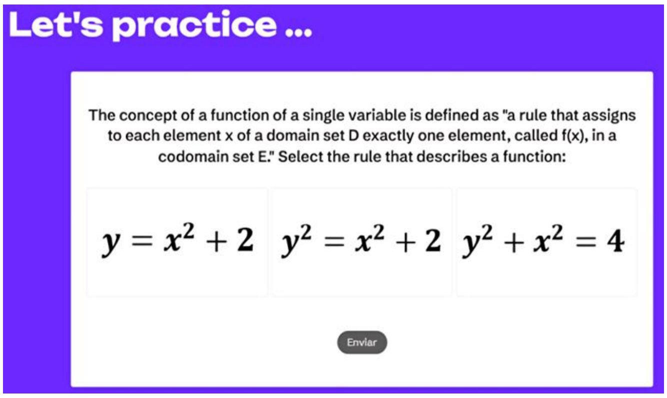


Figure 7. First question in the initial stage of the lesson (Source: Authors' own elaboration)

The OVA includes four mathematical tasks that will guide the lesson development. Students will be able to access the OVA on their mobile devices once a QR code is shared with them. The design of these mathematical tasks involves a conceptual construction process that requires students to engage in analysis, abstraction, generalization, and synthesis. **Table 2** describes the activities to be carried out during the teaching session.

RESULTS

The first objective proposed in this research involved designing a didactic approach consisting of mathematical tasks that utilize the Python programming language to analyze contextualized situations requiring differential and integral calculus, particularly considering the case of the FTC. The lesson development focused on students' interaction with the three designed mathematical tasks and an initial scenario presented in the OVA. The initial stage of the lesson engaged students in the analysis of conceptual questions aimed at connecting their prior knowledge with the new concepts introduced in the session. It is important to clarify that, as the presentation was interactive (see **Figure 6**), students accessed it on their mobile devices and actively interacted with the OVA while the teacher provided specific guidance for each slide of the interactive presentation.

The initial questions of the session were designed to relate the concepts of functions with differentiation and integration. In the first question (see **Figure 7**), students were asked to identify the correct algebraic expression of a function from an algebraic perspective. The answer options were intentionally designed to manage common student misconceptions. Specifically, the options omitted the conventional function notation *f(x)* to encourage students to recognize the uniqueness property of functions and the one-to-one relationship between each element in the domain and the range.

In this sense, the expressions $y^2 = x^2 + 2$ and $y^2 + x^2 = 4$ do not meet the definition of a function but rather represent relations, as their algebraic rules generate two outputs for the same domain value.

In the second question (see **Figure 8**), students were asked to identify the equation that models the geometric interpretation of the derivative concept. The possible answers included mathematical expressions related to the definition of the indefinite integral of a function and the first part of the FTC. However, these options were incorrect, as they did not correspond to the algebraic expression of the derivative of a function.

The third question (see **Figure 9**) aims to connect the concepts of derivatives and integrals through the calculation of the antiderivative of a quadratic function.

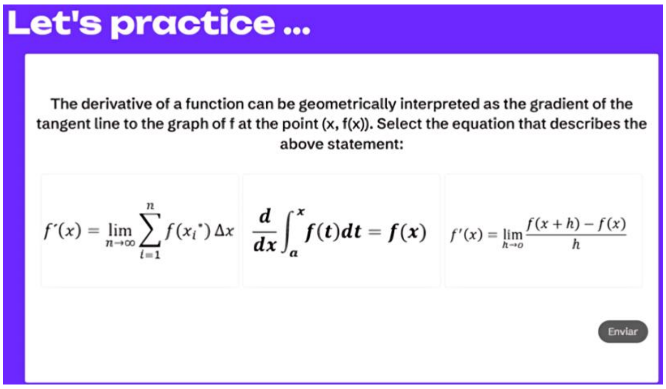


Figure 8. Second question in the initial stage of the lesson (Source: Authors' own elaboration)

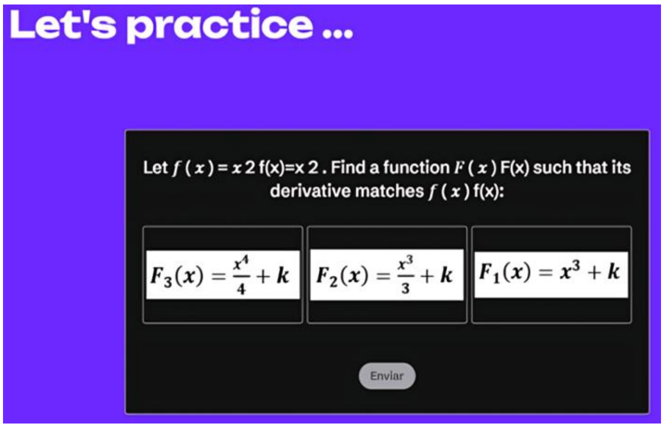


Figure 9. Third question in the initial stage of the lesson (Source: Authors' own elaboration)

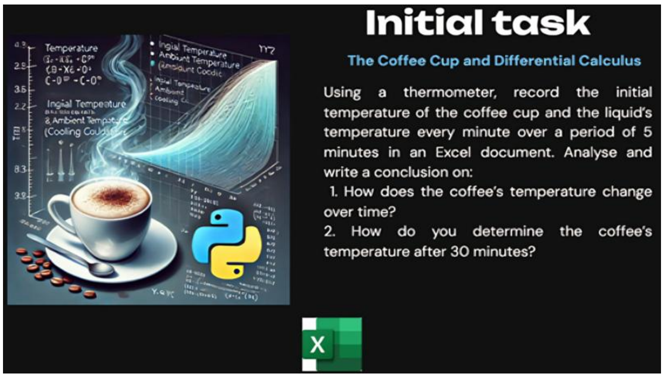


Figure 10. Initial task of the teaching session (Source: Authors' own elaboration)

The possible answer choices are algebraically similar, but the correct option is the antiderivative $F_2(x) = x^3/3$, since differentiating this function returns the original function $f(x) = x^2$.

Regarding the initial task (see **Figure 10**) in the teaching session, an experimental situation involving the phenomenon of cooling, specifically Newton's law of cooling, is introduced. Students will measure the temperature of a cup of coffee using a thermometer to analyze how the temperature changes over time. The objective of this initial task is to encourage students to seek mathematical tools that allow them to model the situation and mathematically predict the temperature after 30 minutes.

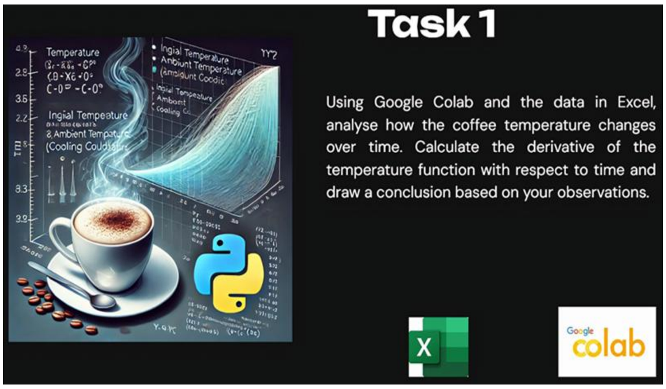


Figure 11. Task 1 of the teaching session (Source: Authors' own elaboration)

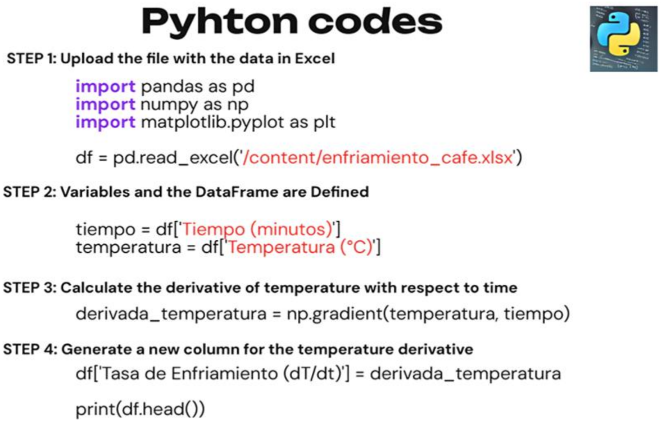


Figure 12. Python codes for implementation in Google Colab (Source: Authors' own elaboration)

Once students have completed the initial task and discussed their findings with the class, they proceed to task 1, which aims to integrate programming language into the modelling and analysis of the rate of temperature change over time (**Figure 11**). Once students have entered the data into an Excel file, they will construct the necessary code for Google Colab to read the data, calculate the derivative (rate of change) of the temperature, and formulate a conclusion based on their analysis.

The codes required for data analysis can be copied and pasted in order from the interactive presentation slides before being executed in Google Colab (see **Figure 12**). This process enables students to visualize the data and interpret the variations in the coffee's temperature over time.

For task 2 (see **Figure 13**), following the same approach described earlier, students are required to analyze the same dataset. However, this time, they will focus on examining the accumulated change in temperature and its relationship with its rate of change. The objective of this task is for students to identify the relationship between differentiation and integration of a function.

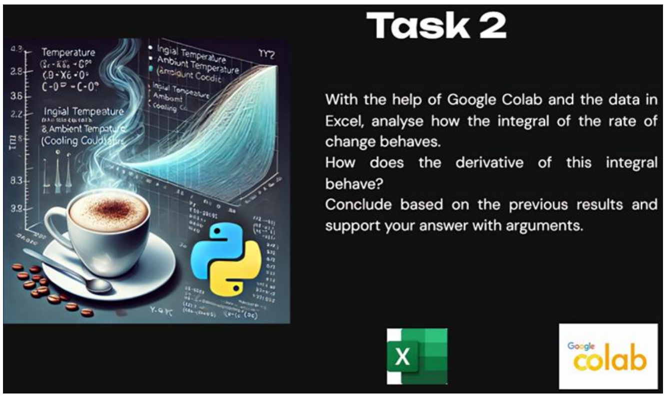


Figure 13. Task 2 of the teaching session (Source: Authors' own elaboration)

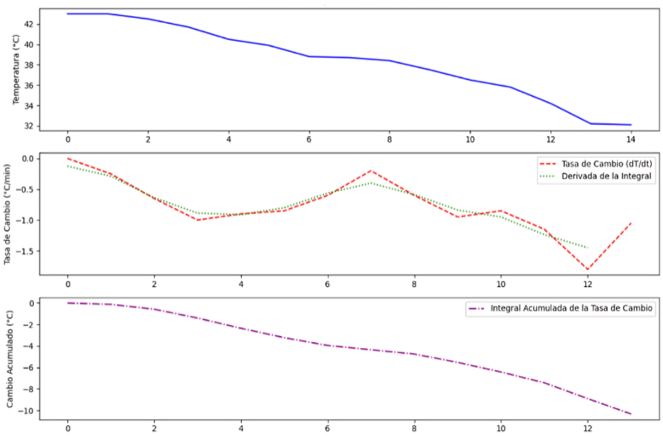


Figure 14. Graphs of temperature analysis, rate of change, and accumulated integral from real experimental data (Source: Authors' own elaboration)

The data analysis conducted in this task enabled students to observe how the temperature function relates to the rate of change and its corresponding derivative. This facilitated a deeper understanding of the FTC. Through the analysis of an experimental situation using real data, the aim was for students to recognize the practical application of university-level mathematical concepts.

In **Figure 14**, the graphs generated using Python display the temperature data, rate of change, and accumulated integral, modelling the phenomenon described by Newton's law of cooling and providing evidence of the relationship between differentiation and integration.

To conclude the lesson, task 3 (see **Figure 15**) was introduced. Here, students were asked to summarize their conclusions based on their analysis of the completed tasks and to provide a synthesis addressing the question: *Is there a relationship between differentiation and integration?* This key question was intended to guide students towards formulating the statement of the FTC and its two parts. The task provided an opportunity to approach the teaching of the FTC within the context of an experimental situation using real data, programming, and the Python language.

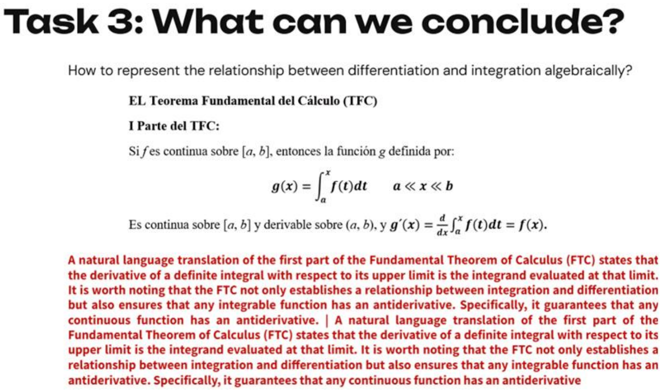


Figure 15. Task 3: Presentation of the FTC and its two parts (Source: Authors' own elaboration)

Figure 16. Photographic record of the cooling law experiment (Source: Authors' own elaboration)

The second objective of this research involved applying the didactic approach, comprising three interactive tasks, to university students enrolled in a differential calculus course. The aim was to promote an understanding of differentiation and integration through the Python programming language in the context of experimental situations, such as the law of liquid cooling. The participants, along with the instructor–one of the authors of this research–developed the lesson following five key stages:

Experimenting Newton's Law of Cooling

In this initial stage, students gathered the required materials (thermometer, plastic cups, hot coffee, mobile phone, and/or laptop) with the aim of analyzing how the temperature of the liquid (coffee) changes over time. This analysis was conducted using prior knowledge of differential calculus, without the use of programming language (see **Figure 16**).

At this stage, students used thermometers to measure the temperature of the liquid every minute, while some groups opted to take measurements every 30 seconds to analyze the results from a different perspective using Excel tables. During this phase of the lesson, students observed a noticeable decrease in temperature, noting that it was dropping towards the ambient temperature of 23 °C. Without yet introducing differential and integral calculus concepts, students sought mathematical relationships to establish connections and draw conclusions about the behavior of the phenomenon (Rodriguez-Nieto et al., 2024).

Tiempo (minutos)	Temperatura (°C)	fluctuaciones
0	43	
1	43	
2	42,5	0,5
3	41,7	0,8
4	40,5	1,2
5	39,9	0,6
6	38,8	1,1
7	38,7	0,1
8	38,4	0,3
9	37,5	0,9
10	36,5	1
11	35,8	0,7
12	34,2	1,6
13	32,2	2
14	32,1	0,1
15		
16		0,838461538

Figure 17. Tabulation and graphs generated in Excel from recorded data (Source: Authors' own elaboration)

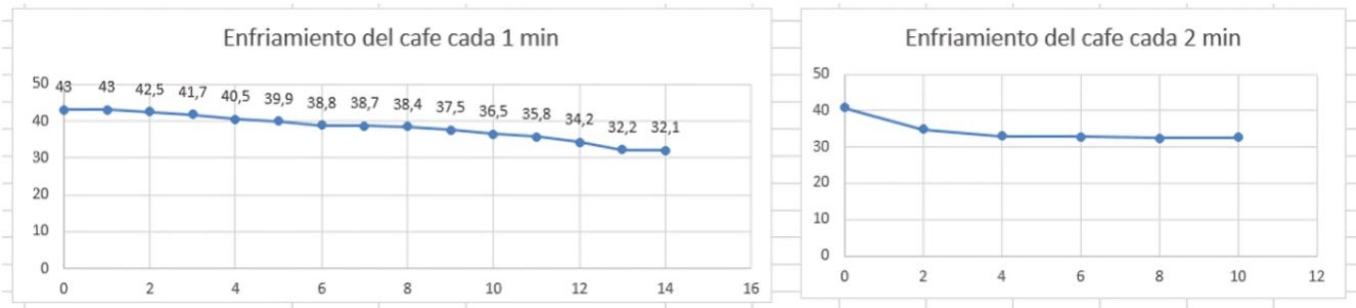


Figure 18. Graphs representing the cooling law of a liquid (Source: Authors' own elaboration)

Recording Temperature Data in Excel Tables

Using an Excel spreadsheet, students recorded the temperature data from the thermometer every minute and every 30 seconds over an average period of 25 minutes. **Figure 17** shows how some groups used their mobile phones to log the data into Excel, creating tables and graphs to visualize temperature trends. This allowed them to validate their initial observations from the lesson–that the coffee's temperature decreases over time.

This experimental stage reflects the first component of Woodard and Lee's (2021) analytical framework: students demonstrated the ability to use technology to create graphs, summarize and interpret datasets, and develop their own methods for analysis while using technological tools.

Data Analysis Using Prior Calculus Concepts

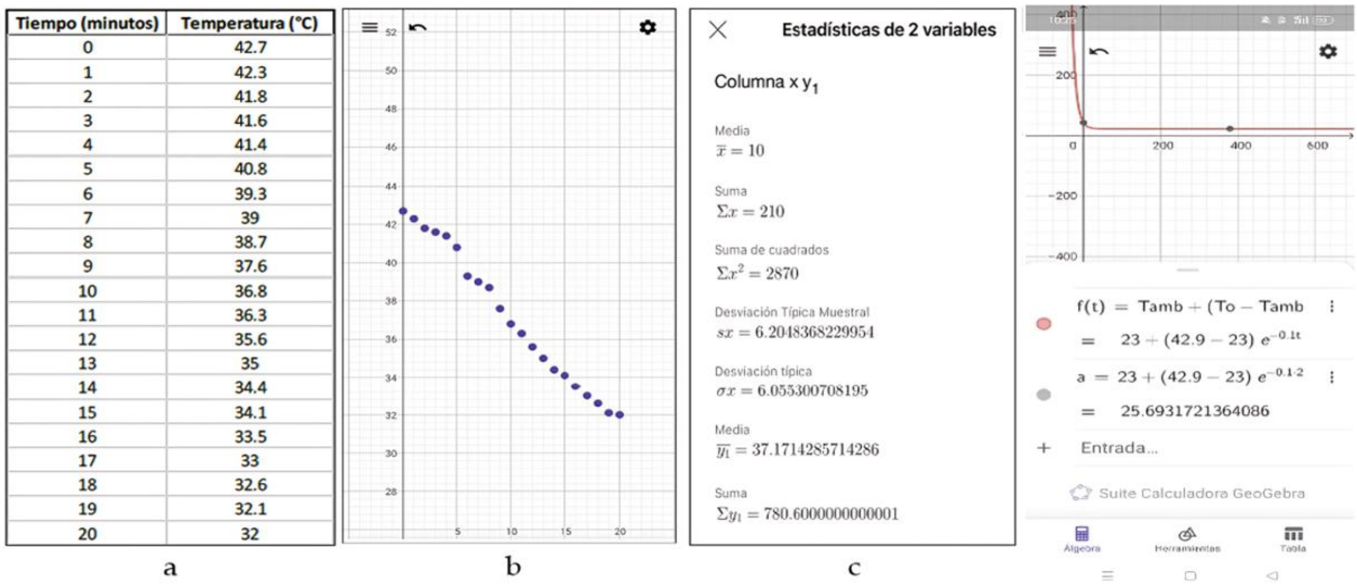
The dataset collected by the five student groups revealed that **the first group** identified that the temperature of the coffee over time follows a decreasing curve, confirming that the coffee cools as time progresses (see **Figure 18**).

> **Group 1.** By comparing the recorded temperatures, it became evident that measuring the coffee's temperature every minute provides greater accuracy, as it allows for a clearer observation of temperature fluctuations.

Below are some of the conclusions drawn by the student groups.

> **Group 2.** As time progresses, the temperature of the coffee decreases due to a process known as thermal equilibrium. The initial temperature was 47 °C. After 5 minutes, the temperature remained within the range of **46.2 °C to 40 °C**. After another 5 minutes, it dropped to a range of **38.2 °C to 30 °C**, and within the next 4 minutes, it decreased further to a range of **28 °C to 25 °C**. From this, we can conclude that
>
> (1) in the first 5 minutes, the temperature decreased by an average of **1.4 °C per minute**,
>
> (2) in the first 10 minutes, the average drop was **1.3 °C per minute**, and
>
> (3) over the last 15 minutes, the temperature decreased by **1 °C per minute**.

> **Group 3.** As time passes, the temperature of the coffee decreases continuously. It is evident that the rate of temperature decline is faster in the initial minutes. However, as time progresses, the cooling process slows down, as shown by the smaller temperature differences recorded after the seventh minute. This observation aligns with cooling theory, which states that cooling is faster when the temperature difference between the liquid and its environment is greater.



Tiempo (minutos)	Temperatura (°C)
0	42.7
1	42.3
2	41.8
3	41.6
4	41.4
5	40.8
6	39.3
7	39
8	38.7
9	37.6
10	36.8
11	36.3
12	35.6
13	35
14	34.4
15	34.1
16	33.5
17	33
18	32.6
19	32.1
20	32

Figure 19. Table, graph, and curves generated using the GeoGebra application (Source: Authors' own elaboration)

Group 4. The temperature of the coffee follows an **exponential pattern**, progressively decreasing over time. According to the temperature data recorded every minute (part a in **Figure 19**) and using the **GeoGebra** app (part b in **Figure 19**), the plotted data revealed a **decreasing curve approaching an asymptote**–in this case, the ambient temperature of **23 °C**.

It is important to note that **synchronizing temperature measurements precisely each minute was challenging**, which introduced **a degree of deviation** in the data and corresponding graphs. To account for this, the **sample standard deviation** was incorporated into the calculations (part c in **Figure 19**).

The responses from the five student groups demonstrated prior concepts related to differential calculus, such as fluctuations and differentials in the context of increments. It was also verified that when the time differential approaches zero, the value approximates a real number. Group 4 applied the second principle of Woodard and Lee's (2021) analytical framework–using technology to perform statistical calculations and applying the results to make informed decisions. Additionally, the algebraic and analytical aspects of the temperature function were implemented by integrating the differential equation that models the physical phenomenon of Newton's law of cooling.

Interpreting Data Using Python Code

In this final stage, students explored how the derivative (rate of change) relates to the original function (temperature) and how the coffee's temperature decreases over time, reflecting Newton's law of cooling through the execution of Python code.

```
[ ] import pandas as pd
    import numpy as np
    import matplotlib.pyplot as plt

    df = pd.read_excel('/content/datos.xlsx')

[ ] tiempo = df['Tiempo (minutos)']
    temperatura = df['Temperatura (°C)']

[ ] derivada_temperatura = np.gradient(temperatura, tiempo)

[ ] df['Tasa de Enfriamiento (dT/dt)'] = derivada_temperatura

    print(df.head(15))

    Tiempo (minutos)  Temperatura (°C)  Tasa de Enfriamiento (dT/dt)
0                  0              43.0                          0.00
1                  1              43.0                         -0.25
2                  2              42.5                         -0.65
3                  3              41.7                         -1.00
4                  4              40.5                         -0.90
5                  5              39.9                         -0.85
6                  6              38.8                         -0.60
7                  7              38.7                         -0.20
8                  8              38.4                         -0.60
9                  9              37.5                         -0.95
10                10              36.5                         -0.85
11                11              35.8                         -1.15
12                12              34.2                         -1.80
13                13              32.2                         -1.05
14                14              32.1                         -0.10
```

Figure 20. DataFrame of collected and tabulated data with the derivative (Source: Authors' own elaboration)

According to Woodard and Lee's (2021) analytical framework, students demonstrated the ability to

(1) engage in critical and abstract computational thinking,

(2) review existing code to improve and adapt it for new tasks, and

(3) develop multiple solutions to statistical problems.

The first four lines of code, shown in **Figure 20**, correspond to importing the necessary libraries to interpret the datasets. The second line of code loads the dataset, while the third declares the study variables. In the fifth line of code, the derivative function of the

```
plt.figure(figsize=(10, 6))
plt.plot(tiempo, temperatura, label='Temperatura (°C)', color='blue')
plt.plot(tiempo, derivada_temperatura, label='Tasa de Enfriamiento (°C/min)', linestyle='--', color='red')
plt.xlabel('Tiempo (minutos)')
plt.ylabel('Temperatura (°C) / Tasa de Enfriamiento (°C/min)')
plt.title('Variación de la Temperatura del Café a lo Largo del Tiempo')
plt.legend()
plt.grid(True)
plt.show()
```

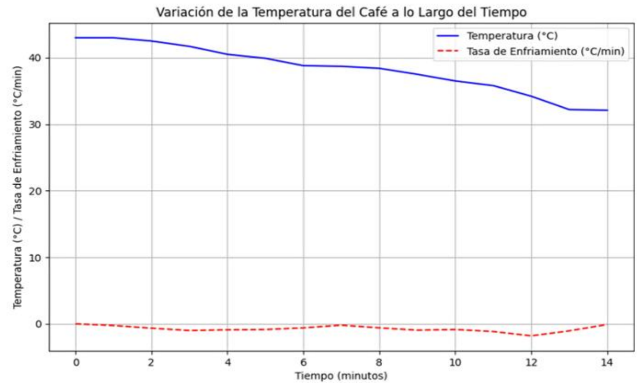


Figure 21. Graph of temperature function vs. rate of temperature change (Source: Authors' own elaboration)

```
# Calcular la derivada de la temperatura respecto al tiempo (tasa de cambio)
tasa_cambio = np.gradient(temperatura, tiempo)

# Calcular la integral acumulada de la tasa de cambio para obtener el cambio acumulado en la temperatura
integral_acumulada = cumtrapz(tasa_cambio, tiempo, initial=0)

# Calcular la derivada de la integral acumulada (debería aproximarse a la tasa de cambio original)
derivada_integral = np.gradient(integral_acumulada, tiempo)
```

Figure 22. Code for calculating the rate of change of temperature, accumulated integral, and derivative of the integral (Source: Authors' own elaboration)

temperature variable with respect to time is applied. Printing the derivative results as part of the DataFrame–tabulated by students–generated a column containing the cooling rate *(dT/dt)*.

This step aligns with the analytical framework principle: *creating a solution strategy and communicating it through software, programming, or coding* (Woodard & Lee, 2021).

Jupyter Notebook template: https://colab.research.google.com/drive/1sp45iiv4CSf-Hx7LJLCstXpfLSAIyLKU?usp=drive_link

The fifth line of code is relatively simple, as it only visualizes the data loaded in the DataFrame under a curve that fits the values provided. **Figure 21** illustrates this curve, where the blue line represents the temperature function, and the segmented lines depict the rate of change of temperature.

The next line of code calculates the rate of change of temperature, the accumulated integral, and the derivative of the integral using the gradient function. **Figure 22** presents the implemented code for this step, demonstrating its relationship with predefined variables and functions from the NumPy (np) library.

The seventh line of code (see **Figure 23**) was executed to generate tables containing values for both the rate of change and the accumulated integral. The objective was to visually compare these data points through a comparative curve and facilitate drawing conclusions.

```
print(df.head())

# Graficar la temperatura, la tasa de cambio, la integral acumulada, y la derivada de la integral
plt.figure(figsize=(12, 8))

plt.subplot(3, 1, 1)
plt.plot(tiempo, temperatura, label='Temperatura (°C)', color='blue')
plt.ylabel('Temperatura (°C)')
plt.title('Análisis de Temperatura y Tasa de Cambio')

plt.subplot(3, 1, 2)
plt.plot(tiempo, tasa_cambio, label='Tasa de Cambio (dT/dt)', linestyle='--', color='red')
plt.plot(tiempo, derivada_integral, label='Derivada de la Integral', linestyle=':', color='green')
plt.ylabel('Tasa de Cambio (°C/min)')
plt.legend()

plt.subplot(3, 1, 3)
plt.plot(tiempo, integral_acumulada, label='Integral Acumulada de la Tasa de Cambio', linestyle='-.', color='purple')
plt.xlabel('Tiempo (minutos)')
plt.ylabel('Cambio Acumulado (°C)')
plt.legend()

plt.tight_layout()
plt.show()
```

```
   Tiempo (minutos)  Temperatura (°C)  Tasa de Cambio (dT/dt)  \
0                 0              43.0                    0.00
1                 1              43.0                   -0.25
2                 2              42.5                   -0.65
3                 3              41.7                   -1.00
4                 4              40.5                   -0.90

   Integral Acumulada de la Tasa de Cambio  Derivada de la Integral Acumulada
0                                    0.000                            -0.1250
1                                   -0.125                            -0.2875
2                                   -0.575                            -0.6375
3                                   -1.400                            -0.8875
4                                   -2.350                            -0.9125
```

Figure 23. Tables displaying values for the rate of change and accumulated integral (Source: Authors' own elaboration)

```
print(df.head())

# Graficar la temperatura, la tasa de cambio, la integral acumulada, y la derivada de la integral
plt.figure(figsize=(12, 8))

plt.subplot(3, 1, 1)
plt.plot(tiempo, temperatura, label='Temperatura (°C)', color='blue')
plt.ylabel('Temperatura (°C)')
plt.title('Análisis de Temperatura y Tasa de Cambio')

plt.subplot(3, 1, 2)
plt.plot(tiempo, tasa_cambio, label='Tasa de Cambio (dT/dt)', linestyle='--', color='red')
plt.plot(tiempo, derivada_integral, label='Derivada de la Integral', linestyle=':', color='green')
plt.ylabel('Tasa de Cambio (°C/min)')
plt.legend()

plt.subplot(3, 1, 3)
plt.plot(tiempo, integral_acumulada, label='Integral Acumulada de la Tasa de Cambio', linestyle='-.', color='purple')
plt.xlabel('Tiempo (minutos)')
plt.ylabel('Cambio Acumulado (°C)')
plt.legend()

plt.tight_layout()
plt.show()
```

```
   Tiempo (minutos)  Temperatura (°C)  Tasa de Cambio (dT/dt)  \
0                 0              43.0                    0.00
1                 1              43.0                   -0.25
2                 2              42.5                   -0.65
3                 3              41.7                   -1.00
4                 4              40.5                   -0.90

   Integral Acumulada de la Tasa de Cambio  Derivada de la Integral Acumulada
0                                    0.000                            -0.1250
1                                   -0.125                            -0.2875
2                                   -0.575                            -0.6375
3                                   -1.400                            -0.8875
4                                   -2.350                            -0.9125
```

Figure 24. Graphical representation of the FTC in an experimental context: The case of Newton's law of cooling (Source: Authors' own elaboration)

In the final stage of this phase (see **Figure 24**), the primary objective of implementing programming code was to intuitively understand the FTC. This was achieved by demonstrating how

(1) the integral of the rate of change provides the accumulated change in temperature and

(2) the derivative of this integral returns the original rate of change function.

The analysis of students' responses before and after the instructional intervention revealed significant conceptual growth. During the pre-interviews, most participants could execute procedural steps related to coding tasks (e.g., plotting a function or computing derivatives using Python) but struggled to explain the underlying mathematical ideas. For instance, one student stated, "I know the code gives me the slope, but I'm not sure why that's useful." In contrast, in the post-

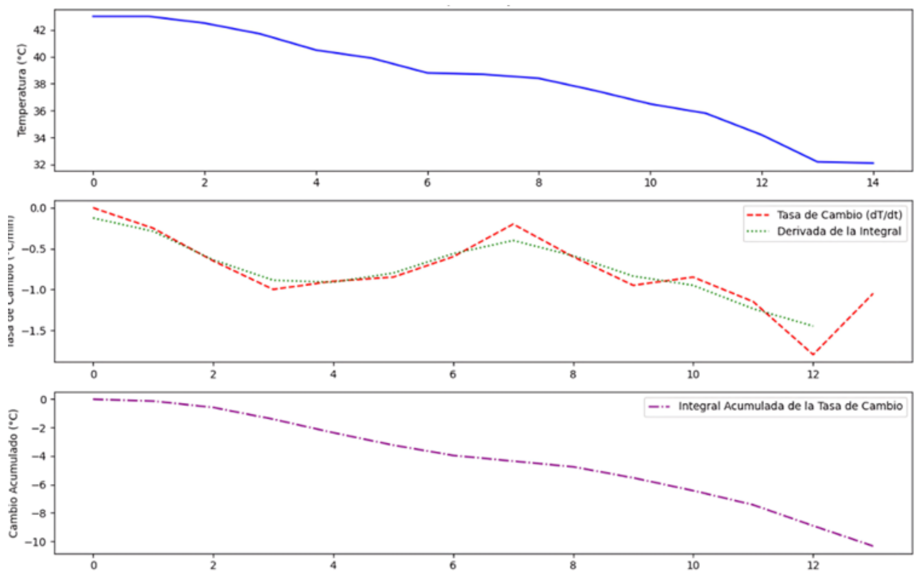


Figure 25. Graphical representation of the function and the FTC in an experimental context: The case of Newton's law of cooling (Source: Authors' own elaboration)

interviews, the same student reflected a deeper understanding, saying, "Now I see that the derivative tells me how the function is changing at each point–it's not just a number; it's describing behavior." This shift illustrates a move from rote application toward conceptual insight into calculus principles.

Furthermore, while procedural fluency improved uniformly among participants, conceptual understanding evolved more variably but meaningfully. Several students began to articulate connections between code outputs and theorems discussed in class. One participant noted, "When I coded the integral approximation, I finally understood what the area under the curve really meant–the fundamental theorem links it all." This ability to link computational procedures with formal mathematical reasoning demonstrates not only increased proficiency but also the development of reflective understanding, indicating a meaningful integration of theoretical and practical knowledge.

Figure 25 visually illustrates the FTC applied to a real-world phenomenon: the change in temperature over time. The first plot shows how the temperature decreases progressively. The second plot displays the rate of change of temperature (derivative), highlighting how the slope of the upper graph varies; it also compares this derivative with the numerical derivative of the integral, showing that they match. Finally, the third plot represents the accumulated integral of the rate of change, that is, the total temperature changes over time. Together, these three plots demonstrate that differentiating an integral returns the original function, thus validating the first part of the FTC. Then, programming allows us to model and visualize data that might otherwise be voluminous and difficult to interpret. In this experiment, a smaller dataset was used due to time constraints for the students. Ultimately, students concluded on the relationship between differentiation and integration, reinforcing their understanding of the FTC.

DISCUSSION

This research contributes to the design of a sequence of mathematical tasks focused on the use of the Python programming language. These tasks enabled participating university students to analyze contextualized situations involving fundamental concepts of differential and integral calculus, specifically considering the case of the FTC. Within this experimental setting, authors such as Cervantes-Barraza and Aroca-Araujo (2023) and Pochulu et al. (2016) argue that designing mathematical tasks grounded in real-life and familiar contexts enhances students' learning processes. Indeed, mathematical tasks designed with well-defined didactic objectives create direct opportunities for students to integrate mathematical reasoning with the mathematical concepts under study.

From the results aligned with the second objective of this research, it was confirmed that experimentation allows students to practically consolidate their understanding of the second part of the FTC, while also serving as a dynamic and innovative approach for implementation in university education. In this regard, Woodard and Lee (2021) assert that when students relate the data collected in the experimental stage, consolidate them for analysis through the lens of prior calculus concepts, and verify their conclusions, they enhance their comprehension of the results obtained from the Python-executed codes.

More specifically, reflecting on the responses from the five groups regarding their prior knowledge of differential calculus–such as fluctuations and differentials in the context of increments–it was verified that when the time differential approaches zero, the value approximates a real number, thus introducing the concept of limits. According to Rodriguez-Nieto et al. (2024), without explicitly addressing differential and integral calculus concepts in earlier stages, students need to connect prior concepts and explore mathematical relationships that allow them to establish formal mathematical connections and draw conclusions about the physical phenomenon under study.

Researchers such as Cervantes-Barraza et al. (2019) argue that university-level courses in differential and integral calculus and differential equations often focus on algorithmic procedures and memorization of rules or formulas, rather than on achieving a deep understanding of conceptual definitions. However, by incorporating Python programming and using the correct coding approach to analyze the derivative and integral of a function in a real-world context, the participating groups demonstrated computational competencies aligned with Woodard and Lee's (2021) analytical framework. These skills enabled students to perform statistical calculations and use the results to make informed decisions by constructing and adapting programming codes. Consequently, they successfully applied algebraic and analytical procedures to the temperature function once they solved the differential equation modelling Newton's law of cooling.

CONCLUSIONS

The research findings demonstrate that designing a sequence of mathematical tasks focused on the Python programming language allowed participating university students to analyze contextualized situations involving fundamental concepts of differential and integral calculus, specifically within the framework of the FTC.

Through this approach, students related the data collected during the experimental stage, consolidated them for analysis using prior calculus concepts, and verified their conclusions by comparing them with the results generated by the Python-executed codes. These codes enabled students to visualize how the derivative of the temperature function represented the rate of change over time, leading to the calculation of the accumulated integral and demonstrating how the derivative of the integral returned the original temperature function. This process reinforced the second part of the FTC in a practical and experimental setting, offering a dynamic and innovative proposal for university-level mathematics education.

Regarding the second research question, it was identified that implementing the sequence of Python-based mathematical tasks facilitated students' comprehension of differentiation and integration. The analysis of the responses from the five groups revealed prior differential calculus concepts, such as fluctuations and differentials in the context of increments. This confirmed that when the time differential approaches zero, the value approximates a real number, introducing the concept of limits.

Furthermore, the participating groups demonstrated computational competencies within the Woodard and Lee (2021) analytical framework, using technology to perform statistical calculations and make informed decisions through the construction and adaptation of programming codes. They also applied algebraic and analytical procedures to the temperature function, solving the differential equation that models Newton's law of cooling.

In conclusion, this study materializes the integration of mathematical task design within real-world scenarios, engaging students in data collection, result analysis, and programming adaptation using Python. This approach fostered collaborative learning, as student groups analyzed and mathematically explained how temperature variations are governed by the negative rate of change experienced by a liquid exposed to ambient temperature.

Programming languages enabled students to model and visualize the data they collected, allowing them to make informed predictions about temperature changes based on Newton's law of cooling. One limitation of this approach is the potential learning curve associated with Python programming. For some students, especially those without prior coding experience, the complexity of writing and debugging code may have overshadowed the primary goal of understanding calculus concepts. This could lead to cognitive overload, where the focus shifts from mathematical reasoning to technical problem-solving. Additionally, the generalizability of this method to non-STEM students is uncertain, as they may lack the foundational skills or interest in programming needed to fully engage with this type of learning experience. As such, while Python can be a powerful tool for visualizing and exploring calculus, its integration must be carefully scaffolder to ensure that it enhances rather than hinders conceptual understanding.

Graph comparisons between rate of change and accumulation played a crucial role in deepening students' understanding of calculus concepts. By visually analyzing how the derivative (rate of change) corresponds to the slope of the original function and how the integral (accumulation) reflects the total change over time, students were able to connect abstract mathematical definitions to concrete visual representations. This side-by-side comparison allowed them to observe, for example, that when the rate of change is negative, the accumulation graph decreases, reinforcing the inverse relationship between these two operations. Such dynamic visualization not only clarified the meaning of differentiation and integration but also highlighted their interdependence, effectively supporting comprehension of the FTC in an intuitive and meaningful way.

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Ethical statement: The authors stated that the participants and the university were informed that this study has educational purposes and not economic or political ones and they agreed to participate voluntarily.

AI statement: The authors stated that no generative AI or AI-based tools were used in any part of the study, including data analysis, writing, or editing.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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