

Prospective Secondary Mathematics Teachers' Perspectives and Mathematical Knowledge for Teaching

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This study investigated the relationship between prospective secondary mathematics teachers' perspectives and their mathematical knowledge for teaching in action. Data from two prospective teachers' practice-teachings, one in geometry and one in algebra, their lesson plans and self-reflections were analyzed with Teacher Perspectives and Knowledge Quartet frameworks. Results showed that prospective teachers who thought of mathematics and mathematics learning as dependent on the knower, corresponding with a progressive incorporation perspective, had demonstrated all the codes in Knowledge Quartet Framework. These results suggest that, once prospective teachers are given opportunities to develop a progressive incorporation perspective on mathematics, mathematics learning, and mathematics teaching during methods and practice teaching courses, this might contribute to the developments in their mathematical knowledge for teaching in action, independent of the particular concept they teach. Suggestions for developing a progressive incorporation perspective during methods and practice teaching courses are given.

Keywords: teacher knowledge, Knowledge Quartet, Teacher Perspectives, prospective secondary mathematics teachers.

INTRODUCTION

Teachers' mathematical knowledge for teaching has been studied to a great extent in mathematics education field (e.g., Ball, Thames & Phelps, 2008; Rowland, Huckstep & Thwaites, 2005). In particular, Rowland, Huckstep and Thwaites' (2005) framework, Knowledge Quartet, has been used to analyze what both prospective elementary and secondary mathematics teachers have and/or lack in terms of their mathematical knowledge during teaching (Thwaites, Jared & Rowland, 2011). By the same token, Teacher Perspectives framework identifying pedagogical principles regarding the nature of mathematics, mathematics learning and mathematics

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teaching (Heinz, Simon, Tzur, Kinzel, 2000; Simon, Tzur, Heinz & Kinzel, 2000; Tzur, Simon, Heinz, & Kinzel, 2001; Jin & Tzur, 2011) has also been used to analyze why elementary inservice teachers reveal what they know in terms of their mathematical knowledge during teaching.

This study attempted at using Knowledge Quartet framework and Teacher Perspectives framework in juxtaposition to each other in order to rather than merely determine what prospective secondary mathematics teachers have and/or lack in terms of their mathematical knowledge for teaching, but also explicate why they reveal such mathematical knowledge. This is important because; first, there is compelling evidence that research mostly focuses on what prospective mathematics teachers have and/or lack (e.g., Leatham, 2006; Wilson & Cooney, 2002). However, what research needs to focus on is the consistencies among knowledge, beliefs, and practices (prospective) teachers might hold (Grossman & McDonalds, 2008; Leatham, 2006; Wilson & Cooney, 2002). Second, Knowledge Quartet framework allows for examining teachers' mathematical knowledge in action 'with particular reference to the subject matter being taught' (Rowland et al., 2005). However, Teacher Perspectives framework 'is an attempt to go beyond understanding particular knowledge and beliefs in the context of practice of teachers in transition' (Simon et al., 2000, p. 580) such that

these...perspectives can be thought about as two paradigms with respect to the development of mathematical knowledge...the term paradigm emphasizes the existence of internally coherent systems... (p. 599)

That is, a teacher's perspective is 'a conglomerate that cannot be understood by looking at parts split off from the whole (i.e., looking only at beliefs or methods of questioning or mathematical knowledge)' (Simon & Tzur, 1999, p. 254), where the term 'perspective' indicates not only what prospective teachers think about, know, believe and do but also everything that contributes to their practice teaching (planning, assessing, interacting with students) (Simon, et al., 2000). In other words, a teacher's perspective goes beyond knowledge and beliefs in a particular context (Simon, et al., 2000) such that the term 'perspective' involves the underlying pedagogical principles prospective teachers might hold regarding the nature of mathematics, mathematics learning and mathematics teaching (Simon, 2006). Third, Jin and Tzur (2011) proposed that during methods and practice-teaching courses establishing the progressive incorporation perspective (PIP) on the part of prospective teachers is an ambitious goal. Thus, knowing the coherency between the perspective with which a (prospective) teacher acts and the kind of

State of the literature

- The literature points to two different frameworks: Teacher Perspectives and Knowledge Quartet.
- Knowledge Quartet provided the field to assess (prospective) teachers' mathematical knowledge for teaching, what they know. Teacher Perspectives framework contributed to the field to examine (prospective) teachers' pedagogical principles (why) underlying their mathematical knowledge during teaching.
- There is compelling data that suggests the need to investigate concurrently both what prospective teachers know and why they know in terms of their mathematical knowledge in teaching. Diagnosing the reasons behind prospective teachers' actions/practices might provide teacher educators with specific ways to assist them prior to the completion of their teacher preparation programs.

Contribution of this paper to the literature

- The main contribution relates to finding a correspondence between prospective secondary mathematics teachers' perspectives and their mathematical knowledge for teaching by revealing simultaneously what they know in terms of their mathematical knowledge for teaching and why they know it.
- Results showed that prospective teachers having a progressive incorporation perspective possessed all the characteristics given in the Knowledge Quartet framework.
- Results requires the need to further investigate the consistency between other teacher perspectives (e.g., perception-based perspective) and the mathematical knowledge for teaching since (prospective) teachers might reveal different levels of mathematical knowledge for teaching depending on the perspectives they might have.

mathematical knowledge s/he possesses might provide the teacher educators with the reasons why prospective teachers possess and/or lack certain mathematical knowledge for teaching. This might assist teacher educators in guiding prospective teachers' actions towards more sophisticated perspectives (see Table 2) by understanding the reasons why they reveal such knowledge; prospective teachers need to "learn to use their knowledge base to provide the grounds for choices and actions" (Shulman, 1987, p. 13). In this study, coordinating Knowledge Quartet and Teacher Perspectives framework, the following research question was investigated:

How do two prospective secondary mathematics teachers, holding a progressive incorporation perspective on mathematics, mathematics learning and mathematics teaching, reveal their mathematical knowledge for teaching in action?

Conceptual framework

This study mainly drew on Knowledge Quartet (Rowland et al., 2005) and Teacher Perspectives (e.g. Simon, et al., 2000; Jin & Tzur, 2011) frameworks. In this section, I explain the aspects of these frameworks that pertain directly to the account presented here.

Knowledge Quartet framework, whose correspondence with Ball et al. (2008) and Shulman (1986, 1987) has been presented (Turner, 2012), is based on four main categories; namely, foundations, transformation, connection and contingency (see Table 1). The foundation dimension relates to beliefs teachers hold regarding the nature of mathematics and mathematics learning and teaching (Thwaites et al., 2010). Also, it is about teachers' knowing 'why' behind the mathematics they teach. Transformation is about teachers' presentation of ideas to learners in the form of analogies, illustrations, examples, explanations and demonstrations. Connection consists of sequencing the material for instruction and awareness of the relative cognitive demands of different topics and tasks. Finally, contingency is the ability to make cogent, reasoned and well-informed responses to unanticipated and unplanned events during the lessons (Thwaites et al., 2010).

As well the field knows about the nature of teachers' mathematical knowledge for teaching, possible reasons behind their knowledge have also been postulated within Teacher Perspectives framework (Simon et al., 2000).

This framework consists of three different perspectives; namely, the traditional (TP), a perception -based (PBP) and a conception-based perspective (CBP). Researchers pointed that PBP is very close to TP and CBP is beyond them on the perspective continuum. Following, Jin and Tzur (2011) put PIP perspective in between PBP and CBP (see Table 2).

Table 1. The Knowledge Quartet: dimensions and contributory codes (Thwaites et al.,2010, p. 86)

Dimension	Contributory Codes
Foundation	awareness of purpose adheres to textbook concentration on procedures, identifying errors, overt display of subject knowledge, theoretical underpinning of pedagogy, use of mathematical terminology
Transformation	Choice of examples, Choice of representation Use of instructional materials Teacher demonstrations (to explain a procedure)
Connection	Anticipation of complexity Decisions about sequencing Making connections between procedures Making connections between concepts Recognition of conceptual appropriateness
Contingency	Deviation from agenda Responding to students' ideas Use of opportunities Teacher insight during instruction

Table 2. Placing PIP within teacher perspectives (Jin & Tzur, 2011, p. 20)

	View of Knowing	View of Learning	View of Teaching
Traditional Perspective (TP)	Independent of the knower, out there	Learning is passive reception	Transmission, lecturing instructor.
Perception-Based Perspective (PB)	Independent of the knower, out there	Learning is discovery via active perception	Teachers as explainer (points out)
Progressive Incorporation Perspective (PIP)	Dialectically independent and dependent on the knower	Learning is active (mentally); focus on the known required as start, new is incorporated in to known.	Teacher as guide and engineer of learning conducive conditions.
Conception-based Perspective (CBP)	Dynamic; depends on the knower's assimilatory schemes	Active construction of the new as transformation in the known (via reflection)	Engaging students in problem solving; Orienting reflection; facilitator.

The CBP on mathematics, mathematics learning and mathematics teaching rests on the basic principles of radical and social constructivism. In this regard, mathematics knowledge is considered "a dynamic, continually expanding field of human creation and invention, a cultural product" (Ernest, 1989, p. 250), the Problem Solving view of mathematics (Beswick, 2005; Ernest, 1989) Aligned with the Problem Solving view of mathematics, then, teachers having this perspective acknowledge that mathematical learning occurs through one's transformation (accommodation) of existing ideas (assimilatory schemes) through their own logico-mathematical mind activities (Simon, 2003; Simon, Saldanha, McClintock, Karagoz Akar, Watanabe & Zembat, 2010). Mathematics teaching therefore requires that, first, the teacher is aware of her current mathematical understandings being qualitatively different from her students' understandings (Jin & Tzur, 2011); and second, the teacher focuses on what students currently know and reason (their assimilatory schemes) rather than what they do not know (Heinz et al., 2000).

Contrary to CBP, PBP is similar to the traditional approach. Teachers' having this perspective acknowledges mathematics as an ontological reality independent of the knower (Tzur, et al., 2001). Thus, for them, mathematics learning means coming to see a first-hand experience of mathematical reality shared by all through discovery. First-hand experience refers to one's engagement in materials while learning mathematics as meaningful and interconnected body of knowledge (Tzur, et al., 2001). That is, although teachers with PBP think of mathematics interconnected, contrary to those holding CBP and also PIP, they are not able to explain how students come to understand a concept; nor can they think "about how such conceptions might be promoted when students do not perceive the intended mathematics despite the teacher's deep understanding of mathematics" (Tzur, et al., 2001, p. 250). Also, in this perspective teachers do not realize that their mathematics is different from their students' mathematics. They focus only on what students do not know rather than focusing on how they reason.

On the other hand, contrary to all the other perspectives, the main characteristic of PIP perspective is that a teacher holding PIP views mathematics as both dialectically independent and dependent on the knower. Dialectically independent means, mathematics concepts have commonalities outside the learner. The dependence on the knower refers to one's ability to do problem solving. That is, this perspective views mathematics learning as an active mental process. In this respect, Jin and Tzur (2011) stated that "...a PIP-rooted teacher's practice can engender students' learning processes envisioned by CBP without requiring the teacher's explicit awareness of such view..." (p. 20). Jin and Tzur (2011) further stated, "because of a PIP-rooted teaching seems to conducive to fostering all students' learning, it can be made a desirable goal for mathematics teacher development" (p. 19).

Different from CBP, PIP requires teacher involvement through frequent questioning as necessary to bring the commonalities/differences in mathematical concepts to the learners' attention. That is, PIP views that teachers can engender students' learning through re-activating the known 'as an anchor to which the new will be linked,..., a learning starting-point' (p. 5). This is because teachers think of the already-known as a mathematical tool to engage students in communicative activities such as explaining and justifying their solutions and attributes of an extended range of problems. In addition, "particular features of the new are gradually added and linked to the old' through students' mental and communicative activities" (Jin & Tzur, 2011, p. 17). In this regard, teachers' involvement is important for "a) introducing purposeful errors and b) taking every opportunity to expose and discuss students' mistakes—a dialectical strategy of negating what is not so students properly distinguish and embrace what is" (p. 17). Also, different from

CBP, a teacher with PIP might focus on both what the students know and reason and do not know (Jin & Tzur, 2011) Jin and Tzur (2011) created the table of differences among these perspectives (Table 2).

The aforementioned explanations point to the fact that prospective teachers holding different perspectives regarding the nature of mathematics, mathematics learning and mathematics teaching might depict different mathematical knowledge for teaching (Thwaites, Jared & Rowland, 2010) (See Table 1). For instance, prospective teachers with PBP might not be able to help students make connections between concepts, anticipate students' difficulties and respond to their students' ideas in spite of the fact that they might specifically know the 'why' behind the concepts (Rowland & Turner, 2009) because they are not able to think from their students' point of view (e.g., Tzur et al., 2001). This is also consistent with Weston (2013) study results. Weston showed that some codes such as anticipation of complexity, making connections between concepts and procedures were not demonstrated consistently by the prospective teachers during their practice-teachings. On the other hand, prospective teachers with PIP might depict the codes such as anticipation of difficulty and deviation from agenda because they purposefully take every opportunity to expose and discuss students' mistakes (Jin & Tzur, 2011). By the same token, even if prospective teachers depict the same mathematical knowledge for teaching they might do so with having different reasons. For instance, prospective teachers with PBP and PIP might depict the codes, choice of different representations /examples/ materials with the former having the reasoning such that students learn through first-hand experiences and the latter having the reasoning that students learn through active mental participation (Jin & Tzur, 2011). This also was consistent with Weston (2013) results. She found that although different prospective teachers demonstrated the same codes in Knowledge Quartet, the nature of such demonstration differed from one prospective teacher to the other in terms of how much of such knowledge they had. In this regard, scrutinizing the coherency between teacher perspectives and the domains in the Knowledge Quartet might help uncover the reasoning behind (prospective) teachers' mathematics knowledge for teaching. Diagnosing the reasons might provide teacher educators with particular steps to follow towards establishing more sophisticated perspectives and a full grasp of mathematical knowledge for the teaching on part of prospective teachers.

METHOD

Participants

Data from two prospective secondary mathematics teachers were chosen for the study. These prospective teachers were in their fifth year in a teacher education program at one of the best English-language universities in Turkey. They were able to construct cognitively high-demand tasks they used in their practice-teaching. I used the data from these prospective teachers because it was representative among data coming from the rest of the participants in the study and provided contexts in algebra and geometry which allowed me to examine the relationship between PIP and the mathematical knowledge for teaching in two different contexts.

Data collection

What is reported here is part of a larger qualitative research study the purpose of which was to investigate how PIP on mathematics, mathematics learning and mathematics teaching could be developed on the part of prospective secondary mathematics teachers and what practices could afford for such development (Karagöz Akar, Delice, & Aydın, 2015). In this regard, classroom teaching experiment methodology (Cobb, 2000) was used in the larger study in which data were collected for a total of eleven weeks during both a methods and the seminar part of a practice-teaching course. This methodology requires the researcher to plan the classes prior to the enactment of the study. In this regard, I designed the teaching sessions prior to the methods and practice-teaching (seminar part) courses, taking previous research into account. However, as one of the important aspects of the methodology obliges the researcher, for each teaching session, the (sub) learning goals depending on the hypotheses about what prospective teachers knew were revised. Again since the methodology requires the researcher to collect data through videotaping, all lessons within the methods and practice teaching courses were videotaped and transcribed right after each teaching session. The design of the courses was developed based on the following criteria: Conjecturing that once prospective teachers realized the nature of mathematics, this could help establish the nature of mathematics learning (with understanding) towards PIP, Thompson's (1994, 2011) quantitative and numerical operations framework was used. Thompson argued that mathematics is built through quantitative operations (non-numerical operations such as subdividing, counting, matching etc.) rather than being built upon numerical operations (evaluation of quantitative operations -- addition, division, etc.). This view aligns with CBP and Ernest's (1989) Problem-Solving category, too. Based on this view of mathematics, then, mathematics learning occurs through the abstraction of the regularities in one's own logico-mathematical activities, such as subdividing, matching, etc. (Simon et al., 2010) through tasks. Thus, prospective teachers were provided with the Simon (2003) framework for the distinction between the tasks focusing on logico-mathematical learning processes and empirical learning processes and the Task Analysis Guide (Stein, Smith, Henningsen & Silver, 2000). Third, conjecturing that prospective teachers need to examine '...all mental operations a teacher needs to carry out' (Thompson, 2008), they were provided with a case study (Stein et al., 2000) to analyze with conceptual analysis framework (Thompson, 2008). In addition, conjecturing that clinical aspects of teaching such as interviewing need to be included in (prospective) teachers' knowledge repertoire (Grossman & McDonald, 2008, Heinz et al., 2000) it was brought to prospective teachers' attention. Finally, the lesson planning model for teaching secondary school mathematics (Wilburne & Peterson, 2007) was introduced to the prospective teachers. They then did their lesson planning for their peer-teachings and practice-teachings and reflected on their teaching (Hiebert, Morris & Glad, 2003). For the

practice-teaching experiences prospective teachers prepared lesson plans one week prior to the actual teaching sessions. During that week, they got feedback on their lesson plans from their peers. They also met the instructor of the course (I) to explain the rationale behind their lesson plans. Then, they taught the lessons.

As part of the larger study, to observe the affordances of the practices done in the methods and the practice teaching courses, for this study, prospective teachers' practice teachings were videotaped and transcribed afterwards. Also, I met with the prospective teachers to talk about their lesson plans and observed their lesson plans. In addition, they wrote self-reflection papers after watching their videotaped lesson. Prospective teachers wrote self-reflection papers based on reflection-tasks (Öner & Adadan, 2011). Reflection-based tasks were detailed enough that allowed for accounting for the prospective teachers' practices and their perspectives. For instance, the reflection-based tasks involved providing evidences on the reasons why prospective teachers think that they accomplished the learning goals for their students; what examples/tasks they used and how they knew that those tasks/examples afforded their students' understanding; what difficulties they could think of regarding their students' understandings, etc.

Data shown in this study, therefore, included prospective teachers' lesson plans, practice-teachings and self-reflection papers. Alex's practice-teaching was held in a public high school for orphans. He taught a 40-minute lesson to the 10th grade students. Sarah's practice-teaching were held in a private high school. She also taught a 40-minute lesson to the 10th grade students.

Data analysis

For the analysis, I examined their i) teachings, ii) lesson plans and iii) self-reflection papers. I focused on two things: first, the meanings these prospective teachers attributed to mathematics, mathematics learning, and mathematics teaching; and, secondly, their mathematical knowledge for teaching in the particular concepts.

I used coded analysis (Clement, 2000) using both frameworks. Clement (2000) stated

...a coded analysis... focuses on observations that are assigned to predefined categories by a coder, usually from relatively small segments of a transcript. A transcript is coded when the analyst formulates criteria for recognizing a phenomenon and then lists the places where the phenomenon occurs in the transcript. The conclusions than may be at the level of observation patterns alone, or, they can be used as data to support or reject theoretical hypotheses that may have been generated by other means. (p. 558)

In this respect, I analyzed the data to support theoretical hypotheses generated by the two frameworks—Teacher Perspectives and Knowledge Quartet-- and provide empirical data to show the coherence between these two frameworks that might yield to hypothesis generation, in the following way: First, I read each of the transcripts from their practice-teachings line-by-line, looking for Alex's and Sara's explanations (within situations) regarding their perspectives on mathematics, mathematics learning and mathematics teaching. Using the characteristics of teachers' perspectives given in Table 2, I looked for their existing meanings. Once I spotted a line of explanation regarding their meanings in any of the data sources, I also checked their lesson plans and reflection papers that could possibly provide further evidence of such meaning. Based on the conjectures, I continued to examine the rest of the data. Then, I went back to the whole data set to challenge my conjectures, and modified them to cohere with the whole data. Secondly, using the codes from Knowledge Quartet (see Table 1), I read each of the data sources line-by-

line, looking for situations regarding their mathematical knowledge during teaching. I also used the code 'emphasizing why the graphs must be as they are, rather than how to arrive at them' under the transformation dimension and the code 'responding to the (un)availability of tools and resources' (Thwaites et al., 2011, p. 228) under contingency dimension for the prospective secondary mathematics teachers' mathematical knowledge in teaching. Then, I went back to the whole data set to challenge my conjectures. When my conjectures were challenged I modified them to cohere with the whole data. I also asked two colleagues to challenge/affirm my conjectures. Finally, I wrote the narratives about these prospective teachers' perspectives and mathematical knowledge for teaching.

FINDINGS

In this section, first, under each Knowledge Quartet dimension, the two prospective teachers' mathematical knowledge for teaching is shown. Secondly, why they might have such mathematical knowledge for teaching was depicted through the analysis of their perspectives on mathematics, mathematics learning and mathematics teaching.

Alex's lesson

Alex created his own task sequence for his practice-teaching. His task included three key mathematical ideas: a triangle's bisector being equidistant from its sides, a triangle's bisectors intersecting at the same point, and the relationship between a triangle's bisector and the inscribed circle tangent to that triangle. Due to space limitations, only data from the first two key mathematical ideas are shared.

Foundation. In his lesson plan, adhering to National Curriculum, Alex pointed to the fact that students would be able to state 'why' any triangle's bisector is equidistant from its sides. He explained (see Figure 1).

Data show that Alex thinks of folding papers through the ray-OP, putting two triangles on top of each other constructing congruent triangles. That is, Alex knows the 'why' behind the concept he will teach and he would like his students to reason about it, too. One might argue that the data seems to point to the fact that Alex might have reasoned on the equality of the triangle's angles while thinking about the congruency between these two triangles. However, data from Alex's lesson will evince that he focused his students' attention on the equality of the sides (side-side-side formation of congruency) to come to the reasoning behind the congruency. Then he asked them to think about the angles to take them to the idea of bisector. This suggests that Alex is aware of his purpose, displays well-established subject matter knowledge and uses mathematical terminology correctly.

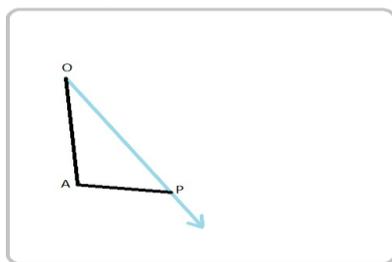
During the lesson, he first asked students the meaning of a triangle, why the three points making a triangle must be non-linear, and if students could show an angle and/or a side of a triangle etc. He explained why he asked such questions during the lesson in his self-reflection paper. This will be shown later. Then he provided the students with different shapes (see Figure 2a).

After this, he wanted them to copy the original triangle to the other side using their pencils (see Figure 2b). The students talked about the shapes they came up with. Some students constructed triangles; others, quadrilaterals. They even discussed the reason 'why' some students constructed triangles. One student reasoned 'Because... because the first triangle is right-angled, when we fold it, I get a triangle again'. Discussion followed.

A: Ok. Good. Now, there are two triangles on the quadrilateral we have.

What kind of a relationship do you see?

Some students: Congruent triangle

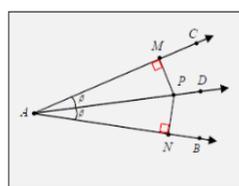


Students will be given parchment papers on which there is arbitrarily drawn triangles like above picture.

They will fold the paper through [OP- ray, and will get a triangle POA' congruent to AOP. This fact will be realized by students and they will discuss *why* those two triangles are congruent.

They will also observe that [OP- ray is actually a bisector of the angle $m(\text{AOA}')$.

They will again discuss *why* that semi-line is an angle bisector.



1. If $P \neq A$ and P is on the $[AD\text{-ray}$ then the point P is equidistant to $[AB\text{-ray}$ and $[AC\text{-ray}$
2. If the point P is equidistant to $[AB\text{-ray}$ and $[AC\text{-ray}$, then $P \neq A$ and P is on the $[AD\text{-ray}$

Findings :

- 1) $m(\text{BAD}) = m(\text{CAD})$
- 2) $m(\text{PMA}) = m(\text{PNA})$
- 3) $PM = PN$

Reasonings:

- 1) the definition of bisector.
- 2) equality of right angles
- 3) Congruency of triangles

Figure 1. Alex's written argument for the first learning goal for his students

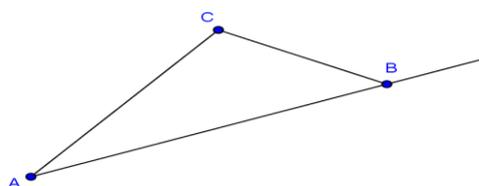


Figure 2a. One triangular shape Alex provided for his students. (Then, he asked them to fold it through the ray-AB and unfold it.)

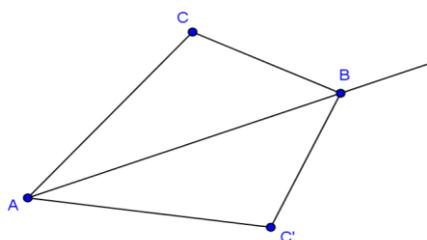


Figure 2b. Students' un-folding the parchment paper on which a triangular shape was drawn

A: Is there anyone who does not agree? As everyone agrees with congruent triangle, can one of you explain why they are congruent?

S3: The distance drawn to the bisectors are equal.

A: What is the bisector?

S3: The line in the middle is the bisector. When we write the degrees, they become equal.

A: Well, let's compare the angles of these two triangles. Are there equal angles?

Chorus: Yes.

A: Which angles are equal?

S3: Angle CAB is equal to angle DAB. When we draw lines with equal angles, the bisector, they become equal.

A: Ok, we don't know the bisector yet. Can anyone explain without using the term bisector? You can start with sides or vertices. Someone is saying symmetric -- let's listen.

S3: AB and BD lengths are equal.

A: Why are they equal?

S3: Because we drew the same line segment.

A: Well, are there any other equal lengths?

S3: AC and CD lengths are equal, too. BC is already common for both, so those sides are also equal.

A: Now, it is time to talk about angles. We said that since sides are equal in these triangles, they are congruent triangles. Well, are there any congruent triangles due to their angles?

S3: Yes, there are.

A: Which angles are equal then?

S3: ABC and CBD

A: Why equal?

S3: They overlap when we fold.

A: Ok, very well. Now, I want you to think about a broader angle. I want you to think about angle ABD. When we think about angle ABD, what does ray BC make here?

S3: It divides the angle into two – in fact, it divides it into two equal parts.

T: What do we say about a ray dividing an angle into two?

S3: Bisector.

A: Now, let's think about point C on the bisector. What can we say about the distance of point C from points A and D?

Some students: The distances are equal.

A: Why?

Chorus: We get a bisector.

A: What else?

Some students: The points on the bisector are always equidistant to the bisector arms.

A: Ok, very good. Now, we move to the next activity.

Further analysis of this excerpt will be left for the transformation dimension. However, the importance of this data together with what Alex has written in his lesson plan is that Alex was aware of his purpose and subject matter knowledge for the lesson: First, he let guided them to focus on the symmetry of the sides so that they could reason on the congruency. Secondly, although some students mentioned the bisector and the congruent triangles, he did not take it as an answer and asked for their reasoning. One student mentioned "they overlap when we fold". This suggests that Alex focused on his students' mind activities such as such as

‘reflecting’ and ‘matching’”, for justifying that the sides and the angles are the same, creating two congruent triangles. In this way they were able to reason that the ray in between the two triangles was the bisector, dividing the bigger angle. Also, since they all had different triangles and therefore different points on the bisector, they could come to the conclusion that “the points on the bisector are always equidistant to the bisector arms” as they stated. In this regard, he created a learning environment for his students to realize ‘why’ the points on a bisector are equidistant to the angles-rays. Alex wrote in his self-reflection paper:

In my lesson plan, my goals were to make student discuss

- 1) why a point on a angle bisector is equidistant to angle-rays
- 2) why all angle bisectors intersects at one point in a triangle
- 3) why the intersection point of angle bisectors is the center of *incircle*

In traditional teaching methods, the above observations are directly given to students without reasoning on why it happens, which in turns lead to rote learning. However, my aim was that students construct these properties which is the core idea of constructivism. Probably some students (since they might hear from somewhere else) would tell some properties which I want them to construct in this lesson, in order to prevent such rote knowledge, I always asked the students to explain what they say regarding the findings during the lesson.

For example, when they combined the papers distributed for second activity, I asked what the shape is look like. All students replied that it is a triangle. I asked why it is a triangle. One of the students said because sum of its interior angles is equal to 180° but I intervened and asked him to take the definition of triangle we recalled at the beginning of the lesson. After we discussed why it is a triangle regarding the vertices and linear edges. I came back to students who told the angle sum to explain his reasoning.

Data above show that Alex realizes the reason why students need to construct their own knowledge; “to reason on why it happens” in his own words. That is, Alex’s reason for why he created such learning environment, the conditions under which students best learn, was that Alex acknowledged that students learn mathematics through their own mind activities, such as reflecting the lengths and the angles through folding and matching them. In addition the fact that Alex asked for justification for why the lengths and the angles are equal based on his students’ mind activities (reflecting and matching) stated by S3, “we drew the same line segment...they overlap when we fold” rather than merely stating deductive propositions suggests that he thinks of mathematics and mathematics learning as dependent on the knower, the characteristics of PIP.

Transformation. At the end of the section above Alex drew a quadrilateral that looks like the shape on the students’ papers and moved the bisector back and forth using GeoGebra (see Figure 3). The students observed that a point (such as B and G) on the bisector is in the same distance from the angle-rays. This data, along with Alex’s providing his students with different examples of triangular shapes, and his choice of parchment papers and colorful shapes for students’ constructions (see Figure 4a & 4b) all indicate that Alex is able to choose different instructional materials and examples. Also, Alex’s use of GeoGebra to show again that the points on a bisector are always equidistant from its angle-rays shows that Alex is able to direct his students’ attention to the key understanding (learning goal) using different representations. In essence, one could argue that Alex’s showing that a point on a bisector is always equidistant from its angle-rays on GeoGebra might

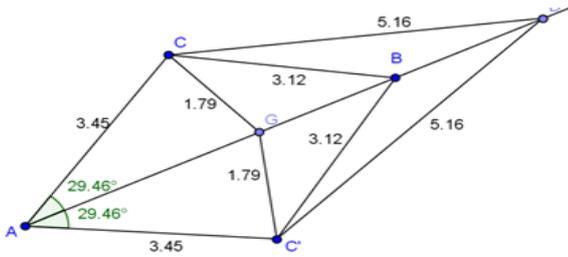


Figure 3. A quadrilateral Alex drew on GeoGebra

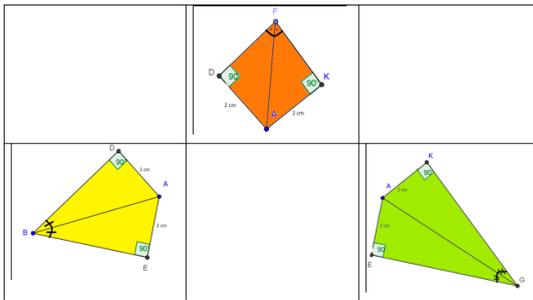


Figure 4a. Some examples of different shapes each student is given

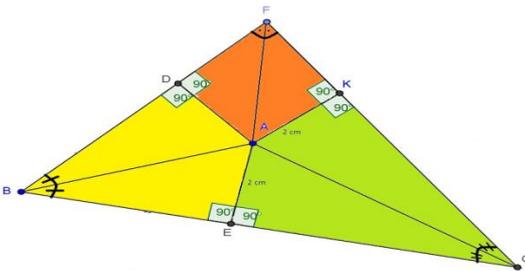


Figure 4b. An example of such construction

indicate that he thinks of mathematics as independent of the knower (that is, everyone seeing a representation observes the same thing). However, his first asking students to do the folding activity individually and explain their justifications on their mind activities (reflecting and matching) might indicate that he viewed mathematics as dialectically independent and dependent on the knower. This also suggests that he emphasizes ‘why the shapes must be as they are’ (Thwaites et al, 2014).

Connection. Students were given colorful shapes and asked to bring them together (see Figure 4b). Then, as Alex pointed to in his self-reflection paper, they discussed why, for instance, the points B, D and F have to be linear, and why the shape is a triangle. Discussion followed:

A: ...What can you say about the bisectors?

S4: All of them intersect at point O.

A: Ok, can you explain it more? Why do they intersect?

S4: If the two intersect, the third will also intersect.

A: Why do the two intersect? You can think about the features of the first pieces I gave you.

S3: When we lay them together, due to the equal lengths of pieces, they intersect at point O, right?

S2: Yes, the other corners (end points) stay on the side of big triangle.

A: Well, do bisectors intersect at one point in every triangle?

Data shows that Alex was not after what his students knew. Rather he was after how they reason so that he could guide them to make connections between the triangle's bisectors being equidistant to its angle-rays and their intersecting at the same point. In particular, Alex's students reasoned about the fact that any point on a bisector is equidistant from the angle-rays and therefore the points B, D, F; F, K, G; and B, E, G (see Figure 4b) have to be linear, creating a triangle. With this conclusion, the same reasoning enabled them to deduce that the line segments AD, AK, and AE have to be the same length, making them intersect at the same point in the triangle. That is, Alex was able to sequence the concepts in the lesson in such a way that students thought the idea of 'When we lay together, due to the equal lengths of pieces, they intersect at point O' to construct the idea of a triangle's bisectors intersecting at the same point.

This suggests that Alex thinks of mathematics as dependent on the knower because he wants his students to reach the newly learned idea (the bisectors in a triangle intersect at one point) from their earlier understandings. This is also why he asks questions to get at his students' reasoning. That is, Alex used gradual- continual questioning to assist students link the known with the newly learned.

Thus, these data suggest that the reason behind his knowledge under the connection dimension was that Alex realized that students made connections on their own and learned mathematics through their own reasoning. All these indicate that he viewed mathematics, mathematics learning and mathematics teaching from PIP.

Contingency and transformation. The last question Alex asked was "do bisectors intersect at one point in every triangle?" At that point, one student said "no. Might it not be true for right triangles?" Then Alex showed it on GeoGebra by moving one vertex of the triangle (See Figure 5a, b). This inquiry helped his students observe the fact that for right triangles also the bisectors intersect at one point. This suggests that Alex was able to deviate from his addenda by responding to his



a



b

Figure 5. Alex's showing that the bisectors of a triangle intersect at one point

students' ideas. He used this as an opportunity to demonstrate and explain that for not only a right triangle, but for all triangles, the bisectors intersect at only one point. That is, he took it as his responsibility to overcome his students' difficulties. This also indicates that he views mathematics teaching from PIP.

Sarah's lesson

Sarah also taught 10th grade students. The task she modified from the textbook included finding the solution set for the trigonometric equation of the form $\sin x = a$. In her lesson plan, she mentioned that students would be able to "state the difference between the

Solution sets depending on the intervals" and 'would know the reasoning behind the solution set for $\sin x = a$ '.

Foundation. Sarah pointed to the difference between solution sets depending on intervals and the set of infinitely many solutions (See Figure 6a).

Data suggest that Sarah was aware of the limitations of the solution sets on intervals. Also, her mention of the 'unit circle' indicates that she wanted to bring her students to the set of infinitely many solutions for the equation at hand. More importantly, she provided the following in her lesson plan (See Figure 6b).

Teacher's activities	Students' activities	Time allocated
<p>Teacher asks how to solve the following equation: $\sin x = \frac{1}{2}$</p> <p>Teacher will ask them to think about whether there can be any other angle which satisfies this equality. She will give them time to think about it.</p> <p>If students cannot find any other or find just some more, teacher can give small clues to them. For example, she can ask them to draw a unit circle and think through it.</p> <p>After enough time, the teacher will ask the students what angles they have found. There can be small discussion about whether there is a limited number which times we can find other angles.</p> <p>Trying to get a conclusion of infinitely many</p>	<p>Most probably all students realize that x may equal to 30 degrees.</p> <p>Students will try to find other angles which satisfy ($\sin x = \frac{1}{2}$)</p> <p>Students may or may not get the conclusion of infinitely many angles.</p>	25 min

Figure 6a. Sarah's explaining the learning goal for her students in her lesson plan

<p>Trying to get a conclusion of infinitely many angles, teacher can say the students to use their TI calculators.</p> <p>Teacher first will ask them to try to draw graph of $\sin x = \frac{1}{2}$ on their TI calculators.</p> <p>If they cannot do, teacher can help them. For example, teacher will say that think about the following question:</p> <p>First let's draw graphs of $y = \sin x$ & $y = \frac{1}{2}$. Then what do the intersection points give us?</p> <p>In order to help students, teacher can use online TI calculator on computer and show the students such a graph we have.</p> <p>After students get the point, teacher will ask them to count the intersection points on the graph.</p>	<p>Students will be realized that the intersection points satisfy the equation $\sin x = \frac{1}{2}$</p> <p>Students will be realized that there can be infinitely many answers for this equation. It depends on the domain interval.</p>	
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Figure 6b. Sarah's plan for using TI calculators during her lesson

The data suggest that, Sarah realizes that the equation $\sin x = 1/2$ represents the two functions'--- $y = \sin x$ and $y = 1/2$ --- intersection points. Together with the data from Figure 8, this suggests that she realizes that one can find solutions to this equation first in the interval $(0, 2\pi)$, then in other intervals, and then in the interval $(-\infty, \infty)$. Using TI calculators further helps students observe that there are infinitely many intersection points for the two functions. This suggests that Sarah is aware of her purpose for this lesson and has a good grasp of subject matter knowledge.

Data from the end of the lesson also pointed to her awareness of purpose.

Sarah: Ok, what did we learn about today?

Some students: Trigonometric equations.

Sarah: Trigonometric equations. Ok. What is important about trigonometric equations? Is there any significant point or is there any points we need to be careful about? Did you realize our solution sets can be changed according to our domain (pointing to $(0, 4\pi)$). When our domain is this interval (pointing to $(0, 2\pi)$), the only angles which satisfies are these (pointing to 30 and 150 degrees). What about this (pointing to $(0, 4\pi)$) interval?

S2: 30 degree, 150 degree, 390 degree and 510.

Sarah wrote it down on the board while S2 was talking.

Sarah: So, as you realize, our solution sets can be changed according to our domain intervals. So, can we say domain interval is important?

Chorus: Yes.

Connection and transformation. When the lesson started, Sarah first asked the meaning of 'equation' and 'trigonometric equation'. Then, she asked if $\sin x = 1/2$ is a trigonometric equation. Following, she asked for solutions to the equation given the interval $(0, 2\pi)$. One student stated 30, 150 and 240 degrees. Then, Sarah asked if the student could draw it on the board (See Figure 7). The student drew it and stated 'but it is not?' Discussion continued.

Sarah: what is the sign of this value? Is this negative or positive?

S1(on the board): It is negative.

Sarah: What do we ask for then (pointing to '1/2' on the equation)

S1: Ohh, yes, so this we cannot accept.

Sarah: So, does this value (pointing to 240) satisfy this equation (pointing to the equation)?

Chorus: No.

Sarah's starting the lesson asking students the definition of 'equation' and 'trigonometric equation' and continuing with a relatively easy example shows her choice of example was within the scope of students' knowledge repertoire. Her asking S1 questions to think about the sign of the value of 240 degrees suggests that she was able to anticipate S1's difficulty in providing a correct answer.

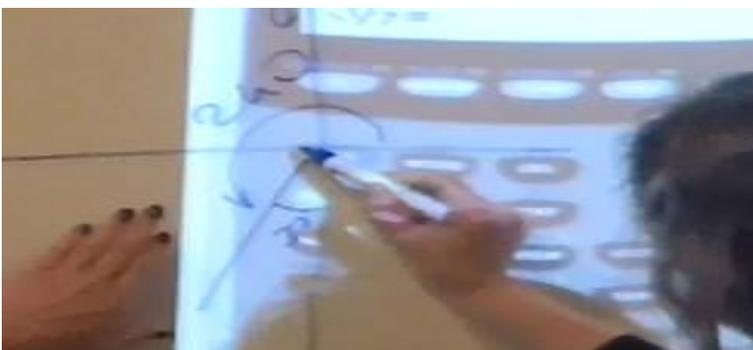


Figure 7. S1's showing 240 as an answer on the board

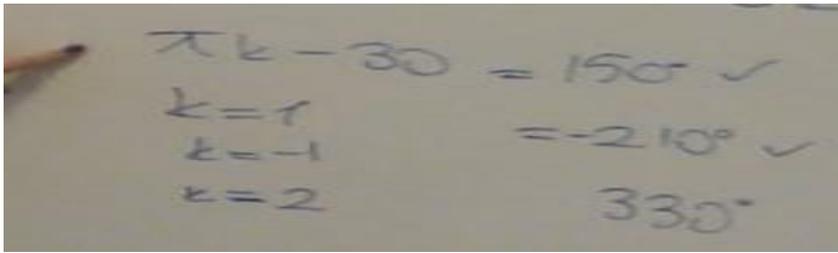


Figure 8a. Sarah writing $\pi k - 30$ and finding different solutions using it

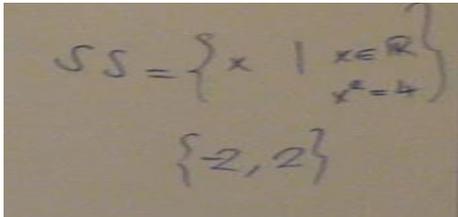


Figure 8b. Sarah's summary of students' responses

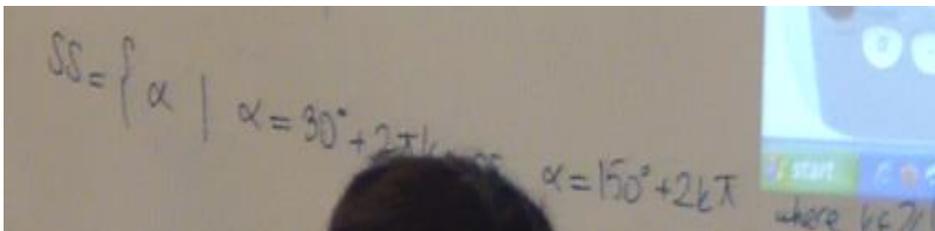


Figure 8c. Sarah's summary of students' responses

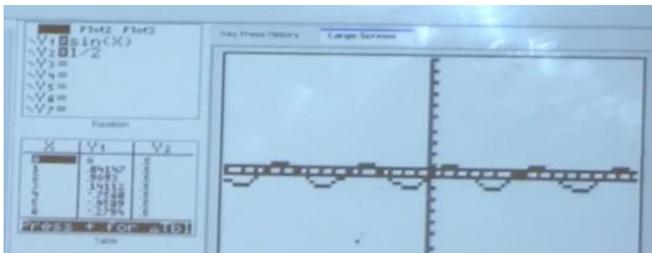


Figure 8d. One student showing $\sin x = 1/2$ on TI84+

The reason behind her knowledge in these dimensions was because she not only wanted to assess what her students knew but also wanted them to be prepared for the task. Her continual questioning to get at how her students reason even when they provided wrong answers, suggests that she took it her responsibility to help overcome her students' difficulties. All these suggest that she views mathematics teaching from a PIP.

Then, Sarah asked her students to think about the solutions in different intervals such as $(0, 4\pi)$ and show them on the board; they found 390 and 510 degrees. Then she asked for the solution set for the interval $(-\infty, +\infty)$. One student said there was a formula for that. Discussion followed:

Sarah: It is okay. But how can we find it?

S2: Every time we add 360 to 30 we can find an angle that is, that is considered as $\frac{1}{2}$, do you understand? If we add 360 to 150 we will find 510 and if we add 360 to 510 then we will see another result so after adding 360 we will find an answer.

Sarah: So, how do you know? After adding 360 how can you be sure you are getting the same angle?

S2: Because, 360 is the degree of the circle and after every full circle, we will come to the same spot, so if we read it, it will be the same result.

Sarah: Is there anybody who does not understand what [S2] said?

Sarah's choice of specific intervals suggests that she was able to sequence the activities in the lesson such that her students could make connections between the procedures (finding solutions within specific intervals) and the concepts behind such procedures (why the rules applies to infinitely many solutions). Also, the planning and the implementation of the lesson sequence indicate that Sarah anticipated the complexity of reasoning about the solution set for the infinitely many solutions for the equation as well as its importance for finding solutions. That is why she also had planned for the execution of the solutions for different intervals.

Data from the excerpt suggests that Sarah focused students' reasoning on how to come to the solution set and why. Rather than accepting one student's answer -- 'there is a formula for that' -- her acceptance of S2 's explanation suggests that she focused her students' attention on their mind activities (such as rotation). This suggests that she acknowledged that students' reasoning on their own mind activities (such as rotation) might have assisted them in reasoning about the formula for infinitely many intersection points. This indicates that the underlying reasoning behind Sarah's knowledge in these domains is her acknowledgment that mathematics learning is dependent on the knower.

Transformation and contingency. Data in this section corresponded both with transformation and contingency dimensions. While talking about the solution set for the interval $(-\infty, +\infty)$, while S2 reasoned as shown in Connection dimension, Sarah wrote down $30+2\pi k$. Discussion followed:

Sarah: Is this the only formula for this equation (pointing to $\sin x=1/2$)?

S3: And, there is another one.

Sarah: There is another one. Ok, what is this?

S3: We added $150+2\pi k$.

Sarah: Did you hear [S3], [S4]?

S4: (silent) (laughter in the class)

Sarah: I am asking whether there is any other formula which satisfies this equation? Or is this formula enough?

S2: I think this is enough.

Sarah: S2 says this is enough. Who agrees this is enough?

S5: $\pi k-30$.

Sarah wrote down it on the board and solved for $k=1$ (see Figure 8a)

S3: What about for $k=-1$?

Sarah asked one student to show it on the board. She found its actual value as 150. Sarah asked if it satisfied the equation and students said 'yes'. Discussion followed:

Sarah: What happens if k is 2? $2\pi-30$. Does this satisfy our equation S5?

S5: No.

Sarah: No. This formula (pointing to $\pi k-30$) satisfies our equation for some angles but not all angles. So this formula...

Students (almost all of them interrupting Sarah): Does not work.

First, data show that responding to her students' ideas, Sarah was able to explain the expression to make it clear to her students Also, Sarah was able to listen to the students' ideas and respond to their reasoning, taking those times as an opportunity to overcome their difficulties. Then, she asked for a formula for finding the values of 150 and 510. After one of the students answered ' $150+ 2\pi k$ ', Sarah explained it on the board one more time. Then, she asked about how to represent the solution set.

One student stated the union of the expressions '30+ 2πk union 150 + 2πk'. Sarah pointed to whether those were sets and one student stated 'no, it is an expression'. Discussion followed:

Sarah: How can we show it?

Since there was still no explanation from the students, Sarah wrote down on the board:

Sarah: Think about this equation (writing $x^2 = 4$) What are our solutions?

Some students: 2 and -2.

Sarah. Yes, so how can we represent the solution set?

Some students talked simultaneously while Sarah wrote on the board (See Figure 8b).

Sarah: Ok. Let's think about this one (indicating $30 + 2\pi k$ union $150 + 2\pi k$ on the board). Let us start similarly, our solution will include all angles let us say alpha such that, how do our angles have to be?

S2: our rule is $30 + 2\pi k$.

Sarah then wrote.

Data above are important in two ways. First, Sarah deviated from her agenda, having the insight that her students had difficulty in coming up with the solution set mathematically for the equation of $\sin x = 1/2$. Also, she asked for whether there were any other solutions for the equation at hand. On the other hand, her use of degree and the radian measures in the same solution set points to lack of knowledge in terms of the mathematical terminology on her part. However, her question regarding whether $\pi/6 + 2\pi k$ is a set or not does show that she had good grasp of subject matter knowledge. Similarly, the reason behind Sarah's using $x^2 = 4$ and letting her students find the solution set for this equation to assist them writing the solution set for the equation, $\sin x = 1/2$, was because she wanted her students to incorporate the newly learned knowledge (the solution set for the equation of $\sin x = 1/2$) into the known (the solution set for the equation, $x^2 = 4$). This suggests that she had PIP.

Sarah also wanted her students to graph $\sin x = 1/2$ on the TI 84+ calculator. This suggests that she was aware of different instructional materials. However, her students had difficulty transferring their knowledge of drawing two functions' intersection on a TI 84+ for trigonometric equations. Sarah asked them to think about another equation, $x^2 = 5$ to show on the TI 84+. This suggests that her insight helped her in terms of anticipating the students' difficulties, such that only after they thought about the $x^2 = 5$ equation on the TI 84+, one student came to the board and showed his classmates how to figure out $\sin x = 1/2$ on the calculator (See Figure 8d).

This suggests that Sarah was able to deviate from her agenda because she took it her responsibility to assist her students' overcome their difficulties and make connections between the algebraic and the graphical representations of the equation $\sin x = 1/2$.

DISCUSSION

Two prospective secondary mathematics teachers' practice-teaching provided data from two different contexts, geometry and algebra. Results of this study showed that prospective teachers having PIP on mathematics, mathematics learning and mathematics teaching (Jin & Tzur, 2011) demonstrated all the aspects of mathematical knowledge for teaching (Rowland et al., 2005; Thwaites et al., 2011) and vice versa. Particularly, data showed the reasons behind the nature of

prospective teachers' foundational knowledge and other domains in Knowledge Quartet.

First, data suggested that both prospective teachers' awareness of purpose (Rowland et al., 2005) was based on their acknowledgement such that students' mathematics learning is an active process dependent on the knower. The particular aspect of their awareness for the learning goals for their students was different from prospective teachers who "...may already think that that they should teach for understanding, they might have a limited notion of what understanding something in mathematics means" (Ball, 1998, p. 16). That is, these prospective teachers' lesson plans involved two cognitively high demand tasks in which the focus was on reflecting, matching and rotating, compatible with the Problem Solving view of mathematics (Ernest, 1989). That is, since they realized that mathematics is a construction on the part of knower (Ernest, 1989) they were able to create/modify tasks prior to their teaching.

Also, both prospective teachers' focus of attention on quantitative operations—reflecting, rotating etc.— showed that they were able to think from their students' point of view. This was also evident in their implementation of the lesson plans. Both of them accepted their students' explaining 'why' behind the concept(s) not when students stated their conclusions but when they made explanations based on their mind activities such as reflecting and rotating. For instance, in Sarah's lesson, S2 stated "Every time we add 360 to 30 we can find an angle that is, [whose sine value] is considered as $\frac{1}{2}$...Because, 360 is the degree of the circle and after every full circle, we will come to the same spot, so if we read it, it will be the same result". For someone to be able to make such statement s/he needs to focus on the full rotation of 360 degrees. Similarly, in Alex's lesson, S3 stated "they overlap when we fold them" and Alex suggested that the student focused on reflecting and matching. In this regard, data showed that these prospective teachers viewed mathematics dependent on the knower (Jin & Tzur, 2011) and mathematics learning as an active process. This stood as the root of the codes in foundational knowledge they depicted.

In addition, both prospective teachers' creating/modifying tasks and the way they engaged their students in the tasks evidenced the reasons behind other domains, too: Rowland (2010) stated that unpacking secondary mathematics knowledge for teaching is difficult since it depends on teachers' subject matter knowledge and connected understanding of the different ways in which such knowledge can be represented. In this respect, both prospective teachers' ability to think in terms of their students' mind activities, such as rotating, matching etc. while creating and/or modifying tasks enabled them unpack their mathematical knowledge for teaching for their students (Ball et al., 2008). As the data from the lesson plans and their teaching showed, their unpacking of such knowledge involved hypothesizing possible students' answers to the teachers' questions. This placed the tasks they created/modified within the scope of their students' cognitive reach, such that they included examples and representations (transformation dimension) appropriate for the gradually sequenced learning goals for their students (connection dimension). For instance, Alex started the lesson re-activating his students' current knowledge on triangles in order to link it to the knowledge that any point on an angle bisector in a triangle is equidistant from the angle-rays (sides of the triangle). He then re-activated this knowledge on their part to deduce why all the bisectors intersect at a point in a triangle. By the same token, Sarah started the lesson by re-activating her students' knowledge of what 'equation' and 'trigonometric equation' mean in order to incorporate the particular equation $\sin x = \frac{1}{2}$ into the already known. These data suggested that both prospective teachers viewed mathematics learning as the old incorporating the new (Jin & Tzur,

2011). These data also suggested that they acknowledge that it is their responsibility to make apparent the commonality among concepts for their students through frequent questioning. These characteristics of PIP stood as the roots of their knowledge in the domains of transformation and connection.

Both prospective teachers listened to their students' reasoning rather than determining what their students' did not know. However, when they spotted gaps in their students' knowledge, they took it as an opportunity to guide them to what they needed to learn. In this regard, through questioning, they both dialectically engaged their students in negating what is not so that they could properly distinguish and embrace what is. This again indicated that they acted based on the fact that mathematics is dialectically independent, but dependent on the knower and mathematics learning is an active process on the part of knower through justifying, explaining the reasons (Jin & Tzur, 2011). These characteristics of PIP stood as the roots of their knowledge in the domain of contingency.

Overall these results showed that there is correspondance between PIP perspective and all of the codes in Knowledge Quartet. These results suggest that determining the perspective with which prospective or inservice teachers act in terms of their mathematical knowledge during teaching might enable teacher educators to analyze the principles (reasons) behind such knowledge.

IMPLICATIONS FOR TEACHER EDUCATION AND RESEARCH

Results from this study were based on data from two prospective teachers. This makes the generalizations bounded in the context. However, the results of the study showed that considering Teacher Perspectives and Knowledge Quartet frameworks together might provide teacher educators with the reasons why prospective teachers reveal in teaching what they know in terms of their mathematical knowledge. In this regard, teacher educators might use both frameworks in juxtaposition to each other while examining prospective teachers' mathematical knowledge in action. Knowing 'why' behind the affordances and limitations of prospective teachers' mathematical knowledge for teaching might assist teacher educators to determine how to design and/or modify methods and practice-teaching courses and/or professional development activities for inservice teachers.

Results also bring about the need to do further research on the coherency between other teacher perspectives and the mathematical knowledge prospective teachers might reveal during teaching. This is especially important because although prospective teachers might reveal the same kind of characteristics in their teaching practices (Watson, 2013), the reasons behind their practices (actions) might differ.

What is reported here did not focus on the effect of the design of the methods and the practice-teaching courses on prospective teachers' development of different perspectives. However, based on the results of this study I propose that scrutinizing i) the nature of mathematics through quantitative and numerical operations, ii) the nature of mathematics learning and therefore the tasks focusing on such learning through distinctions between logico-mathematical and empirical learning processes and, iii) the nature of mathematics teaching through conceptual analysis and clinical interviewing during methods and the practice teaching courses might afford the development of PIP in prospective teachers. As evidenced by the data, situated in one geometry and one algebra lesson, this might help them develop mathematical knowledge for teaching in action regardless of the particular concept they teach.

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