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Reflective Awareness in Mathematics Teachers' Learning and Teaching

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The nature of mathematics teachers' knowledge specific to teaching mathematics [MTK] is of ongoing concern in mathematics education research. This article contributes to our under-standing of this knowledge with particular focus on reflective awareness. It discusses MTK based on ways it has been used in research. It highlights reflective awareness as a central aspect of MTK based on a study of elementary school mathematics teachers' learning and teaching associated with a self-directed professional development initiative to transform their teaching to an inquiry-based perspective. Research questions focused on how this initiative supported reflective awareness in the teachers' learning and teaching. Findings indicated that engaging in self-based and meaning-based questioning and creating pedagogical models were central to the teachers' learning and use of reflective awareness. Their knowledge of reflective awareness as a way of knowing was important to their development of an inquiry stance and knowledge of mathematics for teaching and mathematics pedagogy.

Keywords: Mathematics teachers' knowledge, mathematics teachers' learning, pedagogical models, questioning, reflective awareness.

INTRODUCTION

The nature of mathematics teachers' knowledge specific to teaching mathematics is of ongoing concern in mathematics education research. This knowledge is essential to engage students in meaningful and effective mathematical experiences in the classroom in order to construct deep understanding of mathematics. Thus, understanding it is of significant importance to help prospective and practicing mathematics teachers to enhance the knowledge they hold and improve their practice. This article contributes to our understanding of this knowledge with particular focus on *reflective awareness* in relation to mathematics teachers' knowing. It first discusses teachers' knowledge for teaching mathematics based on ways in which this knowledge has been used in research of mathematics teachers. It then discusses the theoretical perspective of reflective awareness used to frame the study on mathematics teachers' learning being reported. Finally, it discusses this study which is part of a larger project that investigated elementary school teachers involved in a self-directed professional development initiative aimed at transforming their teaching of mathematics to an inquiry-oriented perspective. The focus here is on how reflective awareness was fostered in the teachers' learning and its relationship to their teaching. Specific attention is on two research questions: What aspects of the professional development supported reflective awareness in their learning? What was the impact of reflective awareness on their thinking and its relationship to their teaching and knowledge for teaching?

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MATHEMATICS TEACHERS' KNOWLEDGE

While there is general understanding that mathematics teachers' knowledge specific to teaching mathematics [MTK] consists of knowledge of mathematics for teaching and knowledge of

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State of the literature

- Mathematics teachers' knowledge specific to teaching mathematics is described in the literature in a variety of related ways involving different perspectives, different categories, depth of understanding, and beliefs.
- Category-based perspectives involving components of content knowledge and pedagogical content knowledge have received more attention in research of the mathematics teacher than perspectives involving their ways of knowing and acting mathematically.
- While reflection and noticing have received attention in research on mathematics teachers' learning, reflective awareness has not been considered as a way of knowing and an aspect of mathematics teachers' knowledge.

Contribution of this paper to the literature

- It provides a theoretical perspective for reflective awareness that can be used to broaden our understanding of mathematics teachers' knowledge, learning and teaching.
- Based on a case study of elementary school teachers, it identifies two factors (questioning and creating pedagogical models) that are important to support mathematics teachers' development and use of reflective awareness in their learning and teaching.
- It provides evidence of an important relationship between reflective awareness and mathematics teaching that suggests reflective awareness is central to mathematics teachers' knowledge as a way of knowing and acting mathematically.

mathematics pedagogy, there is less agreement on whether they are the most appropriate ways of specifying MTK or even whether MTK can be specified given its contextual nature. As such, MTK has been described in the literature in a variety of related ways. This section highlights some of these ways that deal with MKT based on perspectives, categories, depth of understanding, and beliefs.

MTK Based on Perspectives

In the edited book on "mathematical knowledge in teaching" (Rowland & Ruthven, 2011), different perspectives of MTK are explicitly or implicitly evident in the various authors' contributions. In an overview of some of these contributions, Ruthven (2011) identified four perspectives to "subject knowledge for mathematics teaching": (i) subject knowledge differentiated, which is concerned with identifying "types of subject-related knowledge that are distinctive to teaching so as to develop a taxonomy of such knowledge" (p.83); (ii) subject knowledge situated, which is concerned with the "use and development of subject-related knowledge in teaching [that] is strongly influenced by material and social context" (p. 87); (iii) subject knowledge interactivated, which is concerned with the "epistemic and interactional processes through which mathematical knowledge is (re)contextualised and (re)constructed in the classroom" (p. 89); and (iv) subject knowledge mathematised, which is concerned with characterizing "those mathematical modes of enquiry which underpin any authentic form of mathematical activity, and to show how teachers employ them to foster such activity in their classrooms" (p. 91). Other researchers have suggested related perspectives of MTK that include: "a way of being and acting" (Watson, 2008, p. 1) that develops and grows through "doing mathematics and being mathematical" (p. 1); how teachers hold their knowledge, in particular, their orientation towards mathematics, e.g., embodying modes of mathematical enquiry (Barton, 2009); "a participatory attitude toward mathematics than as an explicit body of knowledge" (Davis & Renert, 2009, p. 37) and pedagogical content knowing to stress pedagogical content knowledge as a dynamic concept meaning knowing-to-act that is inherently linked to and situated in the act of teaching within a particular context (Cochran, DeRuiter, & King, 1993).

These perspectives provide a landscape of MTK that includes categories of knowledge, discussed in the next section, and ways of knowing, discussed in a later section of this article with a focus on reflective awareness. They suggest the importance of other than a category-based perspective in understanding MTK, in particular, the perspective that "teachers must act mathematically in order to enact mathematics with their students" Ruthven (2011, p. 91).

MTK Based on Categories

Some mathematics education researchers have conceptualized MTK as specific categories of knowledge, influenced by Shulman's (1986) classification of subject matter knowledge, pedagogical content knowledge, and curriculum knowledge. They have developed models or ways of interpreting these classifications to better describe them in relation to teaching mathematics. Table 1 summarizes examples of these categories based on the works of Ball, Thames and Phelps (2008), the Teacher Education Development Study in Mathematics (TEDS-M) (Tatto et al., 2012), Krauss, Baumert, and Blum (2008), and Rowland, Turner, Thwaites, and Huckstep (2009). The focus here is to highlight and not describe or critique these categories. However, there are many similarities among them in terms of their content and the way they are represented as independent components of MTK.

MTK Based on Depth of Understanding

Some researchers highlighted the depth of the mathematical knowledge teachers should hold. For example, Ma (1999), in particular, is known for making the case for teachers' mathematics content knowledge to have breadth, depth and thoroughness, that is, teachers should have profound understanding of fundamental mathematics which is attuned to and usable in teaching. This understanding includes connectedness, multiple perspectives, fundamental ideas, and longitudinal coherence. Similarly, Kilpatrick, Swafford and Findell (2001) suggested that mathematics teachers need specialized knowledge that "includes an integrated knowledge of mathematics" (p. 428). In addition, their concept of mathematics proficiency suggests that MTK should include conceptual understanding, procedural fluency, and strategic competence. Eisenhart, Borko, Underhill, Brown, Jones, and Agard (1993) also suggested considering MTK as procedural and conceptual knowledge to avoid the distinction between subject knowledge and pedagogical content knowledge which they consider to not be clear-cut. However, these features of MTK, while not always explicitly stated, are important to the mathematics content components of the categories of knowledge noted in table 1.

MTK as Beliefs

Beliefs have played a prominent role in studies of mathematics teachers and their teaching, influenced by works such as Thompson (1992) and Ernest (1991). These works have established important relationships between beliefs and practice, which suggest that teachers' mathematical beliefs are important aspects of MTK. These beliefs include: beliefs about what mathematics is, how mathematics teaching and learning actually occurs, and how mathematics teaching and learning should occur (Handal, 2003), beliefs associated with problem solving (Mayer & Wittrock, 2006; Schoenfeld, 1992) and beliefs associated with mathematical proficiency as part of productive disposition (Kilpatrick et al., 2001). However, as can be seen in table 1, beliefs do not form a category in these conceptions of MTK. Beliefs are also not specified as features of these categories except for Rowland et al.'s (2009) knowledge quartet framework that explicitly includes beliefs about the nature of mathematics. Fennema and Franke (1992) also included beliefs as a key component of their category-based model of MTK.

The preceding sub-sections highlighted the nature of MKT in terms of various perspectives, categories, depth of understanding, and beliefs. These ways of conceptualizing MTK are collectively important to our growing understanding of MTK. However, they do not promote some aspects of MTK that may not be uniquely mathematical but are integral to learning and teaching mathematics. In particular, most do not explicitly address the underlying layers of teachers' knowledge involving their knowing or ways of knowing that are central to learning and doing mathematics. This includes "mathematical modes of enquiry which underpin any authentic form of mathematical activity" (Ruthven, 2011, p. 91).

Teachers need to hold ways of thinking that underlie knowledge of mathematics and mathematics pedagogy. For example, they need to understand problem solving and mathematical thinking (Mason, Burton, & Stacey, 2010; Schoenfeld, 1992) and inquiry thinking (e.g., Dewey, 1933; Wells, 1999). A teacher who lacks such ways of knowing is unlikely to be equipped with appropriate knowledge to teach mathematics with deep understanding or to help students to think mathematically. As Ponte and Chapman (2006)

Table 1. Category-based **P**erspectives of MTK

Ball, Thames & Phelps, 2008	Rowland, Turner, Thwaites, & Huckstep, 2009	Tatto et al., 2012	Krauss, Baumert, & Blum, 2008
Common content knowledge	Foundation knowledge	Mathematics content knowledge	Knowledge of mathematical tasks as instructional tools
Specialized content knowledge	Transformation	Mathematics curricular knowledge	Knowledge and interpretation of students' thinking
Horizon content knowledge	Connection	Knowledge of planning	Knowledge of multiple representations and explanations of mathematical problems
Knowledge of content and students	Contingency	Knowledge for enacting mathematics	
Knowledge of content and teaching			
Knowledge of con-tent and curriculum			

suggested: "the teachers' knowledge is not only "knowing things" (facts, properties, if-then relationships…), but also knowing how to identify and solve professional problems, and, in more general terms, knowing how to construct knowledge" (p. 1). Thus, more attention is needed on ways of knowing in relation to MTK in research to provide a broader picture of what is necessary to teach mathematics with depth and understanding. The study being reported in this article offers an example of such research with a focus on reflective awareness as a way of thinking that underlies both mathematical knowledge for teaching and mathematics pedagogical knowledge.

PERSPECTIVES OF REFLECTIVE AWARENESS

The notion of reflection is common in mathematics teacher education, influenced by Schon's work (1983, 1987), in particular, as a process of self-understanding and growth. While reflective awareness [RA] includes such reflection, it is considered here from a perspective with broader pedagogical implications regarding what teachers know or should know.

RA has been used with some variations in the literature. For example, Shulman (1986) associated it with enabling professionals to perform tasks in their particular disciplines and to communicate their thinking, rationales, and judgments as they do so. He explained that "reflective awareness of how and why one performs complicates rather than simplifies action and renders it less predictable and regular" (p. 13). Senge, Scharmer, Jaworski and Flowers (2005) developed a framework for deep change that includes "sensing", that is, a deeper kind of observation that consists of suspending, redirecting, and letting go. They explained that the capacity to do this depends on the ability for RA.

Recent studies on mathematics teachers have also considered "awareness" in the context of noticing (i.e., what teachers are aware of or attend to) and support its importance in teaching (e.g., Sherin, Jacobs, & Philipp, 2010; Star & Strickland, 2008; Scherrer & Stein, 2012; Sherin & van Es, 2005). Many of these studies deal with helping teachers to notice; some focusing on students' mathematical thinking, while others are more open. For example, van Es and Sherin's (2008) learning-to-notice framework included helping teachers to identify noteworthy aspects of a classroom situation, use knowledge about the context to reason about classroom interactions, and make connections between the specific classroom events and broader principles of teaching and learning. They noted that how teachers analyze what they notice is as important as what they notice. Star and Strickland (2008) focused more generally on identifying what teachers do and do not attend to in classroom lessons. However, these studies do not explicitly address RA.

Mason (1998; 2008) provides a perspective of awareness and its importance to teaching and teacher learning that relates to RA. In Mason (2008), he emphasized the importance of not only the attention that teachers give to significant classroom actions and interactions, but also their awareness involving their reflections, reasoning, and decisions based on it. In Mason (1998), he explained that "the key notions underlying real teaching are the structure of attention and the nature of awareness" (p. 244). "Being aware is a state in which attention is directed to whatever it is that one is aware of" (p. 254). He suggested that "to become an expert it is necessary to develop and articulate awareness of your awarenesses-in-action; to become a teacher in the full and most appropriate sense of that word, it is necessary to become aware of your awareness of those awarenesses-in-action" (p. 255). He identified three forms of awareness:

> *awareness-in-action* (the powers of construal and of acting in the material world); *awareness of awareness-in-action, or awareness-in-discipline*, which enables articulation and formalisation of awarenesses-in-action, and is closely linked to one form of shift of attention; awareness of awareness-in-discipline or *awareness-in-counsel*, which is the self-awareness required in order to be sensitive to what others require in order to build their own awarenesses-in-action and -in-discipline. (p. 256)

Such awareness of awareness of awareness involves a reflective process. In relation to RA, this process is being associated with reflective thinking. From Dewey's (1933) perspective, reflective thinking is "central to all learning experiences enabling us to act in a deliberate and intentional fashion … [to] convert action that is merely … blind and impulsive into intelligent action" (p. 212). It involves an "active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it and the further conclusions to which it tends" (p. 9). It begins when one encounters "a state of doubt, hesitation, perplexity, mental difficulty" (p. 12); "an entanglement to be straightened out, something obscure to be cleared up" by thinking (p. 6). It consists of a particular type of questioning. While Dewey's notions are specific to a process of systematic inquiry to resolve problems, awareness and awareness of awareness are important to the process since one must become aware of a situation one considers to be "a problem" and then thinks of and tests ways of resolving it.

The preceding notions of awareness and reflective thinking provided the basis for the theoretical perspective of RA adopted in this study of mathematics teachers' learning and teaching. In this perspective, RA involves a state of curiosity, wonderment, or puzzlement that results in action through questioning and inquiry in order to know, to enhance understanding, and to act or change behavior appropriately. Thus, it is not simply about one seeing something (i.e., instrumental awareness) but, more importantly, about one being able to see a puzzling situation or becoming curious about a situation in that something and acting to understand it. So while instrumental awareness involves one seeing only what one knows and accepting what one sees based on what one knows, RA involves one seeing something in one's experience that is or could be different from what one already knows and results in questioning/inquiry to understand it.

Teachers with knowledge of RA and a RA disposition think about what is happening in their classrooms rather than merely reacting by jumping to conclusions or blindly accepting the situation. They ask questions to understand, to check their thinking and students' thinking, and to consider alternative interpretations of an event or behavior. For example, consider a mathematics teacher who becomes aware of her students thinking that 2a + 2b = 4ab. Without RA, her reaction could be that it is wrong, or the students do not know how to add variables, or she has to re-teach this concept. With RA, she would "wonder" about what they are thinking, or why they are thinking that, or what does "a" and "b" mean to them, or is it because of something she did or said and then question the students not to judge but to understand.

STUDY OF TEACHERS' RA

As part of a larger project involving a group of elementary mathematics teachers, an investigation was conducted to understand the role of RA in their learning and teaching. The teachers participated in a self-directed professional development approach [PD] aimed at transforming their teaching to an inquiry-oriented perspective. The focus of this study was to identify aspects of the PD that supported RA in their learning and the impact on their thinking and teaching.

The participants were 14 grades 1 to 6 teachers from the same elementary school. To fulfill their school's requirement of a professional growth plan, they formed a mathematics study group to work on bringing their teaching more in line with curriculum expectations involving a constructivist/inquiry perspective. The researcher was invited to join the group to provide support, by responding to their needs rather than imposing direction, and not deliberately influencing events by dictating what they should do or how they should do it. Thus, the PD was self-directed in that the teachers made the decisions for every aspect of it. In the first year of the PD, considered here as part of the larger project, the group met in their school once every three weeks for about one and a half to two hours. They also met for one half-day and one full-day during their school's PD days in each school term, had time to observe their research lessons, and sometimes met during lunch breaks to plan and reflect on the lessons.

Although the PD was not based on a predetermined process, it was consistent with current perspectives of teacher learning. These perspectives follow a socially and culturally situated process of knowledge construction, which involves attention to collaboration, discourse, reflection, inquiry and application. Research has also indicated that effective PD requires continuous interactive support over a substantial period of time, should focus on specific educational content under guidance of an expert adopting a hands-off role, and revolve around artifacts that help to foster a sense of ownership with teachers (e.g., Borko, 2004). Communities of inquiry (Wells, 1999) are particularly suited for such activities. Such communities involve dialogical inquiry, that is, "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (Wells, 1999, p. 122). Chapman (2013) provides an analysis of the inquiry orientation of the teachers' PD. The focus here is on the features of the PD that were significant to their development and use of RA.

Data collection for the larger project focused on two aspects of PD: the way it evolved for the teachers and the way it impacted their learning and practice. This included: (1) field notes and audio recording of PD sessions involving their discussions of, for example, what to do and how, when and why to do it; their plans, observations and evaluation of their research lessons; and their students' work. (2) Samples of teaching artefacts (e.g., research lesson plans) and participants' notes (e.g., observations of video lessons) during the sessions. (3) Several classroom observations of each teacher. (4) Three open-ended group interviews and one with each teacher to probe their thinking about the PD, their learning about discourse and inquiry, and the impact on their teaching.

Data analysis for this study was guided by the research questions: What aspects of the PD supported RA in the teachers' learning? How did RA shape their thinking and teaching? Codes were developed based on the theoretical perspective of RA and used to identify the features of the PD that supported the teachers' RA and aspects of their thinking/actions during the PD and their teaching that were characteristic of RA. The coded information was categorized in different ways that included: (1) their questions/prompts that were RA-oriented; (2) what they attended to in students' responses during discourse; (3) their intentions related to RA; and (4) their knowledge of RA. Themes emerging from these categories were used to draw conclusions regarding their learning of RA and use of RA in their learning and teaching.

FOSTERING TEACHERS' RA

Two features of the PD emerged from the analysis as significant to the teachers' use and learning of RA: *questioning* and *creating pedagogical models*.

Questioning

Two categories of questioning were central to the teachers' development and use of RA: self-based questioning and meaning-based questioning.

Self-based Questioning

Self-based questioning involves posing questions that enable one to think about and talk about oneself. In the context of RA, it is triggered by curiosity, puzzlement, or wonderment about one's own or others' thinking or actions. The teachers did not initially demonstrate self-based questioning and had to be prompted to do so. This prompting occurred early the PD during their investigation of video lessons to gain understanding of inquiry-based communication in teaching mathematics. The set of video lessons consisted of inquiry-oriented teaching in elementary mathematics classrooms. The teachers' intent was to use the video study to produce a list of questions and other ideas they could adopt. So they started their video study by focusing on the questions posed to students, recording what they considered to be meaningful and sharing what they recorded. However, there was passive acceptance of what they shared, for example, accepting commonalities as validation of what they noticed and differences as new information of what they all did not notice. So they were prompted to be curious about their own and each other's thinking. For example, it was suggested that they pose questions (e.g., why-type questions) to each other regarding what they observed and recorded to help them to understand their thinking and teaching. While this was initially challenging because their self-directed approach required that they figure out for themselves what this involved, their why questions (e.g., Why did you choose [not choose] ____? Why is ____ important [not important] to you? Why do you like [not like] ____?) enabled them to understand the nature and intent of process. So their questions started to take on new forms as they became more interested in exploring their thinking (e.g., How will ____ help you [or the kids])? What other ways will ____ work for you? How will you use ____? When will you use ____? How is ____ different from what you do?).

This self-questioning enabled the teachers to make sense of their observations in relation to their thinking and teaching, to understand issues with communication in their teaching, and to hypothesize what could be more effective to engage students in inquiry. It enabled them to compare their thinking and teaching with each other and with the lessons they observed and to pose questions for further observation or inquiry. For example, they started to wonder about other aspects of the video lessons. This broadened their focus beyond observing the questions posed in the lessons in isolation of how they were situated in the lesson. So as they observed other video lessons they expanded their focus to include the tasks, students' thinking and actions, teacher's thinking and actions, and inquiry structure of the lessons.

The self-based questioning enabled the teachers to engage in RA as they became curious about their own and each other's thinking and engaged in dialogical inquiry (Wells, 1999) to understand themselves and each other regarding how they made sense of aspects of their teaching and the changes they should pursue based on the video lessons. Their continual use of these questions throughout the PD suggested that RA had become a way of knowing for them. By the end of the video-lessons study they had developed an understanding of RA that enabled them to continue to pose not only self-based questions but also meaning-based questions.

Meaning-based Questioning

Meaning-based questioning involves asking questions that require a search for meaning, that is, to understand or make sense of something through inquiry. In the context of RA, it is triggered by curiosity or puzzlement about something related to one's experience that one wants to understand in order to make changes to it. The teachers were able to engage in meaning-based questioning which was central to defining how the self-directed PD evolved. Their self-directed process was driven by their curiosity or puzzlement about the nature of inquiry teaching in mathematics and what they needed to know and do to adopt it in their teaching. This allowed them to engage in RA as they posed questions such as: What does communication look like in an inquiry mathematics lesson? What attributes make up an inquiry mathematics lesson? What is a meaningful model of inquiry teaching for us? The resulting inquiry involved creating and testing hypotheses emerging from their learning through their discussions, self-based questioning and video-lessons study. For example, from the video-lessons study, they hypothesized a set of attributes of inquiry-based teaching of mathematics and created experimental lessons to test it.

The direct relationship between the self-based and meaning-based questioning was important to support the teachers' engagement in RA as curiosity or

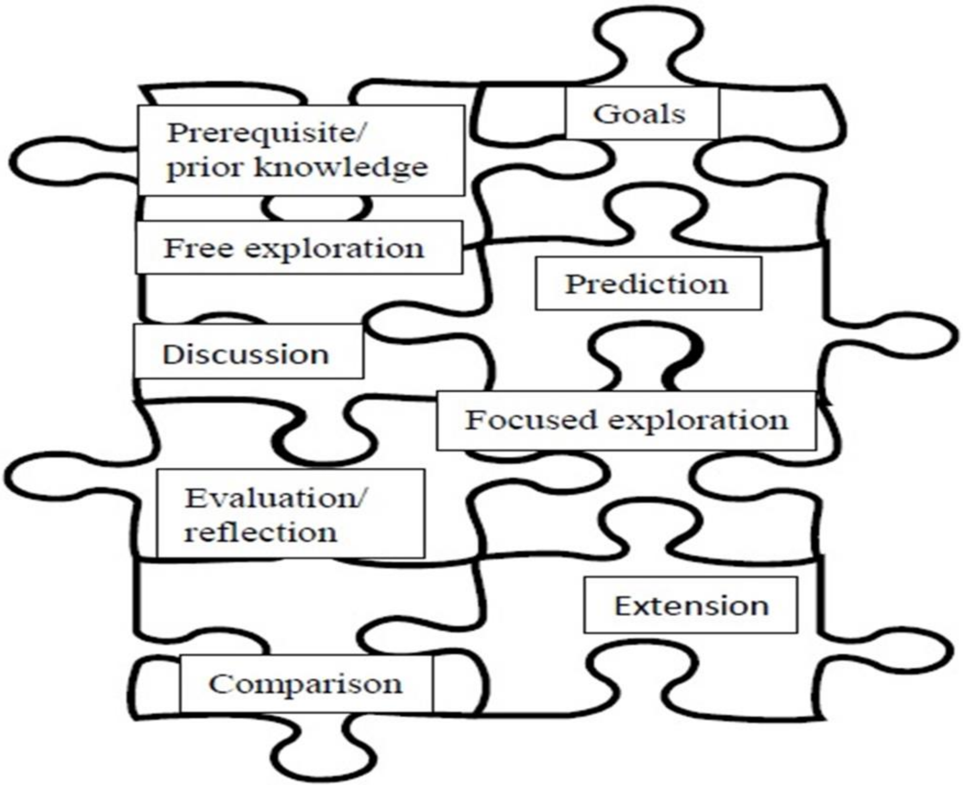

Figure 1. Inquiry-based Teaching Model

puzzlement resulting from one impacted the other and vice versa. Thus, they separately and together contributed to the teachers' development of an understanding of RA and how to use it in their learning.

Creating Mathematical Pedagogical Models

The second aspect of the PD that emerged as central to the teachers' development and use of RA was their engagement in creating pedagogical models, that is, general approaches to guide their learning and teaching. The decision to create these models was an outcome of their meaning -based questioning and provided opportunities for them to engage in RA. For example, RA played a role in creating the models as they puzzled with, posed questions about, hypothesized, and tested what features the models should have to satisfy their goals. In return, creating the models played a role in developing their RA as they wondered about the structure of the models, posed questions about it and engaged in inquiry to make sense of it. RA was also important to understand how to adopt the models in their teaching. Three examples of their models follow.

Model of Inquiry-based Teaching

Figure 1 is the teachers' representation of the inquiry-based teaching model they developed. Based on evidence from their RA, they used the jigsaw background to indicate that the model is not linear and the components could be arranged in different ways depending on the mathematics topic and teacher's goal for the lesson. A major shift in their thinking reflected in the model is the focus on the students (i.e., learner/learning) and not on themselves (i.e., teacher/teaching). For example, the model is about attending to the students' prior knowledge and engaging the students in free exploration, focused exploration, discussions, predictions, comparison, applications, evaluation, reflection, and extension of the concept being taught.

The teachers engaged in RA to create the model and to understand how it worked, what it meant, why it worked in a particular way, and how they could adopt it in their teaching. Table 2 contains one of their abbreviated lesson outlines based on this model.

Table 2. Inquiry-based Teaching Model

Components of teaching model	Grade 3: Place value – representing multi-digit numbers
Goals	To understand that the value of a digit depends on its position in a number. To understand the meaning of regrouping among hundreds, tens and ones.
Prior knowledge	What do students know about regrouping and base-10 materials?
Predictions	What happens if position of a digit in a number is changed? Discuss in groups. Record predictions.
Free exploration	Use base-10 materials to explore what they notice when representing 3-digit numbers.
Focused exploration	Build a 3-digit number with the base-10 materials, represent with picture and symbols. Change position of digits in number and represent resulting number with picture and symbols. Repeat for a number with zero. Whole-class discussion on what did; what noticed; comparison of results to predictions; conclusions about concept.
Application	In groups, create a game to build 3-digit numbers using manipulatives so player with smallest or largest number wins.
Evaluation/ reflection	Reflection and discussion of what they think they now know about representing multi-digit numbers.
Extension	What if you had a 4-digit number?

Table 3. Problem-solving Inquiry Model

Phase	The Key Element
Before	How do you present the problem?
During	Teacher active intervention (When and how do teachers intervene?)
During	Teacher passive intervention (What do teachers hear and notice?)
During	Student action (How do students handle group situations (distractions)?)
During	Student thinking (What questions do students ask?)
After	Mechanics of sharing (When and how does sharing occur?)
After	Purposes of sharing (What do teachers use sharing for?)

Lesson Observation Model

In order to observe their experimental lessons, the teachers developed an observation model to prompt their observations. The focus of the prompts was on the students, for example, what they were able to notice, make sense of, and predict on their own regarding the mathematics concepts through the inquiry-based activities. In creating this model, the teachers used RA to decide on and interpret each prompt. They represented each prompt as a question that indicated what they were curious about regarding the students' sense making of the mathematics concepts and participation in the lesson. Thus the model and its implementation enabled them to further engage in and develop their RA in being curious about students' thinking and behavior when engaging in mathematical activities and following up to learn from it to enhance their teaching and students' learning.

Problem-solving Inquiry Model

The teachers developed a problem-solving inquiry model to make sense of inquiry-based teaching of problem solving. They engaged in RA to understand what they should do before, during, and after a student is engaged in solving a mathematics problem for which the solution method is not known in advance. Table 3 summarizes the key elements of this model. These elements include self-based and meaning-based questions that supported RA in their learning.

IMPACT ON TEACHING

All of the teachers made significant changes to their teaching that included the use of RA. The focus here is on the relationship of RA to their teaching with emphasis on their use of questioning. This is illustrated by the case of one of the teachers, Lena, who taught grade 3. Her teaching reflected RA consistently as a result of the PD.

Lena's teaching shifted from a focus on telling to engaging students in inquiry-oriented, learner-focused discourse on an ongoing basis. Her questioning shifted to mirror what she learned from the PD. She explained:

> *I have changed my question techniques after our work in the study group and the questions that we've come up with based on what we know that will promote good conversation. So I do use the techniques like "what have you noticed" … I use that term "make sense" a lot. So my kids now know you have to explain the why … how you make sense of this.*

One of Lena's goals for questioning was her own learning. She was now curious about students' thinking and wanted to learn from it. As she explained:

> *It's a little bit selfish, but I want to learn something. So I want to be aha'd! and surprised. … I almost get a rush … a high when they teach me something. I'm not afraid to take risk, so I put myself out there to see what I can learn too.*

She was often "amazed" by the students' thinking and what she learned from them that improved her mathematical thinking and teaching as in the case of this task she gave to her students.

> *At [our] school this many books are read each month: September 200, October 279, November 193, December 151, January 307, February 233, and March 302. How many more books were read in March than in October? Explain whether a correct answer is 23, 123, 31, or 177.*

She explained that in the past she approached a problem like this by getting students to set up

$$\begin{array}{r} 302 \\ \underline{-\ 279} \end{array}$$

and expected them to use the standard algorithm. In this example, she provided "answers" to focus them on the process. She was curious to see how they would approach it. The students had no problem seeing that the solution was the difference between the two months. However, they had alternative ways of doing the computation because some of them found the standard algorithm to be challenging or did not think of it because they had their own way that worked and made sense. Their approaches consisted of the following, summarized based on how they described them during whole-class sharing.

> *(a) 279 plus 20 is 299 then add 3 [counts on with finger] and the answer is 302, so 23.*
> *(b) 279 plus 1 is 280 and 302 take away 2 is 300, so take away 280 from 300 is 20 and add back 1 and 2 to get 23.*
> *(c) 302 take away 279 is too difficult, so we tried all the numbers to see which add up to 302. We started with 123. 123 plus 279 is 402 so it is not this, so it is also not 177. Then try 279 plus 31 which is 310, so not that too. Try 279 plus 23 so that is 279 plus 10 and 10 is 299 plus 3 is 302. So it is 23.*
> *(d) 9 from 12 is 3 because you take one from three and change zero to 10 then, one from ten to change 2 to 12, then take away seven from nine and you get 23.*

Instead of directing students to use her method as being most efficient, as she did in the past, Lena now learned from them and engaged them in discourse about what they noticed about the approaches, when and why to use them, and the importance to use what made sense to them.

Another shift for Lena was in her planning. She started to think deeply about questioning and tried to imagine possible scenarios.

> *So if I put this [question] out there, what direction could it go? And if it went that direction, what would I do, and if it went that direction, how would I help them?*

Lena also engaged in self-reflection during whole-class discourse. For example;

> *So I'm always attending to: have I met their needs and where do I need to take this now? What do I need to do next with it? Does that make sense? … I'm thinking: so what they said; can I come up with a question based on that to promote more thinking and oral discussion?*

While her learning was important, the key goal of Lena's questioning was her students' learning. She encouraged them to be curious and to ask questions, which were central to their whole-class discourse. She explained:

> *What sets the direction for it [discourse] now is the math questions that the kids are asking, because they were given freedom to say, tell me what you want to learn. … So what is important for it [discourse] is the interest of the kids and questions that they have.*

When students wanted to know what was a good question to ask, she told them:

> *It should be something you want to learn. Something that you might have seen or heard and you wonder about. ... So that's how we left it: wonder, curiosity, what if?*

Lena also challenged students' thinking with meaning-based and self-based questions, such as: What do you mean by that? How do you know it is a rectangle? What could you do to prove that? How do you know a number is even? Why did you give each student one dollar first? Is that math? What do you know about this? Have you ever wonder ...? She often posed self-reflecting questions that allowed students to think about what they knew based on prior knowledge or their experiences and what they wanted to know. She started lessons on new concepts by "always trying to find out what they are bringing to the lesson, before just bringing what I think in to know." For example, "if we're going to do patterns, then I would start with … what do you want to know about patterns? Or, what have you noticed about patterns? Or can you tell me about patterns in your world?" She often encouraged students to think about "Did that make sense?" for themselves. She would tell them "Ask yourself in your head, did that make sense?" She encouraged them to "see the math." She posed questions for them to think about and identify examples of mathematics in their out-of-school, real-world experiences, for example, "Where is math in your world?" "Did anything happen in your life that involves math?" "What math situation did you experience since we met in class yesterday?" "Where would you find the number one million used in your world?" "What is something in your life that has a growing pattern?" She also required students to think about their problem-solving processes or strategies and to reflect on their mathematical learning experience such as affective aspects of their problem-solving experience or their choices of tools to aid learning. For example, "Who had a little bit of difficulty trying to solve it [the problem]? … What was challenging for you?" "Who used the place-value mat? … Can you tell us why you chose to use that?"

The preceding discussion of Lena's questioning approach embodies her knowledge and use of RA. Questions asked by her and her students had a personal component of acting on a curiosity or perplexity that was resolved through inquiry of real-world experiences or mathematical tasks. Lena's inquiry orientation in her teaching was supported by her RA, that is, her ongoing curiosity and desire to learn from students, to grow in her understand and teaching and to help her students to grow in their understanding and learning.

CONCLUSIONS

The case of this group of teachers suggests that there is an important relationship between RA and mathematics teacher learning and between RA and teaching mathematics. RA was central to the teachers' development of an inquiry stance (i.e., a disposition to question in order to understand and take "intelligent action" (Dewey, 1933)), knowledge of mathematics for teaching (e.g., through questioning students' thinking) and knowledge of mathematics pedagogy. They developed understanding and knowledge of RA through the PD that impacted their teaching in meaningful ways. Self-based questioning, meaning-based questioning and creating pedagogical models were central features of their PD that enabled them to develop this understanding. They demonstrated inquiry stance for learning and in teaching through their ongoing questioning and investigating to gain knowledge to enhance their teaching. With RA and this inquiry stance they were able to sustain and continue their ongoing leaning.

Lena held knowledge of RA and used it to help her students to enhance their learning of mathematics. The shift in her thinking and teaching resulting from the PD showed depth in her RA. This shift was directed to her own learning and her students' learning. Her teaching approach included self-based and meaning-based questioning for herself and her students. Lena's case shows that RA is central to a teacher's ability to shape events in the classroom by being aware of and questioning the phenomena around which the discourse of the classroom is organized (e.g., students' thinking). Her RA was important to promote curiosity and questioning in students and to help them to also develop their RA in learning mathematics.

The study suggests that for teachers to engage in RA or engage students in RA in the mathematics classroom, the intent behind their questioning of students' mathematical thinking and actions should embody curiosity or perplexity, a desire to know or learn something new, and a desire to understand "why." Without such intent, both teacher's and students' learning could be limited to what they already know, what they expect, or what they are willing to accept. RA enables them to transcend such boundaries into a world of possibilities to enhance learning and teaching.

RA, then, is an important component of teachers' knowledge and knowing and should explicitly be treated as such in teacher education. Teachers should learn of the importance of RA in their own and students' learning and how to engage in RA and engage their students in it. They should learn to treat students' thinking not just as a source of information, but also as a means to engage in RA and engage them in RA. Future research should explore ways of facilitating prospective teachers' RA development and the impact on their teaching and students' learning. It should also study practicing teachers to understand RA from a practice perspective for different contexts and levels of school (i.e., primary, middle, and high school) regarding their engagement in RA during mathematical activities in the classroom.

To conclude, MTK is more complex than discrete categories of content knowledge and pedagogical content knowledge when considered from a broader perspective of what teachers should know to teach mathematics. Teachers need to learn to think in different ways that support mathematical thinking and meaningful mathematics pedagogy. Focusing on RA could significantly enhance their knowledge and teaching and their students' learning.

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