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Semiosis of conceptual learning of mathematical inequalities through semiotic meaning triads

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Abstract

The process of semiosis for the conceptual learning of inequalities allows the student to revitalize the interpretation, understanding, and solution of problems both in mathematical contexts and in everyday contexts. This research designed and applied a didactic strategy based on the semiotic theory of semiotic treatments and conversions to develop the semiosis processes of conceptual learning of inequalities through the methodology of the semiotic meaning triad and its three phases applied in class sessions focused on the solution of inequalities, defining the domain and range of functions and interpreting the lipid profile of a person.

Keywords: semiosis, semiotic meaning triads, mathematical inequalities

INTRODUCTION

The concept of mathematical inequality is frequently used in the everyday life of students and its learning process in schools takes place at a very early age. Every day people make numerical estimates to define measurement intervals, conceiving connotative meanings for decision making; for example, the expressions "she cannot work because she is between 14 and 16 years old", "the economic growth of the country by 2024 will be around 1.3% and 1.8%", "her triglyceride level is higher than 150 mg/dl", and "the speed of the car before the crash was between 180 and 220 km/h" are semantic representations that allow the interpretant to create an idea or knowledge that can be transformed into symbolic and graphic representations. Now, the formal process of teaching and learning the concept of mathematical inequality begins curricular at a very early age, with the numerical relations "greater than" and "less than" and continues throughout schooling, expanding its conceptual field level by level until reaching more abstract and complex applications, such as the analysis of functions and calculus; for example, the expressions "the domain of the function $f(x) = \sin(x)$ is (-∞, ∞)" and "If $f(x)$ is continuous on $a \leq x \leq b$, then $\int_a^b f(x)dx = F(a) - F(b)$, where $F(x)$ is any antiderivative of $f(x)$, that is, a function such that $F'(x) = f(x)$" are symbolic representations denoting meanings proper to mathematical functions, which can be transformed into graphical and semantic representations. According to the above, the teaching of inequalities is a great challenge for mathematics education, since students, when they come into contact with procedures and problem-solving with inequalities, experience both ontogenetic and epistemological obstacles (Brousseau, 2007) due to the lack of meaning and understanding in the conceptual learning process of inequalities (Iori, 2017). Therefore, this research proposes a didactic strategy to improve the conceptual learning of inequalities with meaning by means of the triads of semiotic meaning; for this, the study of inequalities followed the following order: first, characterizing the difficulties of learning inequalities studied by Almog and Ilany (2012) and Blanco and Garrote (2007); second, analyzing epistemologically the conceptual learning of inequalities taking as a reference the studies of semiotic representations, treatments and conversions by D'Amore (2003) and Duval (2006); and third, proposing and applying a didactic strategy for the conceptual learning of inequalities with meaning by means of Durán Salas' (2022) triads of semiotic significance.

THEORETICAL FRAMEWORK

Mathematics is a formal body of ideal, abstract, linguistic, and conceptual objects that are externalized by means of semiotic representations such as definitions,

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Contribution to the literature

- This research characterizes the different types of semiotic representations of inequalities for the teaching and learning of the concept of inequality that can be used in the learning of linear inequalities.
- This research presents the concepts of treatment and semiotic conversion to develop the processes of semiosis in the learning of mathematical concepts.
- This research proposes the triads of semiotic signification as a methodology to design a didactic strategy for the conceptual learning of meaningful inequalities.

axioms, collunariums, theorems, graphs, notations, and symbols. This is why the conceptualization processes of mathematical activity must necessarily pass through semiotic representation registers in order to be communicated and learned (D'Amore, 2003). Indeed, Weiskopf (2008) states that concept learning is understood as the ability to represent different properties of concepts; therefore, the activity of semiotic representations of mathematical concepts recognizes two functions:

(a) semiotic representations, which allow the learner to come into contact with the mathematical concept (Iori, 2017) and

(b) semiotic representations, which put the mathematical concept on a learning level, so that the processing and conversion of semiotic representations provide meaning to the concepts (Fandiño, 2010).

This is why semiotic representations play an important role in the development and transformation of mathematical concepts applied to both every day and mathematical contexts.

PROBLEM STATEMENT

The concept of inequality, like any mathematical concept, has ideal, abstract, and linguistic characteristics, and therefore requires specific semiotic representations in order to be defined, denoted, and communicated (D'Amore et al., 2010), which is why the cognitive construction of learning inequalities is closely related to the ability to use various registers of semiotic representation for their understanding, comprehension, and meaning (Balomenou et al., 2017). Indeed, the multiplicity of semiotic representations that characterize the concept of inequality mathematically makes it difficult to learn and its teaching becomes more complex with the wide field of its application with other disciplines of knowledge. This is evident in classroom practices, since "our teaching experience has allowed us to observe the difficulties that high school students have and the errors they make when they are studying inequalities. Many of these problems re-occur year after year " (Blanco & Garrote, 2007, p. 221). Thus, Blanco and Garrote (2007) identified two causes that affect students' learning of inequalities:

(a) they point out that the teaching of inequalities is reduced to mechanical tasks and not to the appropriation of the semantic content of each property of inequalities and

(b) likewise, students establish limited relationships between the different systems of semiotic representation of inequalities.

Similarly, Almog and Ilany (2012) identified three other causes that hinder the operative learning of inequalities, as students generally make the same mistakes when developing inequalities arithmetically; for example:

(a) they do not change the sign when multiplying or dividing by a negative number;

(b) they do not relate the results to their graphical representations when solving inequalities, and

(c) the teaching and learning of inequalities is focused on the logical formalism of their properties.

These five factors that hinder the conceptual and operational learning of inequalities are the focus of attention in this research. Therefore, it is necessary to design a didactic strategy that allows students to learn multiple semiotic representations and their conceptual interrelationships to solve problems of their environment, otherwise, any rote or mechanical learning can lead students to make mistakes frequently (Blanco & Garrote, 2007).

SEMIOSIS OF CONCEPTUAL LEARNING OF INEQUALITIES WITH MEANING

Identifying the errors that students frequently make when trying to solve problems with inequalities allows us to propose a didactic alternative to improve the learning of inequalities with meaning; for this reason, it is pertinent to distinguish the concepts of semiosis and noesis. To begin with, Duval (2017a) defines semiosis as the production of semiotic representations and noesis as the conceptual apprehension of an object. That said, semiosis is understood as the activity between signs, symbols, graphics, and semantic expressions that evoke, represent and refer to an abstract or real object; and noesis as "the conceptual acquisition of any object has to go through the acquisition of one or more semiotic representations" (D'Amore, 2003, p. 77). Hence, there is

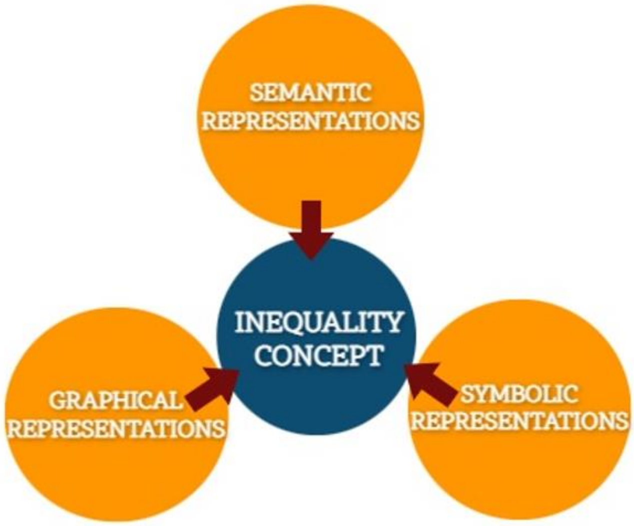


Figure 1. Graphical, semantic, & symbolic representations reference concept of inequality (Source: Author's own elaboration)

a close relationship between semiosis and noesis for the learning of concepts, as Duval (2017b) states that "there is no noesis without semiosis" (p. 47). This means that the learning of mathematical concepts necessarily occurs through the activity between the semiotic representations that characterize the concept. However, the learning of mathematical concepts with meaning refers to the capacity of an individual or interpretant to internalize and externalize the semiotic representations that characterize the concept (Godino & Batanero, 1994). Thus, the interpretant internalizes the meaning of the mathematical concept every time they make transformations to the semiotic representations that characterize the concept; that is, by the action of semiosis; in the same way, the interpretant externalizes the meaning of the mathematical concept whenever they solve problems correctly by making transformations to the semiotic representations of the concept. Therefore, by integrating the processes of acquisition and development of mathematical concepts through the internalization and externalization of semiotic representations, the following can be affirmed: the learner has achieved conceptual learning with meaning whenever they

(a) characterize the mathematical concept by evoking its semiotic representations,

(b) perform semiotic treatments within the same register of semiotic representation, and

(c) perform semiotic conversions between different semiotic representations (D'Amore, 2006; Duval, 2017b; Fandiño, 2010; Vergnaud, 1998)

When it comes to the conceptual learning of inequalities with meaning, we have to

(a) characterize concept of inequality in its semantic, symbolic, and graphical representations,

(b) perform semiotic treatments between symbolic properties, and

(c) perform semiotic conversions between semantic, symbolic and graphical representations to solve everyday problems.

In the following, the three aspects that account for the conceptual learning of meaningful inequalities are explained in detail.

Characterization of Semiotic Representations of the Concept of Inequality

Duval (2017b) states that mathematical concepts are invariant once they are defined, however, in order to access mathematical concepts, multiple semiotic representations are required. For the specific case of inequalities, it is possible to characterize them in three groups of semiotic representations, as shown in **Figure 1**. **Figure 1** illustrates that each group of semiotic representations refers to a part of the concept of inequality and is not related to each other because each of them has characteristics that differentiate them from one another.

Table 1 details the different registers that are part of each group of semiotic representation.

Table 1. Semiotic representation registers of concept of inequality

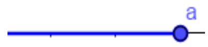
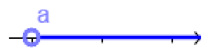
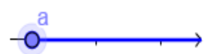
Semantic representations		Graphical representations	Symbolic representations	
			Notation	Set builder notation
	Open interval		(a, b)	$A = \{x \in \mathbb{R}/a < x < b\}$
	Closed interval		$[a, b]$	$B = \{x \in \mathbb{R}/a \leq x \leq b\}$
Semi-open intervals	Half-open interval		$[a, b)$	$C = \{x \in \mathbb{R}/a \leq x < b\}$
Semi-closed intervals	Half-closed interval		$(a, b]$	$D = \{x \in \mathbb{R}/a < x \leq b\}$
Infinite intervals	Right-open		$(-\infty, a)$	$H = \{x \in \mathbb{R}/x < a\}$
	Right-closed		$(-\infty, a]$	$G = \{x \in \mathbb{R}/x \leq a\}$
	Left-open		(a, ∞)	$F = \{x \in \mathbb{R}/a < x\}$
	Left-closed		$[a, \infty)$	$E = \{x \in \mathbb{R}/a \leq x\}$

Table 1 shows that the characteristics of the semantic representations are written and spoken. Conceptually, “open interval” means that both endpoints of the inequality are excluded; whereas “closed interval” means that both endpoints of the inequality are included. Likewise, the semantic representations “semi-open intervals” and “semi-closed intervals” include one endpoint and exclude the other endpoint of the same inequality, therefore, these inequalities can be rewritten as half-open intervals or half-closed intervals, and in some cases, the sector of the endpoint inclusion is specified as left half-closed interval or the other sector of the endpoint exclusion is specified as right half-open interval. In this way, infinite inequalities have only one endpoint either inclusion or exclusion as the other sector is unbounded. Graphical representations are a very important didactic element, which is usually used as a starting point in classroom practice, as students can easily visualize the possible values to understand the inequality by means of the number line. In this way, the two endpoints of inclusion or exclusion of the inequality are represented graphically on the number line by two types of dots ○ and ●. The meaning of the dot “○” corresponds to the exclusion of the bounded and the dot “●” corresponds to the inclusion of the bounded of the mathematical inequality. On the other hand, symbolic representations are written in two ways, notation representations, and set builder notation representations. Notation representations are characterized by the use of punctuation marks such as “( ), [ ], ( ], [ ], [ )”. The parenthesis represents the exclusion of the endpoint in the inequality, while the bracket includes the endpoint of the inequality. Notation representations are often used to define numerical values or conditions in theorems and definitions in mathematics. As for set builder representations, these are based on the linguistic concepts “... is greater than or equal to ...”, “... is less than or equal to ...” and “... is less than or equal to ...”, which symbolically correspond to >, <, ≥, and ≤ are used to establish the endpoints of an inequality defined in the set builder. The meaning of the symbols “<“ and “>“ exclude the endpoint, while the symbols “≤” and “≥” mean the inclusion of the endpoints. Thus, the characterization of the semantic, graphical, and symbolic representations that constitute the first part of conceptual learning of inequalities ends.

Semiotic Treatments of Inequalities

Semiotic treatment is defined as the transformation of an initial semiotic representation into another terminal semiotic representation of the same type (Duval, 2006, 2017b). Act of transformation between representations is performed by the interpretant (Duval, 2017a; Peirce, 1986), as shown in Figure 2. Figure 2 shows the semiosis process of semiotic treatments, which starts from an initial semantic, graphical, or symbolic representation and the interpretant transforms it, applying the rules of operation of semiotic treatments, in order to create another terminal representation of the same semantic, graphical or symbolic type (Duval, 2006, 2017a).

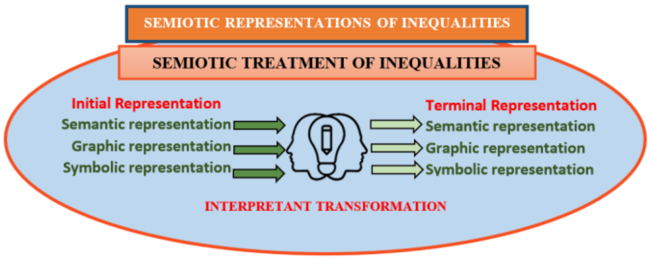


Figure 2. Treatment between semiotic representations (Source: Author’s own elaboration)

Table 2. Arithmetic properties of inequalities

Property	Definition
Property of addition	If $a<b$ then $a+c<b+c$
Property of subtraction	If $a<b$ then $a-c<b-c$
Property of multiplication	If $a<b$ & $c>0$ then $ac<bc$
	If $a<b$ & $c<0$ then $ac>bc$
Property of division	If $a<b$ & $c>0$ then $a/c<b/c$
	If $a<b$ & $c<0$ then $a/c>b/c$

Note. a, b, & c real numbers

For the case of inequalities, the rules of operation of semiotic treatments follow the following arithmetic properties described in Table 2. Learning the properties of inequalities should be an essential and consistent classroom practice, since semiotic treatments strengthen memory, affirm logical processes and develop strategic thinking (Fandiño, 2010). Thus, semiotic treatments apply the operations of addition, subtraction, multiplication, and division to inequalities, as shown in Table 2. The arithmetic properties of addition and subtraction do not alter the sense of the inequality; however, multiplication and division change the sense of inequalities when they are operated by a negative real number; this last case is relevant in the study of the conceptual learning of inequalities, due to the fact that students frequently make mistakes when applying these properties (Balomenou et al., 2017).

Now, functionally, inequalities are used to model inequalities in both mathematical and everyday contexts; and the semiotic treatments allow the solution to the inequality to be applied step by step in the model. An example of the application of semiotic treatments of inequalities is shown in Table 3. Table 3 shows four semiotic treatments referenced in Table 2 necessary inequation $-4x+5>25$ until the set of numerical solutions $x<-5$ is obtained. Table 3 shows the main rule of semiotic treatments, which consists of carrying out transformations within the same group of semiotic representations (Duval, 2017b); that is if the initial representation is a symbolic register, the result of the semiotic treatment is another symbolic register. It is necessary to clarify that the development of semiosis for the treatment of the semiotic registers of inequalities must be carried out with great care and precision in

Table 3. Application of properties of inequalities to solve inequalities

Inequality	$-4x + 5 > 25$	Explanation of treatments
Treatment 1: Subtraction property	$-4x + 5 - 5 > 25 - 5$	Sense of inequality does not change.
Treatment 2: Subtraction result	$-4x > 20$	
Treatment 3: Property of division	$\frac{-4x}{-4} < \frac{20}{-4}$	Sense of inequality changes.
Treatment 4: Result of division	$x < -5$	

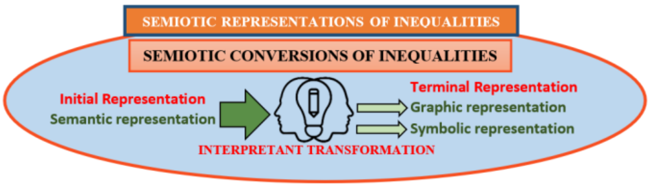


Figure 3. Conversion between semiotic representations (Source: Author's own elaboration)

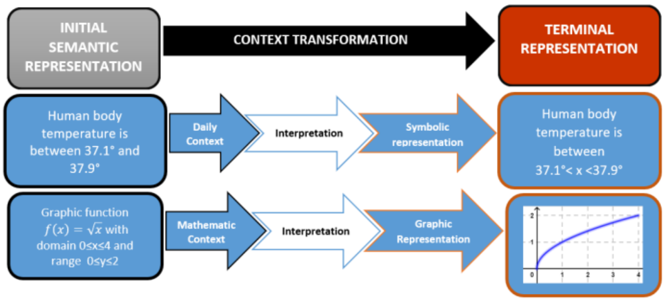


Figure 4. Case (a) of conversion between semiotic representations of inequalities (Source: Author's own elaboration)

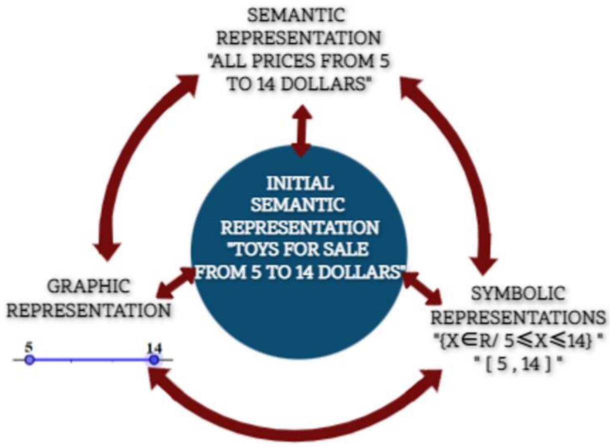


Figure 5. Case (b) of conversion between semiotic representations of inequalities (Source: Author's own elaboration)

order to avoid errors in their applications, as it is evident in classroom practice as the main difficulty in the learning of mathematical concepts in the treatment since many students lack the tools or the mastery of the properties to relate and construct new semiotic registers from the initial register (D'Amore, 2004).

Semiotic Conversions of Inequalities

Semiotic conversion is defined as the transformation of an initial semiotic representation into another terminal semiotic representation of a different type (Duval, 2017a, 2017b). This means that the interpretant coordinates the semiotic conversion between a semiotic representation of inequality and other representations, as shown in **Figure 3**. **Figure 3** shows the interpretant carrying out the semiotic conversion in two ways:

(a) converting an initial representation into another terminal representation (**Figure 4**) and

(b) converting an initial representation into two terminal representations (**Figure 5**).

An example for case (a) of semiotic conversion applied to inequalities is shown in **Figure 4**. Case (a) shows two semiotic conversions; the first one is the conversion of an everyday context represented semantically and transformed into a symbolic representation; the second one is the semiotic conversion of a mathematical context transformed into a graphical representation with the domain and range conditions.

Now, case (b) is the conversion of an everyday context transformed semantically, symbolically, and graphically, as shown in **Figure 5**. **Figure 5** shows the possible semiotic conversions that the initial semantic representation has when transformed into the three types of representations of inequalities, as well as the fact that the graphic, semantic, and symbolic representations have their own relationship without being connected to the initial register. Therefore, the coordination between equivalent representations allows the learner to select the most relevant representation to interpret mathematical or everyday contexts of inequalities and to give meaning to problem situations. To conclude, given the relevance of inequalities inside and outside the classroom, this research proposes a didactic strategy for the conceptual learning of inequalities, with the aim of understanding the meaning that students give to the treatment and conversion to semiotic representations of inequalities applied in different contexts, taking as a reference three phases of triads of semiotic signification.

METHODOLOGY

This research studied the semiosis process of conceptual learning of inequalities by analyzing the integration between multiple transformations of semantic, graphical, and symbolic representations made by students to solve problems of inequalities used in both every day and mathematical contexts in didactic strategies applied in class.

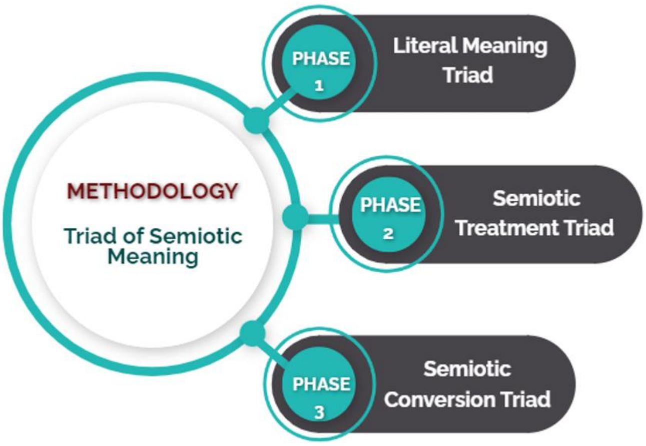


Figure 6. Phases of triads of semiotic meaning (Source: Author's own elaboration)

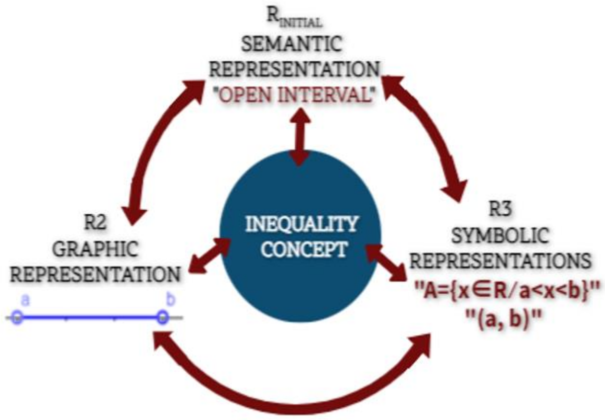


Figure 7. Literal meaning triad of concept inequality (semantic, graphic, & symbolic)-2 (Source: Author's own elaboration)

For this reason, this research is framed within the qualitative approach, as qualitative studies aim to carry out in-depth studies of interpretative phenomena based precisely on the practices carried out by students (Hernández et al., 2010). Thus, the methodological development of this study follows the three phases of the triad of semiotic meaning: the literal meaning triad, the semiotic treatment triad, and the semiotic conversion triad, as shown in **Figure 6**.

The triad of semiotic meaning methodology is a semiotic macrosystem that develops processes of semiosis by integrating three different types of semiotic triads for learning mathematical concepts (Durán Salas, 2022); each of the three triads of the methodology is a semiotic system made up of a semantic, symbolic and graphic representation, so each phase develops the learning of the mathematical concept in a different way (Durán Salas, 2022). The three phases of the triad of the semiotic meaning methodology for the conceptual learning of meaningful inequalities are described below:

Phase 1–Literal Meaning Triad

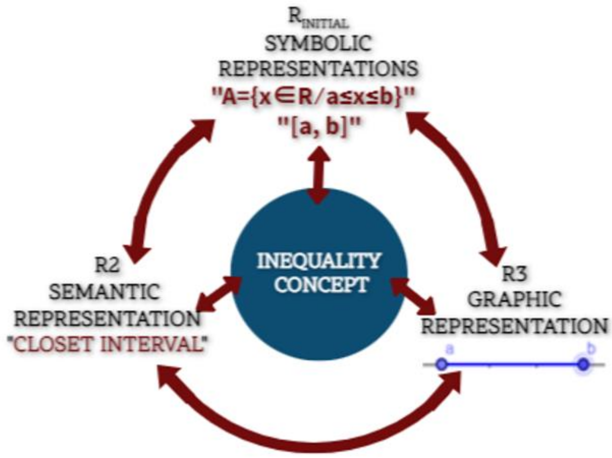
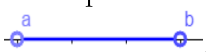


Figure 8. Literal meaning triad of concept inequality (symbolic, semantic, & graphic)-2 (Source: Author's own elaboration)

It is an ordered triadic semiotic system (R_I, R_2, R_3), which establishes an initial representation R_I that integrates another representation R_2 and another representation R_3, all three of different type (Durán Salas, 2022). As an example, R_I is defined as the semantic representation, R_2 as the graphic representation, and R_3 as a symbolic representation, thus forming the triad (semantic, graphic, and symbolic) (Durán Salas, 2022). However, the literal meaning triad for conceptual learning of inequalities takes as a reference the semiotic representations characterized in **Table 1**. Two examples are shown in **Figure 7** and **Figure 8**.

As can be seen in **Figure 7** and **Figure 8**, different types of literal meaning triads can be formed, as it is possible to permute the order of the initial representation, the order of R_2 and R_3 (two-way arrows). In **Figure 7**, literal meaning triad (semantic, graphic, symbolic), which integrates the semiotic representations according to **Table 1** [open interval, , (a, b)] is defined; in **Figure 8** the literal meaning triad (symbolic, semantic, and graphic) is defined with reference to **Table 1** ([a, b], closet interval, ). Literal meaning triad strengthens the development of semiosis for conceptual learning of inequalities by integrating multiple semiotic representations between intervals.

Phase 2–Semiotic Treatment Triad

Like literal meaning triad, it is an ordered triadic semiotic system (R_I, R_2, R_3), but it is the symbolic representation that develops the semiosis process by performing semiotic treatments to the inequalities applying the properties characterized in **Table 2**. Example: Solve the inequality $2x + 4 < 6$ and graph your solution on the straight line.

Figure 9 shows that, in order to solve the inequality $2x + 4 < 6$, four semiotic treatments were necessary,

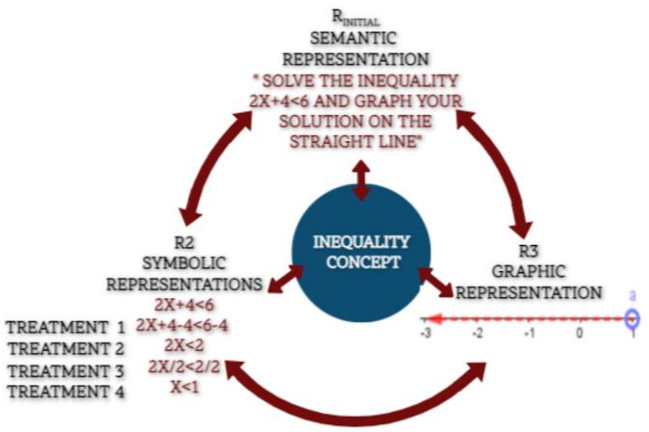


Figure 9. Semiotic treatment triad of concept inequality (semantic, symbolic, & graphic) (Source: Author's own elaboration)

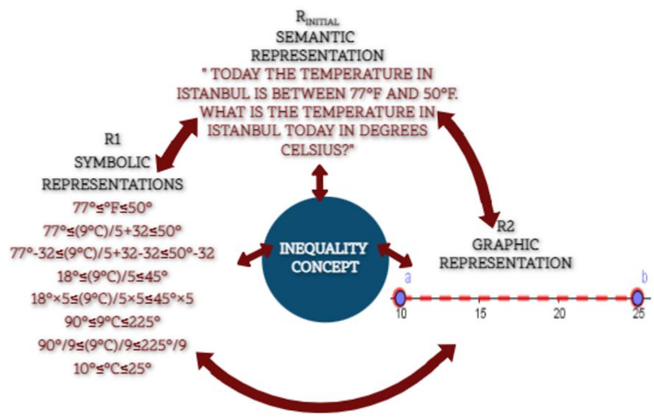


Figure 10. Semiotic conversion triad of concept inequality (semantic, symbolic, & graphic) (Source: Author's own elaboration)

Table 4. Magnitude of Cohen's *d*

Cohen's *d* coefficient intervals	Effect size interpretation
0.2<*d*	Negligible effect
0.2≤*d*<0.5	Small effect
0.5≤*d*<0.8	Medium effect
d≥0.8	Large effect

which are integrated to the semantic and graphic representation of its solution to give meaning to the inequality, thus forming semiotic treatment triad (semantic, symbolic, and graphic).

Phase 3–Semiotic Conversion Triad

As in the previous phases, semiotic conversion triad is an ordered triadic semiotic system (R_I, R_2, R_3) that integrates literal meaning triad and semiotic treatment triad to solve problems with meaning; its main characteristic consists of permuting R_I with the semantic, symbolic and graphic representations to define different semiotic conversion triads. The permutable triadic order leads to the development of semiosis and for this reason, multiple answers can be given to the problem to be solved, enhancing conceptual learning (Durán Salas, 2022). Example: Today the temperature in Istanbul is between 50°F and 77°F. What is the temperature in Istanbul today in degrees Celsius?

As shown in Figure 10, the semantic representation defines a problem of conversion of temperature measurements; whose solution lies in the symbolic representations by developing semiotic treatments until the answer is obtained, which is represented graphically in the straight line as a closed interval.

In synthesis, the triads of semiotic signification as a research methodology are successive, and cohesive and integrate the necessary characteristics to acquire the conceptual learning of inequalities in classroom practices.

Sample

The didactic strategy to develop semiosis in the conceptual learning of inequalities was applied to two tenth-grade classes with 37 students aged between 15 and 16 years from an educational institution in the city of Barrancabermeja (Colombia).

RESULTS

Statistically the conceptual learning progress of the inequalities of the 37 students is measured with the "Cohen's *d*" coefficient. The statistical interpretation of Cohen's *d* defines the effect size intervals, as shown in Table 4 (Cohen, 1988).

Cohen's *d* coefficient allows us to measure the didactic effect of learning in each of the methodological phases by contrasting the results of the means and standard deviations between pre- and post-test after didactically intervening the classroom practices with the group of students.

The didactic strategy was designed and applied following the methodology of triad of semiotic meaning (Figure 6) for the conceptual learning of inequalities whose semiosis process is developed in three consecutive phases; literal meaning triad, semiotic treatment triad, and semiotic conversion triad whose results are the following:

Literal Meaning Triad

The students performed multiple transformations between the three groups of semantic representations, symbolic and graphical representations, as shown in Figure 11.

Part (1) in Figure 11 shows that the symbolic representation $-2 \leq x < 5$ is the R_I representation, which the students transformed into four equivalent representations, as follows: R_1 graphical representation, R_2 set builder notation representation, R_3 notation representation, and R_4 semantic representation. Part (2)

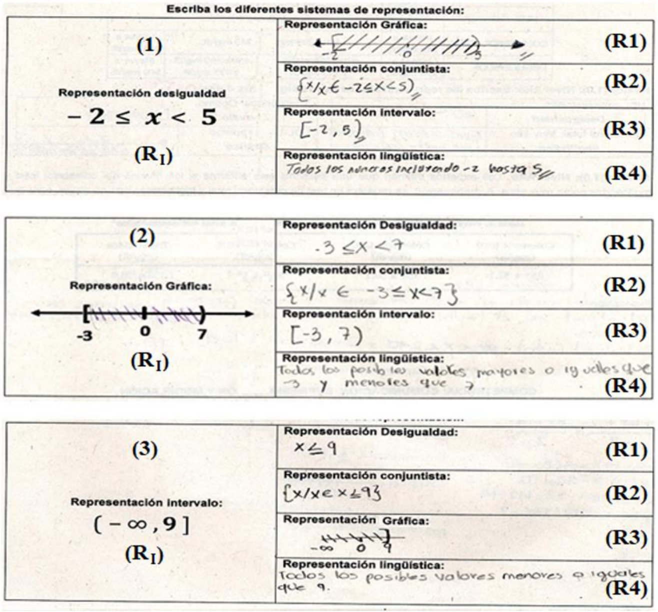


Figure 11. Three transformations of literal meaning triad (Source: Student 8 from field study)

Table 5. Pre-test results: Literal meaning triad of inequality representation

Inequality representation	Transformations	
	Correct	Incorrect
-2≤x≤5	Graphic representation	
	10	27
	Set builder notation representation	
	12	25
	Notation representation	
	13	24
	Semantic representation	
	8	29

in **Figure 11** has, as the initial representation R_I, the graphical representation, which the students transformed into R_1 set builder notation (short) representation, R_2 set builder notation representation, R_3 notation representation, and R_4 semantic representation. Part (3) in **Figure 11** has, as the initial R_I representation, the notation representation (-∞, 9], the students transformed it into, R_1 set builder notation (short) representation, R_2 set builder notation representation, R_3 graph representation, and R_4 semantic representation.

As it can be seen, literal meaning triad is a semiotic system that integrates the transformations of an initial representation of inequalities into equivalent representations that have the same meaning (see **Table 1**); thus, the processes of semiosis are generated for the conceptual learning of inequalities.

Table 5 shows the progress of conceptual learning of inequalities by contrasting the results of the pre- and post-test obtained in the phase 1–literal meaning triad. **Table 5** shows that when applying the pre-test, the students had a greater difficulty in the semiotic transformation of inequality representation into Semantic representation, since only eight out of 37 managed to relate these two representations, while the most effective literal transformation was with notation representation, since 13 out of 37 managed to carry out the transformation correctly.

Table 6. Post-test results: Literal meaning triad of inequality representation

Inequality representation	Transformations	
	Correct	Incorrect
-2≤x≤5	Graphic representation	
	22	15
	Set builder notation representation	
	24	13
	Notation representation	
	24	13
	Semantic representation	
	23	14

Table 7. Comparison of pre- & post-test results inequality representation (Cohen's d=0.56)

Results	Frequency ranges	
Pre-test	Not perform any correct transformation	9
	1 correct transformation	10
	2 correct transformations	11
	3 correct transformations	5
	4 correct transformations	2
	Total	37
	Mean	4.98
	Standard deviation	3.70
Post-test	Not perform any correct transformation	3
	1 correct transformation	3
	2 correct transformations	10
	3 correct transformations	12
	4 correct transformations	9
	Total	37
	Mean	5.20
	Standard deviation	4.15

After the didactic intervention, it can be observed in **Table 6**, the increase of students who correctly transform inequality representation with notation representation retaining the highest number of correct literal transformations together with semantic representation, which significantly increased its pass rate.

The results of the pre-test show that nine students did not perform any semiotic transformation correctly or did not write a possible transformation (**Table 7**). But two of students performed four correct literal transformations of inequality representation. After didactic intervention, the post-test results show that nine students performed four correct literal transformations of inequality representation and 12 students performed three correct semiotic transformations, i.e., 56.7% of sample managed to elaborate networks between semiotic representations to give meaning to the concept of inequality. Semiosis process of literal meaning triad strategy for learning concept of inequality representation showed a progress of 0.56 as observed in Cohen's d.

Table 8. Pre-test results: Literal meaning triad of graphic representation

Graphic representation	Transformations	
	Correct	Incorrect
	Graphic representation	
	14	23
	Set builder notation representation	
	18	19
	Notation representation	
	21	16
	Semantic representation	
	20	17

Table 9. Post-test results: Literal meaning triad of graphic representation

Graphic representation	Transformations	
	Correct	Incorrect
	Graphic representation	
	28	9
	Set builder notation representation	
	26	11
	Notation representation	
	33	4
	Semantic representation	
	30	7

It is observed from the results in **Table 8** that the highest number of correct literal transformations of graphic representation is among notation representation with an index of 21 out of 37 while, among graphic representation and inequality representation shows that 14 out of 37 is the lowest index of correct transformations.

Didactic intervention modified conceptual learning of the graphical representations of inequalities, since, as seen in **Table 9**, the number of correct transformations increased in all their equivalent representations.

The literal transformation of the graphical representation with the highest correct response rate is notation representation with 33 out of 37; and the one with the lowest correct response rate is set builder notation representation with 26 out of 37.

Table 10 contrasts the frequencies of correct transformations obtained by the students in the pre- and post-test. In the pre-test it is observed that three students do not perform any correct literal transformation or do not write any literal transformation of the graphic representation and only four students performed all the semiotic transformations correctly.

On the other hand, it is notorious that after the didactic intervention most of the students manage to establish semiotic connections of the semiosis process between the graphic representation and its equivalent representations, which is observed in the post-test results, as all students perform at least one correct transformation and 20 out of 37 students correctly performed all transformations of the graphic representation of the inequality. The effect of the intervention of the literal significance triads shows a Cohen's *d* coefficient of 0.51, which measures progress in the conceptual learning of graphical representations.

Table 10. Comparison of pre- & post-test results graphic representation (Cohen's *d*=0.51)

Results	Frequency ranges	
Pre-test	Not perform any correct transformation	3
	1 correct transformation	9
	2 correct transformations	12
	3 correct transformations	9
	4 correct transformations	4
	Total	37
	Mean	7.40
	Standard deviation	3.78
Post-test	Not perform any correct transformation	0
	1 correct transformation	5
	2 correct transformations	4
	3 correct transformations	8
	4 correct transformations	20
	Total	37
	Mean	9.25
	Standard deviation	7.36

Table 11. Pre-test results: Literal meaning triad of notation representation

Notation representation	Transformations	
	Correct	Incorrect
(-∞, 9]	Graphic representation	
	17	20
	Set builder notation representation	
	10	27
	Notation representation	
	14	23
	Semantic representation	
	13	24

The results of the pre-test of the semiotic literal transformations of notation representation show that inequality representation and graphic representation have the highest rate of correct answers and set builder notation representation has the highest rate of incorrect answers (**Table 11**).

Conceptual learning of notation representation improved the rate of correct answers in all its semiotic transformations after the didactic intervention. **Table 12** shows that set builder notation representation had a significant increase in the post-test and Inequality and graphic representations maintained the highest rate of correct answers.

It is observed in **Table 13** that nine out of 37 students do not perform any semiotic transformation of the representation (-∞, 9], i.e., 24.0% of the sample and 35.0% of the students, i.e., 13 out of 37 correctly performed only one semiotic transformation.

Table 12. Post-test results: Literal meaning triad of notation representation

Notation representation	Transformations	
	Correct	Incorrect
(-∞, 9]	Graphic representation	
	29	8
	Set builder notation representation	
	26	11
	Notation representation	
	32	5
	Semantic representation	
	23	14

Table 13. Comparison of pre- & post-test results notation representation (Cohen's *d*=2.54)

Results	Frequency ranges	
Pre-test	Not perform any correct transformation	9
	1 correct transformation	13
	2 correct transformations	7
	3 correct transformations	5
	4 correct transformations	3
	Total	37
	Mean	7.40
	Standard deviation	3.84
Post-test	Not perform any correct transformation	0
	1 correct transformation	4
	2 correct transformations	8
	3 correct transformations	10
	4 correct transformations	15
	Total	37
	Mean	9.25
	Standard deviation	4.57

The result obtained in the post-test, after the didactic intervention the semiosis process, calculates that 15 out of 37 students performed four correct transformations and 10 out of 37 performed three correct semiotic transformations of the representation (-∞, 9] i.e., 68.0% of sample correctly relate notation representation with at least three equivalent representations. Results mean that effect on conceptual learning of notation representation by means of literal meaning triad has been progressive considering Cohen's *d* coefficient of 2.54.

Semiotic Treatment Triad

Students solved different types of linear inequalities by applying the arithmetic properties described in **Table 2**, transforming the symbolic representation, by means of successive treatments until the set of solutions of the inequality was calculated. The result of the class activity is shown in **Figure 12**. **Figure 12** shows the succession of semiotic treatments employed by the student to solve the inequality $\frac{4x+1}{3} \leq \frac{12x-3}{7}$ (R_I), which has as answer $2 \leq x$. The students transforms the symbolic answer $2 \leq x$ to its equivalent representations: (R_1) set builder notation, (R_2) notation, (R_3) graphic, and (R_4) semantic following the characteristics of the literal meaning triad.

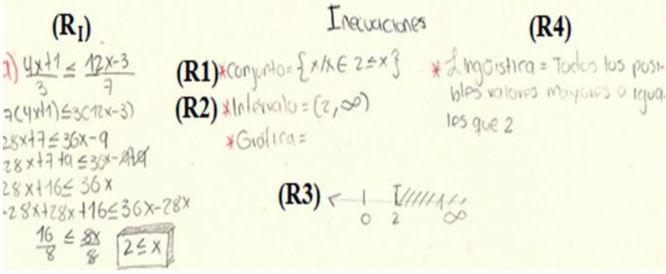


Figure 12. Semiotic treatment triad of linear inequality (Source: Student 15 from field study)

Table 14. Comparison of results between pre- & post-test of linear inequality triad semiotic treatment (Cohen's *d*=0.66)

Results	Frequency ranges	
Pre-test	Did not solve inequality	14
	Solves inequality & does not perform any transformation	4
	Solve inequality & perform 1 transformation of result	5
	Solve inequality & perform 2 transformations of result	8
	Solve inequality & perform 3 transformations of result	3
	Solve inequality & perform 4 transformations of result	3
	Total	37
	Mean	4.60
	Standard deviation	2.07
Post-test	Did not solve inequality	3
	Solves inequality & does not perform any transformation	0
	Solve inequality & perform 1 transformation of result	3
	Solve inequality & perform 2 transformations of result	11
	Solve inequality & perform 3 transformations of result	7
	Solve inequality & perform 4 transformations of result	13
	Total	37
	Mean	6.80
	Standard deviation	5.40

Table 14 shows the progress of conceptual learning of inequalities by analyzing the results between the pre- and post-test of didactic phase 2–semiotic treatment triad. **Table 14** shows the progress of the conceptual learning outcomes of inequalities when applying their numerical properties to solve inequalities measured with a Cohen's *d* coefficient of 0.66. The results of semiotic treatment triad phase measure the relationship between the correct solution of the inequation $\frac{4x+1}{3} \leq \frac{12x-3}{7}$ and the number of semiotic transformations the student performs on his numerical answer.

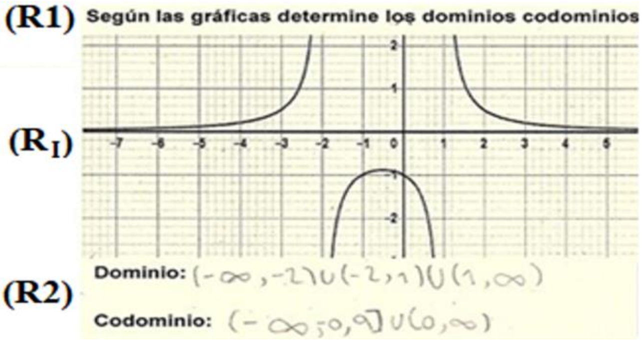


Figure 13. Analysis of domain & range of a rational function (Source: Student 27 from field study)

In **Table 14**, it is observed in the results of the pre-test that 14 out of 37 students do not solve the inequation; therefore, they do not perform any semiotic transformation of the answer 2≤x likewise, four students solve the Inequality correctly, but do not manage to semiotically transform their answer. Also in the pre-test it is calculated that the majority of students eight out of 37 who solve the inequality correctly transform their solution into two equivalent representations. After the didactic intervention, the processes of semiosis are extended and the students are more accurate in their transformations, which can be observed in **Table 14**, since only three students do not solve the inequality while, 34 out of 37 students solve the inequality correctly and at least perform a correct semiotic transformation of their answer, highlighting that 13 out of 37 solve the Inequality correctly and transform their answer into four equivalent representations.

Semiotic Conversion Triad

The main application of this semiotic system is to solve problems or questions from both mathematical and everyday contexts. Two semiotic conversion triad activities carried out in class are shown below. The first one refers to solving a question in a mathematical context, it consisted of analyzing graphically the rational function $f(x)=\frac{2}{(x+2)(x-1)}$ and determining its domain and range, as shown in **Figure 13**.

The second one is developed in an everyday context, it consisted of analyzing the table of numerical values of the lipid profile to define if a person is ill due to high levels of cholesterol and triglycerides, as shown in **Figure 14**.

The graphical representation of the rational function $f(x)=\frac{2}{(x+2)(x-1)}$ is the initial representation (R_I); however, the semantic representation (R_1) defines the task that the student must perform "According to the graph determine its domain and range" and the symbolic representation (R_2) is the result of semiosis between the semantic representation and the graph to obtain the domain and range of the function forming the semiotic conversion system. The results of the process of semiotically transforming graphical and semantic representations into notation representations are shown in **Table 15**. Before didactic intervention, 14 students transformed the domain and range correctly; after the didactic intervention, the post-test showed that 31 students correctly performed the semiotic graphical and semantic transformation of the domain and range in notation representation. This means that the use of the semiotic representations of inequalities was effective in the application of the didactic strategy of triads of semiotic meaning since Cohen's d coefficient is 3,89.

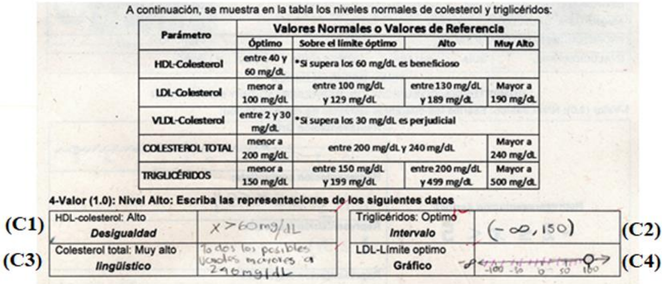


Figure 14. Semiotic analysis of lipid profile (Source: Student 3 from field study)

In the second part of the semiotic conversion triads, the students carried out the analysis of a lipid profile of a person, as shown in **Figure 14**.

The semiosis of the lipid profile analysis had two moments; the first one was the recognition of the "optimal", "optimal limit", "high" and "very high" levels of cholesterol and triglycerides from which four conversions were made: the semiotic conversion (C_1), which consists of transforming HDL-cholesterol: high level to a set builder notation representation; the semiotic conversion (C_2), that transforms the numerical data of triglycerides: optimal level to a notation representation; the semiotic conversion (C_3), which transforms the numerical data of total cholesterol very high level to a semantic representation; and the semiotic conversion (C_4), that transforms the numerical data of LDL-cholesterol: optimal limit level to a graphic representation.

Table 15. Results of semiotic transformations of domain & range of a rational function (Cohen's d=3.89)

Analysis of graphical representation of function		Notation representation domain & range	
		Pre-test	Post-test
$f(x)=\frac{2}{(x+2)(x-1)}$	Correct	14	31
	Incorrect	23	6
	Mean	0.37	0.83
	Standard deviation	0.49	0.37

Table 16. Results of semiotic transformations of lipid profile (Cohen's *d*=1.42)

			C1	C2	C3	C4
Analysis of symbolic representation of lipid profile	Pre-test	Correct	23	21	22	23
		Incorrect	14	16	15	14
		Mean	0.60			
		Standard deviation	0.49			
	Post-test	Correct	34	33	34	35
		Incorrect	3	4	3	2
		Mean	0.92			
		Standard deviation	0.26			

Note. C1: Inequality representation; C2: Notation representation; C3: Semantic representation; C4: Graphic representation

In the second moment, students had to define if a person was sick or not according to the result of their lipid profile. The task says: "experts define that a person is sick if the levels of total cholesterol and triglycerides are very high. Below is a total cholesterol and triglyceride test of a woman, for you to explain whether she is sick or not according to the data".

A student answers (Ans), as follows: "The woman is not sick because she has cholesterol values at the optimal or high limit and triglycerides at optimal levels", furthermore, the student converts the cholesterol and triglyceride levels into symbolic representations of set builder notation. The overall results of the second part of the semiotic conversion triads focused on assessing the comprehension of the lipid profile table by performing the C1, C2, C3, and C4 conversions, as shown in **Table 16**.

Cohen's *d* coefficient 1.42 shows that the didactic strategy of the semiotic meaning triads practiced in class had a positive effect, as contrasted in the averages of correct answers between the pre-test with 0.60 modifying and the 0.92 of the post-test; it is for this reason that a considerable decrease in incorrect answers can be seen when performing semiotic conversions to the lipid profile table after the didactic intervention.

Finally, the students achieved positive results since the progressive development of the semiosis for the conceptual learning of inequalities by means of the semiotic meaning triad was in accordance with the learning times because multiple classroom activities were harmoniously integrated into each phase.

DISCUSSION & CONCLUSIONS

This study investigated the processes of semiosis that students develop when conceptually learning about inequalities, using the triads of semiotic signification as a methodological reference. Thus, the results of this research were obtained at two points in time: the pre-test before the didactic intervention and the post-test after the didactic intervention. The analysis of the results obtained in the pre-test showed that students frequently made the following three mistakes:

(a) error in the interpretation of inequality sign: students give the meaning of equality to the signs <, >, ≤, and ≥,

(b) errors in solving inequalities algorithmically: the students do not perform skillfully the four arithmetic operations between fractions, mainly addition and subtraction, and

(c) error in changing the sense of inequality: the students multiply correctly on both numerical sides of the numerical inequality, but do not effectively transform the sense of inequality sign, they keep it the same.

These three errors are consistent with Almog and Ilany's (2012) and Blanco and Garrote's (2007) research on inequalities. These errors limit the creation of new meanings of inequalities and confuse the processes of semiotic treatments of inequalities that are quite necessary for problem solving.

Therefore, it is necessary to create didactic alternatives and sequentially structured methodologies that allow students to correct their mistakes and strengthen the learning of inequalities (Balomenou et al., 2017).

In fact, the methodology of semiotic meaning triads allowed to intervene didactically in the conceptual learning of inequalities in three sequential phases:

(a) literal meaning triads,

(b) semiotic treatment triads, and

(c) semiotic conversion triads.

The findings in the phases of the semiotic treatment and semiotic conversion triads are similar to the findings of Blanco and Garrote (2007) who agree that the ideal would be the use of more than one semiotic representation system, which would benefit the understanding of inequalities, since the different systems provide alternative and complementary strategies to the students, as they only use arithmetic and algebraic language to tackle the different problems they have as a task.

Hence, the triads of semiotic meaning are relevant for the conceptual learning of inequalities, as they manage to integrate graphic, semantic and graphical representations in the same inequality and in the same

Table 17. Findings from literal meaning triads phase

Initial representation	Pre-test 4 correct transformations	Post-test 4 correct transformations
Inequality representation: $-2 \leq x \leq 5$	2 students	9 students
Graphic representation: 	4 students	20 students
Notation representation: $(-\infty, 9]$	3 students	15 students

Table 18. Findings from semiotic treatment triads phase

Initial representation	Pre-test: Solve inequality & perform 4 transformations of result	Post-test: Solve inequality & perform 4 transformations of result
Inequality representation: $\frac{4x+1}{3} \leq \frac{12x-3}{7}$	3 students	13 students

Table 19. Findings from triads phase of semiotic conversion

Initial representation	Pre-test problem-solving skills	Post-test problem-solving skills
Analysis of domain & range of function: $\frac{2}{(x+2)(x-1)}$	14 students	31 students
Analysis of symbolic representation of lipid profile	21 students	35 students

problem with inequalities. In effect, semiotic meaning triads methodology provides a new perspective for the conceptual learning of inequalities in students by developing semiosis processes integrating multiple graphic, semantic and symbolic representations to solve problems with inequalities, and whose effectiveness in learning is observed in the results obtained after the didactic intervention. When comparing the results between the pre- and post-test, the following advances by phases are found.

Literal Meaning Triads Phase

Table 17 shows that the number of students performing four correct semiotic transformations in the pre-test was low, but after the didactic intervention, the number of students performing four correct transformations of the same semiotic representation increased considerably.

Semiotic Treatment Triads Phase

Table 18 compares the number of students who solved the inequation correctly and who semiotically transformed their answer into four different types of representations. In the pre-test only three students out of 37 were able to do so, however, after the didactic intervention the number of students who were able to perform assigned task correctly increased significantly.

Semiotic Conversion Triad Phase

Table 19 shows that the number of students who solve inequalities correctly in the pre-test has increased in relation to two previous phases. This is since semiotic conversion triads phase is the final phase of the didactic methodology; therefore, the students have already corrected some of their previous obstacles in phase (a) and phase (b). Likewise, the increase in the number of students who effectively solve the problems in the post-test stands out for having the highest effectiveness rate regarding the previous phases. It means that semiotic meaning triads is a methodology that positively impacted the conceptual learning of inequalities through didactic strategies applied in class.

The process of semiosis of conceptual learning of inequalities is slow and allows us to observe the personal progress of each student because, as the didactic strategies were developed, the interweaving between semantic, graphic, and symbolic representations solidified and widened the field of conceptual application in both mathematical and conceptual contexts.

Graphical representations are key in the literal meaning triad phase of inequalities as students easily integrated symbolic and semantic representations; therefore, it is recommended to start the conceptual learning of inequalities with graphical representations.

Most of the students focused the semiosis of their learning on the transformation of the extreme values of the inequalities and thus managed to give meaning to the graphical, inequality and notation representations; however, this caused great difficulties in learning the meaning of set builder notation representation as the symbolism {x/x …} and its semantics focuses its understanding on the numerical content that the "x" in the set of inequality may have, which varies its semiotic interpretation.

When examining the students' performance in both the pre- and post-test, it is evident to note that notation representation had the best rates of correct answers, since it was clear to the students the symbolic relationship of the brackets and braces with the graphical representation of empty point and full point; however, during the didactic intervention in the three phases of the semiotic meaning triads the students had difficulties in interpreting infinite intervals as $(-\infty, 9]$.

The semiosis process of the semiotic treatment triad phase allowed for a precise didactic analysis of the successes and errors that the students wrote when applying arithmetic properties to solve inequalities, thus planning a new didactic activity to strengthen the semiotic treatments in class.

The students stated that the difficulty in solving inequalities was related to arithmetic between fractions but not to the application of the arithmetic properties of inequalities. For this reason, it is recommended to dedicate some class sessions to deepen the arithmetic of fractions prior to the semiotic treatment triad phase.

Semiotic conversion triad revitalizes the conceptual learning of inequalities from a mathematical and everyday perspective. Proof of this was the mathematical understanding of the table of cholesterol and triglycerides, as the students had already heard about it, and this aroused their interest in learning.

Semiotic meaning triads methodology for concept learning develops students' processes of semiosis and noesis, broadening the spectrum of opportunities for teachers and schools to design didactic strategies to be employed in their classroom practices and include them in their curricula.

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Data sharing statement: Data supporting the findings and conclusions are available upon request from the author.

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