MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2023, 19(12), em2368
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/13862

Slope conception as rate of change starter kit: Malaysian pre-service secondary mathematics teachers' subject matter knowledge in rate of change

Nurul Atiqah Talib [1*], Suzieleez Syrene Abdul Rahim [1], Hutkemri Zulnaidi [1]

[1] Department of Mathematics and Science Education, Faculty of Education, Universiti Malaya, Kuala Lumpur, MALAYSIA

Received 18 April 2023 ▪ Accepted 03 October 2023

Abstract
Past research on calculus has shown that students often struggle to understand derivatives due to an overemphasis on algebraic manipulation and procedures rather than grasping the underlying concept. A key factor contributing to this challenge is the lack of a solid understanding of slope as a rate of change. Conceptualizing slope as a rate of change is essential, as it serves as the basis for comprehending derivative concepts. To address this gap, our research explored subject matter knowledge of rate of change, specifically focusing on the slope concept, among Malaysian pre-service mathematics teachers. Our research followed a qualitative methodology, conducting task-based interviews with two pre-service mathematics teachers, Zheng and Amitha, who are majoring in mathematics. The interviews included seven tasks related to slope and ratio concepts. The findings revealed inconsistencies in their notion of rate of change, as they have a loose connection between rate of change and slope concepts, along with its multiplicative property. While they recognized that equivalent values of rate of change indicate a constant rate, they did not grasp that slope also represents a rate of change. Their knowledge appeared limited to viewing derivatives as a method to calculate rate of change, without conceiving the changes occurring between quantities and their multiplicative relationship.

Keywords: slope, constant rate of change, multiplicative comparison, ratio, rate of change

INTRODUCTION

In recent years, there has been growing interest in researching the teaching and learning of calculus in both high school and college settings. This is because calculus plays a fundamental role in real-world applications across disciplines, including engineering, economics, physics, biology, and medicine. It is a dynamic field in mathematics, dealing with quantifying "how things change, the rate at which they change (derivative), the way in which they accumulate (integral)" (Tall, 2009, p. 1). This study specifically focuses on the challenging concept of rate of change, identified by both teachers and university students, who have encountered difficulties in achieving a comprehensive understanding of it in calculus (Byerley & Thompson, 2017; Cetin, 2009; Teuscher & Reys, 2012; Weber & Dorko, 2014).

To progress towards understanding derivatives as an instantaneous rate of change, it is essential to first grasp the concept of slope as a rate of change. Many students and teachers, across different age groups, are unable to acknowledge that the slope of linearity also represents a rate of change, specifically a constant rate of change (Byerley & Thompson, 2017; Nagle & Moore-Russo, 2013; Nagle et al., 2019; Stump, 1999, 2001; Teuscher & Reys, 2012). Past studies in calculus have shown that both students and teachers struggle to understand the real meaning of derivatives, which is closely tied to the concept of rate of change (Desfitri, 2016; Lam, 2009; Orton, 1983; Tyne, 2016; Zandieh, 2000). University research also suggests that high school teachers play a crucial role in instilling an understanding of basic calculus concepts in students (Ayebo et al., 2017; Estonanto & Dio, 2019; Pitt, 2015). Qualitative studies focusing on the topic of rate of change have further

This work is derived from a preliminary PhD thesis.

✉ nurulatiqahtalib@yahoo.com ✉ suzieleez@um.edu.my **(*Correspondence)** ✉ hutkemri@um.edu.my

Contribution to the literature

- Slope conception is important in understanding the essence of rate of change in calculus. There is a lack of studies addressing the importance of slope conception in understanding the rate of change under derivative concept.
- The present study emphasized slope conception as changes between quantities, including their multiplicative relationship as a knowledge of subject matter that pre-service secondary mathematics teachers should have.

revealed that poor performance can be attributed to deficits in understanding the meaning and notion of rate of change (Byerley & Thompson, 2017; Herbert & Pierce, 2012; Weber & Dorko, 2014).

The concept of rate of change, represented by $\frac{dy}{dx}$, varies depending on the context, such as marginal cost for the total cost as a function of the product produced, acceleration for speed as a function of time, and electrical circuit for current flow as a function of time. At the university level, students encounter more advanced calculus that requires them to conceptualize derivatives in various real-life contexts. However, most students only know that $f(x)$ is different from $\frac{dy}{dx}$ due to the differentiation process (derivatives), indicating a procedural understanding of how to obtain $\frac{dy}{dx}$ rather than understanding the conceptual contexts of $\frac{dy}{dx}$ (Hashemi et al., 2014; Makonye & Luneta, 2014; Ubuz, 2007; Weber & Thompson, 2014). Common mistakes reported in past studies show that students tend to verbalize $\frac{dy}{dx}$ as amount of quantity, despite knowing that it represents a rate of change, as a result of memorizing derivatives as a rate of change (Byerley & Thompson, 2017; Kertil & Dede, 2022; Mkhatshwa & Doerr, 2018). This further suggests that students do not conceive $f(x+h)$ and $f(x)$ as quantities associated with specific inputs (Weber et al., 2012), leading to their inability to understand $f(x+h)-f(x)$ as the quantity change relative to the change in input (Weber et al., 2012). To address this issue, it becomes essential to revisit the previously introduced concept of slope. Tall (2019, p. 4) stated, "The derivative now is not introduced as a limit; it is the slope of the graph itself. It is possible to look along the graph to see its changing slope". Therefore, fostering a deeper understanding of derivatives requires grasping the prerequisite concept of slope as a rate of change.

Students need to understand slope as a relationship between two changing quantities, known as a rate of change. While slope has been conceptualized in 11 ways, both in-service and pre-service teachers often use common physical situations to teach slope rather than functional situations (Nagle & Moore-Russo, 2013; Nagle et al., 2019). If students fail to connect different conceptualizations of a slope, they not only struggle to grasp the concept of rate of change but also struggle to understand the derivatives as an instantaneous rate of change. This is because derivatives involve the construction of slope as a functional property, geometric ratio, and algebraic ratio (Nagle et al., 2019). Stump (1999) emphasized the significance of a well-developed understanding of slope for achieving a better understanding of rate of change. In his study, he observed that the majority of pre-service and in-service teachers had limited knowledge, as they only considered physical situations for slope instruction. These findings aligned with studies conducted by Byerley and Thompson (2017) and Nagle and Moore-Russo (2013) among in-service mathematics teachers, indicating that the issue persists. Instead of recognizing slope as a rate of change, teachers commonly used real-life examples such as mountain roads, ramps, and ski slopes, which revealed their weakness in understanding concept of rate of change (Byerley & Thompson, 2017; Stump, 1999).

In Malaysia, students learn about slope in the context of straight lines and linear functions, where the gradient term is introduced. The slope has been taught as the difference in y divided by the difference in x, as in the Cartesian coordinate system. Students often view the algebraic ratio of slope using y_2, y_1, x_2, and x_1 as a mere formula to plug in for calculations, rather than understanding it as a ratio conception (Nagle et al., 2019). This leads students to see slope as a number obtained through rigid arithmetic computation, rather than recognizing it as a relationship between changed quantities (Nagle et al., 2019). While this approach is commonly used for introducing slope, it neglects the interpretation of slope as a rate of change (Teuscher & Reys, 2012). As a result, when students encounter calculus or pre-calculus, they may conceive the concept of rate of change as something new and unrelated to the slope of a line they had learned earlier (Teuscher & Reys, 2012).

Most students and teachers understood ratio as a comparison of quantities in quotient form or numerically as a division between numbers (Byerley & Thompson, 2017). However, they fail to see the vinculum (horizontal division bar) as having a multiplicative meaning (Byerley & Thompson, 2014, 2017). For instance, most teachers interpreted the quotient "over", referring to the division and symbolizing two amounts, while others related "over" to duration in a context (Byerley, 2019; Byerley & Thompson, 2017). Such interpretations lead to problem-solving without conceptualizing a multiplicative relationship between the quantities (Byerley, 2019). Consequently, this limited understanding undervalues the meaning of slope, as they cannot grasp the use of division in the slope formula. For instance, when teachers were asked the meaning of a slope of ¾, they responded with "up three and over four" (Stump, 2001; Weber & Thompson, 2014; Weber et al., 2012). They struggled connecting the division role because it was different from division meaning they learnt during school days (Byerley, 2019). The teachers struggle to conceive it as a relative size, indicating that each change in x corresponds to ¾ times the changes in y. This situation reveals that teachers with weak knowledge cannot conceive the quotient form as a multiplicative comparison, making them less likely to consider slope as a rate of change and more likely to view it as a physical interpretation (Byerley, 2019).

The significance of the multiplicative relation lies in defining the relationship between changing quantities as it exists in the slope formula. This comprehension needs to be established before understanding rate of change since the two concepts are interconnected (Orton, 1983). However, most teachers conceived ratios as additive comparisons, rather than multiplicative comparisons (Byerley & Thompson, 2017). The additive comparison does not belong to the ratio concept, resulting in the misconception that ratios involve comparing the quantity of one item to another in terms of being less or more. Consequently, teachers and students might develop an understanding of slope without linking it to rate of change, as they are not required to compare the quantities multiplicatively and are hence unable to coordinate the multiplicative changes that occur (Byerley, 2019; Byerley & Thompson, 2017). The importance of connecting topics has been emphasized by Liping Ma in her study as a Profound Understanding of Fundamental Mathematics (PUFM) (Fuadah et al., 2002). This connection entails being aware of the conceptual structure and establishing a solid foundation for it. PUFM involves the capacity to thoroughly connect various topics, irrespective of their conceptual complexity (Ma, 1999, as cited in Fuadah et al., 2022). It is vital to emphasize subject matter knowledge (SMK) as a PUFM, besides pedagogical knowledge concerning learning and teaching strategies (Ma, 1999, as cited in Fuadah et al., 2022).

SMK plays a central role in teaching, especially in mathematics education. It is defined as "the amount or organization of knowledge per se in the mind of the teacher" (Shulman, 1986, p. 9). This refers to the knowledge a teacher possesses in their subject area. SMK is important because effective teaching demands the teacher to know the content of their subject matter very well before delivering it to the students. The transfer of SMK from teacher to student, known as pedagogical content knowledge (PCK), requires the teacher to be well-versed in their subject area. PCK sets teachers apart from subject-area experts. Shulman (1986) explained that PCK involves transforming the teacher's understanding of the subject matter. This transformation involves reflection, interpretation, and redefining the subject matter to tailor it to the students (Gudmundsdottir & Shulman, 1987). It is an ongoing process of adapting teaching methods to the student's abilities and needs (Gudmundsdottir & Shulman, 1987). Therefore, for this diversity of teaching plans and techniques to develop under PCK, the teacher needs to have a strong foundation of knowledge in their subject matter.

Teachers cannot effectively teach students something they do not know. The importance of teachers' SMK lies in their ability to instill students with a deep understanding, which requires teachers to be well-equipped with knowledge in the first place. SMK encompasses knowledge of facts and their connections among mathematical ideas (Ball et al., 2008). Mathematical propositions and concepts are often interconnected, hence, teachers need to recognize and grasp these connections to enhance knowledge delivery. In other words, having this knowledge enables individuals to understand more than isolated facts and methods. They can comprehend why a mathematical idea is important and how it relates to its function, rather than merely knowing how to execute it. Unfortunately, most high school and university students lack the understanding of the meaning of derivatives and rely solely on formulas and rules to explain it (e.g., Brijlall & Ndlovu, 2013; Carlson et al., 2015; Makonye & Luneta, 2014). In Malaysia's context, studies by Li et al. (2017) and Shamsuddin et al. (2015) found poor performance among university students in calculus, stemming from their limited understanding of derivatives. Therefore, it is highly encouraged for high school teachers to introduce students to the concept of rate of change, which is constituted by the concept of slope.

Based on the literature, the issue with rate of change revolves around its deep meaning, elaboration, and relation with subordinate topics. Understanding slope as a functional property is crucial as it forms the foundation for understanding rate of change (Byerley & Thompson, 2017; Teuscher & Reys, 2012; Tyne, 2016). Fostering rate of change from slope conception involves considering variation between two variables, requiring functional thinking. This cognitive ability can be approached and developed using different mental representations. Slope conceptualization involves transitional levels of functional property, geometric ratio, and algebraic ratio (Nagle et al., 2019). A comprehensive understanding of slope as a rate of change is important to establish a solid foundation of knowledge, as slope is essential for conceptualizing derivatives at a point as an instantaneous rate of change (Nagle et al., 2019). However, teachers were found to have limited knowledge of slope, mainly viewing it as a physical interpretation, contributing to students' low understanding of rate of change (Byerley & Thompson, 2017; Nagle & Moore-Russo, 2013; Stump, 1999). Past studies (e.g., Byerley & Thompson, 2017; Stump, 2001) revealed that teachers' problematic knowledge of functional slope stemmed from not understanding ratio as a multiplicative comparison. Ratios, rates, and proportions are fundamental concepts that need to be understood beforehand, as they deal with the relationship between two quantities (Orton, 1984; Thompson & Thompson, 1992). Past studies (e.g., Byerley & Thompson, 2017; Nagle & Moore-Russo, 2013; Stump, 2001) have discussed teachers' limited knowledge on rate of change from aspect of slope. Therefore, the attention of this issue should also be brought into mathematics teacher education program, aiming to investigate the knowledge had by pre-service mathematics teachers.

There is a lack of thorough research focusing on pre-service mathematics teachers' knowledge of rate of change. Existing studies (e.g., Dane et al., 2016; Fothergill, 2011; Kertil, 2014; Kertil & Dede, 2022) among pre-service mathematics teachers primarily focus on broader calculus topics such as derivatives, limits, or integrals. These studies have consistently shown weak knowledge of derivatives conception and a lack of awareness of its connection to rate of change. A local study also revealed that Malaysian pre-service mathematics teachers scored lower in teacher education and development study in mathematics, with only 6.9% having higher content knowledge compared to 57.1% at a lower level, ranking below the international mean (Leong et al., 2015). Therefore, we employed a qualitative research methodology to thoroughly investigate Malaysian pre-service mathematics teachers' SMK in rate of change. The research question was derived from the literature and the identified problem related to rate of change: What kinds of SMK of rate of change do Malaysian pre-service mathematics teachers have?

METHODOLOGY

Research Design

A qualitative study was employed to explore SMK of rate of change among pre-service mathematics teachers, specifically focusing on slope conception. This research method was chosen as we sought to understand the knowledge that pre-service teachers attributed to this topic. Since individuals acquire knowledge differently, we adopted the interpretive perspective, which views knowledge as a distinctive event and employed an inductive approach (Creswell, 2013). To achieve a holistic description, we limited the study to one particular participant, a pre-service mathematics teacher, turning it into a case study. This approach is more suitable for educational research as it can enhance the knowledge base and potentially influence and improve practice (Merriam, 2009).

Method of Data Collection

Interviews are found to be the most suitable technique to gain insights into pre-service teachers' knowledge, which is classified as unobservable entities in their thoughts (Merriam, 1998). We employed task-based interviews, widely used in mathematics education, to explore existing mathematical knowledge in individuals and groups (Goldin, 2000; Maher & Sigley, 2014). Unlike conventional paper-and-pencil tests that limit knowledge investigation and do not address the actual knowledge of an individual (Goldin, 2000), task-based interviews offer a deeper understanding of pre-service teachers' knowledge by directly examining their mathematical solutions. This approach allows comprehensive insights into their knowledge, moving beyond the mere patterns of correct and incorrect answers they produced.

This clinical task-based interview used both document analysis and direct-observation techniques. In this context, 'clinical' refers to researchers directly observing the subject's behavior (Piaget, 1929). This allows us to observe how participants behave and their body language when engaging with the tasks, as well as how they explain and clarify their responses. Direct observation during the interview helps the researchers adjust the interaction pace and relevant questions and probes accordingly. Participants' written answers on the task sheets provide valuable insights into their SMK. The data collected from the interviews, including direct observations and document analysis, was recorded through audiotapes, videotapes, participants' written sheets, and the researcher's field notes.

Trustworthiness of the Study

The researcher plays a vital role in data collection and analysis, which can introduce biases and sensitivity to the study. To address this, the trustworthiness of the study was ensured by focusing on its validity, reliability, and generalizability. Qualitative researchers maintain trustworthiness through credibility, dependability, transferability, and confirmability.

In this study, data triangulation was used, incorporating various sources such as video and audio recordings, the researcher's field notes, and participants' written task sheets. This approach was adopted to secure consistency in data validation and strengthen credibility and dependability. Member checking ensures credibility and confirmability (Merriam, 2009), hence, it was used to ensure data validity, allowing participants to review and clarify their interpretation after the interview session. It rules out the possibility of misinterpretation by allowing the participants to discuss and clarify their interpretations. Follow-up interviews were conducted to

Table 1. Tasks

Task	Sub-instrument	Description
Task 1	Rate of change	Task 1 was designed by the researchers to assess the participant's meaning of rate of change.
Task 2	Ratio	Task 2 was modified from Byerley and Thompson (2017) to change the question context from distance-time to number of oranges. Aim was to explore role of ratio as a multiplicative comparison without influence of distance-time context. Goal of this task was to assess participant's interpretation of ratio in a specific manner by giving them answer choices, followed by their justifications.
Task 3	Ratio	Task 3 was designed by the researchers to assess the participant's knowledge of ratio as a multiplicative comparison. The task aimed to assess whether the participants recognize that ratio is not only expressed as a value in quotient form but also as a relative size of two quantities, indicating how one quantity is a multiple of another. It helps gain insights into how participants interpret multiplicative relationships within a given context.
Task 4	Slope	Task 4 was modified from Byerley and Thompson (2017) to include the measurement of a slope. The aim was to assess the participant's meaning of slope and whether they interpret it as a functional property or merely as a physical interpretation, such as an index of steepness. Interpreting slope as a functional property is crucial for grasping the concept of rate of change as a relationship between two changing variables. It also helps to clarify whether participants' interpretation of slope aligns with their notion of rate of change.
Task 5	Rate of change (equation form)	Task 5 was designed by the researchers to assess the participant's knowledge of rate of change in a context of equational form. The aim was to gain insights into how the participants interpret rate of change in terms of equation form within a given context.
Task 6	Rate of change (graphical form)	Task 6 was designed by researchers to assess participant's knowledge of rate of change in graphical form, specifically focusing on interpreting instantaneous rate of change in a motion context, which was chosen for its familiarity with topic of rate of change, allowing task to explore participants' knowledge as they read & interpret data from a graph. This provides insights into participants' knowledge of rate of change shown graphically in a given context.
Task 7	Rate of change (tabular form)	Task 7 was modified from Orton (1983). Graphical representation was removed, & an additional question for another time, $t = 5.34$, was included to assess participant's knowledge of constant rate of change in tabular form. Aim was to assess participants' interpretation of rate of change on given table. This provides insights into if participants recognized that a constant increment in variables indicates same rate of change at any point. Hence, task 7 provides insights into participants' knowledge of rate of change in tabular form within a given context.

verify and confirm any confusing or vague explanations, promoting data accuracy and eliminating biases. It also ensures the internal validity of the findings and adds to the confirmability of the results.

Another strategy employed was peer review, where the data from this study was presented to an expert in the field. The purpose was to assist the researcher in locating any errors or vague explanations that could cause bias. The expert provided valuable recommendations on how to prevent such issues using suitable probing questions, thereby enhancing the internal validity and consistency of the data.

Instruments of the Study

The tasks are attached in **Appendix**. Supplementary questions were included based on the participants' answers and responses to gather clear information on their SMK in rate of change. The content validity of the instruments was established by a panel of experts, which consisted of three university lecturers specializing in mathematics education. Each expert rated the tasks on a scale of three to five points. Tasks that received a score of three were rephrased with improved clarity based on the experts' feedback. As a result, some tasks were modified according to the experts' comments. Some instruments were adapted from past research (e.g., Byerley & Thompson, 2017; Orton, 1983) and modified accordingly to fit the objective of this study. The tasks were designed based on the theoretical perspectives of this study and aimed to answer the research question. Different kinds of representation were used including real-life contexts to assess participants' ability to apply their knowledge of rate of change in practical situations. Past qualitative studies (e.g., Bezuidenhout, 1998; Byerley & Thompson, 2017; Desfitri, 2016; Kertil, 2014; Orton, 1983; Teuscher & Reys, 2012) also encountered similar issues in learning and teaching rate of change, which were addressed by developing tasks using the same approach. **Table 1** presents a description of each task to aid in understanding this study.

Data Analysis

The data were analyzed using the content analysis technique, specifically directed content analysis. This technique involves analyzing the contents of interviews, field notes, and documents to gain insights into the meanings or nuances that contribute to understanding the discussed issue (Merriam, 2009). The process involves simultaneous coding of raw data and the construction of categories to identify meaningful data.

Participant Selection

The pre-service mathematics teachers were selected through purposive sampling, targeting participants who

C: in general, an element or a variable, for me maybe volume or what influenced by small elements, it will be placed above while elements which affecting will be placed below.
R: ok, anything about rate, about rate that you know, an example?
C: oh have learnt, many, like volume influenced by radius, dv per dr, something like that, after that, it needs to do differentiation then will get its change, that what I remember.
…
C: if rate of change, it relates with time.
R: relate with time?
C: yes, and its more specific, it seems like example that I took, dv per dr and its not just relation v with r that influenced, but it is likes v divided by r but it seems relate with time, for me.
…
R: means that rate of change has time, must with time?
C: yes, for me.
…
R: ok, anything about rate of change that you want to add?
C: one more, the concept that I remember, like dv over dr, it can be dv over dt multiply with dt over dr, that is what I remember.
R: is it like a cross multiplication?
C: yes, yes, that is what I remember the most.

Figure 1. Excerpt 1 (Source: Authors' own work)

could provide the most valuable insights into the central issue at hand. The selection criteria comprised the following:

(1) pre-service teachers majoring in mathematics,

(2) pre-service teachers who had completed all courses related to calculus content,

(3) pre-service teachers who agreed to participate in all interview sessions, as required, and

(4) pre-service teachers who consented to being video and audio recorded for the study.

Participants were selected to represent a diverse range of mathematics majors and performance levels in calculus-related courses, from average to high. This aimed to gather rich and informative data from individuals who could contribute the most and had significant insights into the issue. Therefore, their cumulative grade point average (CGPA) and grades in calculus-related courses were considered. Their grades in mathematics and additional mathematics from the Malaysian certificate of education (SPM), a national examination, were also considered. This was important as the study focused on the mathematical knowledge acquired within the context of Malaysian secondary mathematics education. These background details are presented to illustrate the participants' academic achievements, as the study's main focus was on the mathematical knowledge of calculus they had acquired.

Two pre-service mathematics teachers, pseudonymously named Zheng and Amitha, were selected for the study. Zheng, who majored in mathematics with a minor in Chinese language, achieved an A- in calculus and an A in pre-calculus courses. He attained a CGPA of 3.86. He scored an A+ in both mathematics and additional mathematics in SPM. While Amitha, who majored in mathematics with a minor in English language, achieved a B in calculus and an A- in pre-calculus courses. She attained a CGPA of 3.48. She scored A+ in mathematics and A in additional mathematics in SPM.

RESULTS

Case Study: Zheng

In task 1 (see **Appendix**), Zheng defined rate of change as an element or variable influenced by another small element in quotient form (excerpt 1, lines 1-3) (**Figure 1**). He gave an example of a volume influenced by radius and denoted it as dv per dr (excerpt 1, line 5). He explained that differentiation is required to obtain the change (excerpt 1, line 6). He specified that rate of change is dv per dr and it is not merely just v (volume) with r (radius), but also related to time (excerpt 1, line 11-13). When asked if rate of change must include time, he confirmed that it must have time based on his understanding (excerpt 1, lines 15-16). He also mentioned a concept he had learned, where dv per dt was multiplied with dt over dr (excerpt 1, line 19-20).

Rate of change is also characterized by multiplicative comparison between two quantities. Therefore, task 2 (see **Appendix**) was designed to explore pre-service teachers' understanding of multiplicative comparison. For task 2, Zheng initially selected d for the answer, indicating that Sofia has more $k - l$ oranges than Daniel, before changing his answer to c, stating that Sofia has more $\frac{k}{l}$ oranges than Daniel (excerpt 2, line 1) (**Figure 2**).

C: I think answer maybe c.
…
C: answer maybe c, why hmmm… if I use elimination method, is it can?
R: how?
C: ok, because I compare all the four then I cross out answer that less rational.
R: ok, the first one?
C: first one I cross b, k multiply l, k multiply l must be a lot so, I think its illogical for k multiply l. So, for a, Sofia has wait… Sofia has k right then k multiply l then divide with k, for me the ratio seems little bit weird because its not more than but its equal with, so I think the words not suitable or less accurate, so a with b I cross. For d, just now I think I mistaken because I thought it directly minus but after I think, for ratio I cannot directly minus, because we do not know their multiple, ok, so for me d I cross, so I choose c.
R: … in general, what do you understand from ratio, about ratio?
…
C: in general, how much or quantity for something, ok, at above, then, per quantity at bottom, meaning that if bottom has this much, let say bottom has 5, upper has 7, so this is their relationship, for me.
R: … can you give example?
C: let says I have 5 apples, 7 oranges, 7 per 5, so 7 oranges 5 apples, ok, in this situation I can know their relationship is like this, so, I can do like when there are 10 apples, there will be 14 oranges, this is ratio for me.

Figure 2. Excerpt 2 (Source: Authors' own work)

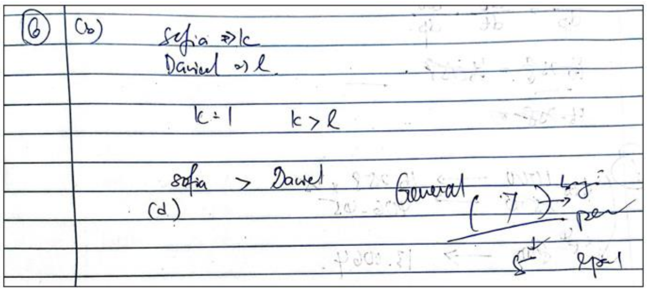

Figure 3. Example (Source: Authors' own work, from Zheng's written steps)

Zheng used the elimination method to narrow down the options (excerpt 2, line 3). He crossed out option b, which is Sofia has k.l as many oranges as Daniel, considering it illogical (excerpt 2, lines 7-8). He crossed out option a, as he considered the ratio to be uncanny since it resulted in the same amount for both Sofia and Daniel (excerpt 2, lines 8-9). Initially, he thought option d was correct but later realized that ratio does not involve subtraction, leading him to choose option *c* (excerpt 2, lines 10-12).

Zheng defined ratio as "how much quantity for something at above per with quantity at the bottom" and gave an example, seven over five, shown in **Figure 3**, representing their relationship (excerpt 2, lines 15-17). When prompted to clarify his example, he used five apples and seven oranges to illustrate their relationship: if there were 10 apples, then there would be 14 oranges (excerpt 2, lines 19-21).

In task 3 (see **Appendix**), Zheng expressed that he did not know how to answer the question (excerpt 3, line 1) (**Figure 4**).

When asked to explain his understanding of the given statement, "the drawing measurement of a car model is m over n millimeter of a real car dimension", Zheng replied that he understood the concept of

C: ok… this one… do not know how to do.
R: so, can you find it?
C: I cannot.
R: so, from the statement, 'the drawing measurement of car model is m over n millimetre of a real car dimension', what do you understand from that sentence?
C: ok, the drawing measurement, it will be per with real car dimension, for me, this is what I understand but I do not know their relation.
…
C: I understand, understand but I do not know how to do.
…
C: yes, I do not know their relation, relation I referring to is how to calculate… I definitely understand what it was trying to say but it seems lacking something, maybe, do not know what to differentiate, or what function to write, so I definitely do not know how to do it.

Figure 4. Excerpt 3 (Source: Authors' own work)

C: do not know, not really know.
R: not really know, what else you know from word slope?
C: change.
R: change, what change?
C: what change, hmm… do not know.
R: … what else do you know?
C: hmm… I just know this, only this.

Figure 5. Excerpt 4 (Source: Authors' own work)

R: … describe the meaning of your answer in a based on the given context…?
C: ok, can, wait, I look over the question, hmmm… that 5.8 represent dollar, for me, for each x gallon cooking oil if it is added.

Figure 6. Excerpt 5 (Source: Authors' own work)

C: -0.667
R: how do you calculate?
C: take three right, three I took it, 2 divides with 3 second
R: so, what do you know from term instantaneous velocity?
C: hmmm… velocity at that time, maybe.
…
R: … what is the instantaneous speed at $t = 3$?
C: hmm… I think it same, maybe.

Figure 7. Excerpt 6 (Source: Authors' own work)

drawing measurement per real car dimension but was unable to figure out their relation (excerpt 3, lines 6-7). Researchers tried to help him by translating the task into Malay, but he said he understood the question but did not know how to solve it (excerpt 3, line 9). He added that he was not sure how to calculate the relationship, if through differentiation or as a function, so it seemed like something was missing for him (excerpt 3, lines 11-13).

Zheng's knowledge of ratio as a relationship between two quantities, where one is "how much quantity for something above" and the other is "quantity at the bottom". In task 2, he correctly identified the need for multiplicative comparison based on his chosen option and meaning of ratio. He also demonstrated by giving an example, where he recognized the role of multiplication by multiplying each term of the ratio 7:5 by two, resulting in 14:10. Although Zheng mentioned the ratio 7:5 as a relationship, he did not refer to it as a multiplicative comparison between quantities, showing how one quantity is a multiple of the other. This lack of interpreting of multiplicative comparison was also evident in task 3, where he struggled to interpret the relationship between the measurement of a car model and the real car dimension.

In task 4 (see **Appendix**), Zheng initially stated that he did not know the answer (excerpt 4, line 1) (**Figure 5**). When asked what he knew from the word 'slope,' he mentioned 'change' (excerpt 4, line 3). After further probing Zheng for clarification, he reiterated that he did not know the answer (excerpt 4, line 5). Researchers asked once more to confirm his knowledge, inquiring if he knew anything else about the slope besides what was given in the statement, and his answer remained the same, he only knew the slope as it was presented (excerpt 4, lines 6-7).

In task 5 (see **Appendix**), Zheng gave an incorrect description of the constant rate of change for the function. He described the rate of change for function $f(x)$ as a production cost of 5.8 dollar for each increment of x gallon cooking oil (excerpt 5, lines 2-3) (**Figure 6**). His description lacked the change in output, as he only mentioned the increment of input x.

In task 6 (see **Appendix**), Zheng gave an incorrect answer, despite understanding that the term 'instantaneous' refers to finding velocity and speed at that exact moment in time. He calculated it directly from the graph, where the horizontal movement is two meters and the time is three seconds, resulting in a value of 0.667 meters per second (excerpt 6, line 1 & line 8) (**Figure 7**). This indicates his inability to recognize that the velocity and speed at a specific time, $t = 3$, are obtained by finding slope of graph, as they remained constant from $t = 2$ until $t = 4$. This also reflects his misunderstanding of the term 'instantaneous', although he interpreted it correctly as the velocity and speed at that exact moment. As a result, he calculated the distance at $t = 3$ without understanding correct formula for velocity and speed.

In task 7 (see **Appendix**), Zheng initially provided water depth difference, 2.5 meters as answer for rate of increase in water depth, as shown in **Figure 8** (excerpt 7, lines 1-3). However, he later realized that the values in the table can be represented by a function $d = 5t$, where d represents the depth of water, and the rate of increase of water depth can be obtained through the derivatives

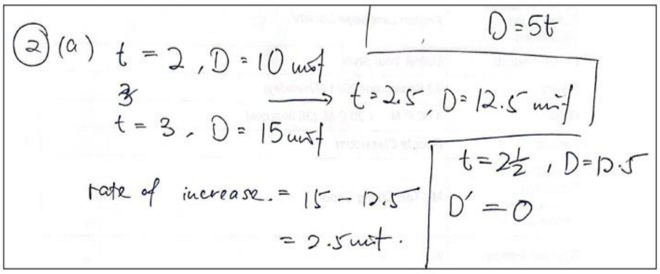

Figure 8. Answer-1 (Source: Authors' own work, from Zheng's written steps)

of the function (excerpt 7, line 12). Consequently, he answered five for the rate of increase of water depth at both $t = 2\frac{1}{2}$ and $t = 5.34$ (excerpt 7, line 13-14) (**Figure 9**). His calculation shows that he recognized that the rate of increase in water depth remains constant at five, as determined by the function he created.

To summarize, Zheng first expressed meaning of rate of change as a relationship between two variables, where one variable influence another. He emphasized the significance of time as an independent variable that must be associated with rate of change. This association may be attributed to the multiplication cross method he mentioned, which requires time as another associated variable. His initial description of rate of change did not include changes in variables, even though he presented a symbolic representation of dv per dr. This limited meaning might explain why Zheng was unable to conceive the slope of a straight line as a rate of change, viewing it solely as an index of steepness. The formula for the slope of a straight line considers changes in both variables, the independent and dependent variables, unlike his description of rate of change. Zheng was unable to conceive the multiplicative property of rate of change as a ratio attribute. He conceived ratio as a comparison between quantities rather than recognizing it as a multiplicative comparison. In a task involving a linear equation, he only considered the increment in the independent variable when describing its rate of change. He described rate of change between the production cost and the amount of cooking oil by only considering the change in the amount of cooking oil without considering the change in the production cost. Nonetheless, Zheng did demonstrate the ability to recognize a constant rate of change when presented with numerical values in the tabular task. Zheng initially conceived the rate of increase in water depth as a simple increment of water depth. However, he later changed his approach and devised a function to represent the values. He then computed the rate of increase using derivatives and asserted that the rate of change remained constant at any time. In the graphical task, Zheng used an incorrect formula to find instantaneous velocity and instantaneous speed. His misunderstanding of "instantaneous" led him to believe it referred to an instant occurrence at a specific time. Consequently, he calculated the values by only considering the distance and time at that exact moment.

Case Study: Amitha

In task 1, Amitha defined rate of change as a change in something over a specific period of time, expressed as "per time" (excerpt 8, lines 1-5) (**Figure 10**). When asked for an example, she mentioned acceleration as rate of change of speed (excerpt 8, line 7). The researchers asked if rate of change should be related to time, and Amitha confirmed, explaining that she understands rate as something divided by time (excerpt 8, line 9).

In task 2, Amitha demonstrated additive comparison by choosing d as the answer, which stated that Sofia has $k - l$ more oranges than Daniel. Amitha understood the difference between them can be represented by subtracting the number of oranges that Daniel had from the number Sofia had (excerpt 9, lines 3-4) (**Figure 11**). When asked about the meaning of ratio, she replied that ratio is a comparison of numbers between two things

1 C: rate of increase… 2.5.
2 …
3 C: I think, maybe, 2.5, I take like 15 minus with 12.5 the one I calculated and get 2.5.
4 R: when t equal with 5.34 hour? Can you find rate of increase in water depth?
5 C: wait, sorry, I want to tell for (a) I change wait, change the answer, that rate of increase
6 I get 0, equal to zero.
7 R: rate of increase for t …?
8 C: t, two and half…
9 R: zero?
10 C: zero
11 R: how do you get? How do you calculate?
12 C: … I find a function, function $d = 5t$, then I take t to differentiate, getting zero.
13 R: if t equal 5.34?
14 C: same function, I get $d = 5t$, still use the same one.
15 …
16 C: … that supposed to be five not zero.

Figure 9. Excerpt 7 (Source: Authors' own work)

A: rate of change, it is like per time, per specific time, what is the change.
…
A: change in something, depend on change what changing in a value.
R: over time?
A: during… particular period of time.
R: so, can you give example of rate of change that you may know?
A: for example, acceleration is rate of change of speed, something like that.
R: so, it must be something that associated with time? Must be?
A: yes, because rate means we dividing something with time.

Figure 10. Excerpt 8 (Source: Authors' own work)

A: it would be d, donkey.
R: … why?
A: because k minus l you will get the different between both of them, so, you know, for example, the different is 5, so Sofia has more 5 oranges than Daniel had.
…
R: … what is ratio?
A: ratio is like a comparison of the numbers between two things, for example, the content of water, so we comparing two glasses of water and two glasses have different level of water, then we can tell that half of the glass is equal to, maybe, we are comparing.

Figure 11. Excerpt 9 (Source: Authors' own work)

A: I think it will be 150 m over n.
R: … can you explain your calculation?
A: because m over n is like… what enlargement or something like that, the model's height is 150, means it times by m over n millimetre to get the real car.
R: …what do you understand from the statement?
A: the drawing of the car model is m over n, ratio millimetre of a real car.

Figure 12. Excerpt 10 (Source: Authors' own work)

A: 4.75 is a gradient.
R: can you explain?
A: ok, 9.5 is actually difference in y and then divided by 2, number 2 is difference in x, so, differences in both you divided it, so, y divided by x, you will get 4.75 which is the gradient.
R: ok, is there anything else besides gradient, what else 4.75 means?
A: As I know as the gradient only.

Figure 13. Excerpt 11 (Source: Authors' own work)

and gave an example by comparing two glasses of water with different water levels (excerpt 9, lines 7-9).

In task 3, Amitha answered $\frac{150\,m}{n}$ because she believed that $\frac{m}{n}$ represented an enlargement (excerpt 10, lines 1-3) (**Figure 12**). According to her understanding, the model's height, 150, needed to be multiplied by $\frac{m}{n}$ to determine the height of the real car (excerpt 10, lines 3-4). When asked about her interpretation of the given statement, she replied that the drawing of the car model is $\frac{m}{n}$ of the real car (excerpt 10, line 6).

Amitha's knowledge of ratio is based on additive comparison. This is evident in her answer choice d in task 2, where she explained the difference in the number of oranges Sofia and Daniel had, concluding that Sofia had five more oranges than Daniel. Her general meaning of ratio also involved comparing two things, as seen in her example of comparing the difference in water level between two glasses of water. Similarly, in task 3, Amitha misinterpreted the statement, leading to an incorrect answer. She seemed to understand the ratio $\frac{m}{n}$ as an enlargement, resulting in the multiplication of 150 millimeters by $\frac{m}{n}$ to obtain the actual car's height. This indicates a lack of knowledge in understanding how one quantity is related to another as a multiple of another quantity and expressing them in ratio form.

In task 4, like Zheng, Amitha stated the gradient as 4.75. When asked to explain, she mentioned that 9.5 is the difference in y and two is the difference in x, and by dividing these values, she obtained 4.75 as the gradient (excerpt 11, lines 3-5) (**Figure 13**). When asked if she knew anything else about 4.75 besides being a gradient, she replied that she only knew it as a gradient (excerpt 11, line 7).

R: … describe the meaning of your answer in a based on the given context?
A: ok, that means… maybe one gallon of cooking oil, the production cost will be 5.8 dollar.

Figure 14. Excerpt 12 (Source: Authors' own work)

A: because formula velocity, meter per second, so, the point is two, sorry, so two meters divided by … wait, is 2 ms^{-1} because the total distance from zero to three is 6 and the time is three, so 6 divided by 3.
R: what do you understand from instantaneous velocity?
A: what is the velocity at the point three, third second.
R: … what is the instantaneous speed at $t = 3$?
A: also same because it does not affect time, I mean direction.
…
R: what is difference between instantaneous velocity with instantaneous speed that you know?
A: direction, the only difference.

Figure 15. Excerpt 13 (Source: Authors' own work)

R: …what is the rate of increase in water depth when t equal two half?
…
A: wait, the rate of increase is five unit per hour.
R: 5 unit per hour, how do you calculate?
A: ok, first I find out what is the depth of the water during 2.5 and then I find the difference in depth of water divide with depth, sorry, depth of water divides with time, difference in time, so I get 5 unit per hour.
R: oh, so what do you understand from term rate of increase in the water depth?
A: how much the water increases in the particular time.
…
R: $t = 5.34$?
…
A: same, 5 unit per hour.
R: … the reason?
A: because we just change time only, the rate will be same when you calculate.
R: so, do you mean that the rate will be constant?
A: I calculate and also observe, it is only same, ok, then, the change is only time, so, the rate of depth of water over time does not change, will be constant.
R: … means, you say at any time, the rate is same?
A: yes, because just now we already calculate for rate of increase in water depth, so, each hour, water is going to increase at that particular amount only.

Figure 16. Excerpt 14 (Source: Authors' own work)

In task 5, Amitha stated the corresponding rate of change, 5.8, as a production cost for one gallon of cooking oil without mentioning any changes for both variables (excerpt 12, lines 2-3) (**Figure 14**).

In task 6, Amitha calculated the instantaneous velocity using the formula for a total distance from time zero until three seconds, then divided it by the total time, resulting in two ms^{-1} (excerpt 13, lines 1-3) (**Figure 15**).

When asked about instantaneous velocity, she referred to it as the velocity at the third second. For instantaneous speed, she believed it was the same as instantaneous velocity because it was not affected by direction (excerpt 13, line 7). When asked about the difference between instantaneous velocity and instantaneous speed, she replied that the only difference is the direction (excerpt 13, line 11). When asked about instantaneous speed, she said it was the speed at that exact time, $t = 3$. This suggests that Amitha had difficulty recognizing that the velocity and speed at the exact time, $t = 3$, could be obtained by finding the slope of the graph since the velocity and speed remained constant from $t = 2$ until $t = 4$.

In task 7, Amitha answered that the rate of increase in water depth at $t = 5.34$ is 5 units per hour (excerpt 14, line 13) (**Figure 16**).

She calculated the water depth at $t = 5.34$ as 26.7 units and then calculated difference in water depth over time intervals shown in **Figure 17**. She stated that it is the same as the rate of increase in water depth at $t = 2\frac{1}{2}$ (excerpt 14, line 15). According to her, the rate of water depth remains constant at any time since it increases the

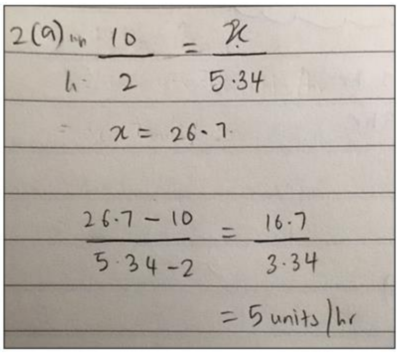

Figure 17. Answer-2 (Source: Authors' own work, from Amitha's written steps)

same amount every hour at five units per hour (excerpt 14, lines 17-21).

To summarize, Amitha defined rate of change as a change in the variable over time, subjecting the independent variable to a time period. The formula for the slope of a straight line given in task 4 did not include time as a variable. This might have led Amitha to view it solely as a gradient or index of steepness. However, she was able to correctly compute rate of change when time was required as an independent variable in the tabular task. While she correctly computed rate of change in tabular tasks when the question is presented numerically and noticed constant rates, she struggled with graphical tasks. She used an incorrect formula, which involved the total distance travelled, resulting in incorrect answers, even though she understood the meaning of 'instantaneous' to reflect the exact time. She also had another notion of rate of change as a variable over another variable, which was shown in a non-temporal task involving a linear equation. Amitha treated rate of change between the production cost and the amount of cooking oil as a functional relationship, overlooking changes in both variables. Her weak rate of change conceptions demonstrated in the tasks may result in her inability to grasp the multiplicative property of rate of change because her interpretations of ratio is as an additive comparison rather than a multiplicative comparison.

DISCUSSION

Zheng initially expressed rate of change as a relationship between two variables, with one influencing the other, and recognized that the importance of associating time with rate of change. This is possibly due to the multiplication cross method he mentioned that involves time as an associated variable. However, he did not consider changes in variables, though he used the symbol dv per dr as an example. Similarly, Amitha emphasized the role of time and conceived rate of change as a variable changing over time, subjecting the independent variable to time. This tendency to involve time might be due to their familiarity with time-based contexts, such as motion..

This pattern is consistent with the findings of a study by Jones (2017) and is frequently observed among calculus students, who tend to consider time as a variable even when a task does not explicitly require it.

Another common mistake is considering rate of change as a variable without any change (amount of quantity), made among students at both the secondary and tertiary levels (Mkhatshwa, 2019; Mkhatshwa & Doerr, 2018). This happens especially when students were asked to reason in application contexts (Mkhatshwa, 2019; Mkhatshwa & Doerr, 2018). Zheng did not consider changes in both variables in his meaning of rate of change as in task 1 and it was a similar situation in the application task involving equation form. He described the rate of change between the production cost and the amount of cooking oil by only considering the change in input, without accounting for changes in the output. While Amitha showed the misconception by treating it as a variable over another variable, without considering any changes in both variables, as if it was a regular functional relationship. These misinterpretations portray the inconsistency of both participants regarding the occurrence of variables or quantities involved in rate of change within the situated context. This problem is similar to past studies (e.g., Byerley & Thompson, 2017; Kertil & Dede, 2022; Mkhatshwa & Doerr, 2018), where teachers and students often confused "amount" with "amount of change". In one study, a student named Brian interpreted rate of change as an amount of quantity rather than the changes between quantities. His misconception was also illustrated by his example of speed as a speedometer point instead of understanding the relationship between distance travelled and hours elapsed (Person et al., 2004 as cited in Byerley & Thompson, 2017). The persisting presence of this issue in studies indicates that the problem still exists and remains unresolved. Therefore, it emphasizes the importance of highlighting the difference between the amount, the amount of change and rate of change, as suggested by Verzosa (2015), who illustrated this difference using an example like forest area and the rate of deforestation.

In the graphical-based task, Zheng used an incorrect formula for velocity and speed. He appeared to focus solely on the distance at the calculated time, disregarding the amount of distance travelled. This might be due to his interpretation of the term 'instantaneous', which led him to only consider the quantity value at a specific time. This suggests that he may not fully grasp how two variables change simultaneously in a graphical context. Amitha also struggled to grasp the constant relationship between changes in quantities on the linear motion graph at the required point. She considered the total distance travelled from the starting point, leading to an incorrect answer. These may be due to their familiarity with motion contexts, which influenced the ability to provide an accurate formula for velocity and speed. Nagle et al. (2019) observed that students may memorize slope interpretations, such as velocity being represented by a graph of distance versus time, without fully understanding the underlying dynamic relationship. This indicates that they tend to understand velocity and speed algebraically as distance over time, without fully grasping the visual information. This finding is consistent with past studies (e.g., Amit & Vinner, 1990; Brijlall & Ndlovu, 2013; Nasir et al., 2013; Tall, 1993), which found that students often show more interest in the algebraic form as it involves working with formulas numerically, while visual representations are perceived as less mathematical. This suggests that the participating pre-service teachers perceived a graph based on its shape rather than the covarying quantities. They mistakenly assumed that both instantaneous speed and instantaneous velocity were positive values simply by observing the shape of the graph situated on positive axes without further interpreting the relationship between the two quantities graphically. A similar error was also observed in a study by Byerley and Thompson (2017) with secondary mathematics teachers who focused on the graph's shape rather than the covarying quantities. However, the core reason behind this observation is likely a misunderstanding of the term 'instantaneous' or a lack of understanding of the concept of rate of change, specifically, in the context of graphical representation.

Both pre-service mathematics teachers unable to conceive the constant rate of change in the equation and graphical contexts, but they able to do so when the values of quantities were provided numerically, as seen in the tabular-based task involving water increment and time interval. In this task, Zheng created a function to represent the values and then computed the derivatives of the function to obtain rate of change. He knew that the rate of change remained constant at any given time, thanks to the function he created. On the other hand, Amitha was able to notice the constant rate of change by calculating the quotient of difference between quantities and concluded that the rate of change remained constant throughout time. This demonstrates how the numerical form made it easier for them to conceive the constant rate of change through algebraic computation and using the derivatives as a rate of change operator. Nagle et al. (2019) mentioned how students can immediately recognize a fixed numerical difference value in the input and its relation to the numerical difference in the output, allowing them to justify using symbolic and geometric arguments. Conversely, students can also notice when the numerical quantities do not align with the relationship between them, indicating a non-linear property (Nagle et al., 2019).

Amitha can compute and understand rate of change when time was involved, like in the tabular task. However, she struggled with the concept of rate of change in task that did not explicitly involve time, such as the equational task. Her focus on time as a key variable might explain why she could not see slope as a rate of change and only saw it as an index of steepness. The same reason may also apply to Zheng, as he emphasized the need for time as another variable for rate of change. Consequently, he struggled to see the slope of a line as a rate of change and only perceived it as an index of steepness. His interpretation and meaning of rate of change were lacking in various ways, which were evident in tasks with different representations. It is common for students and teachers across different educational levels to view the slope as an angle for measuring the steepness of a line (Byerley & Thompson, 2017; Nagle & Moore-Russo, 2013; Stump, 1999, 2001). Numerous studies (e.g., Ball, 1990; Nagle et al., 2019; Nagle & Moore-Russo, 2013; Stump, 1999, 2001) including large-scale research (e.g., Byerley & Thompson, 2017), have focused on characterizing how students and teachers conceive the meaning of slope and its connection to their notion of rate of change. Numerous studies have revealed a consistent pattern, where the slope and rate of change do not align with their measurement schemes for related quantities. This leads to their inability to interpret and grasp the coherent connection between these two concepts (Byerley & Thompson, 2017). This could be attributed to the way slope is taught, where students are often encouraged to plug numbers into formula without sufficient emphasis on interpretation within the context (Teuscher & Reys, 2012). This type of learning, known as rote learning, involves students memorizing the formula for slope as the change in y over the change in x without understanding its mathematical significance, using it only for computing the angle measurement of a slanted line (Nagle & Moore-Russo, 2013). Consequently, the division bar used in the slope formula, representing a multiplicative comparison, is often overlooked and not considered (Byerley & Thompson, 2017). Byerley and Thompson (2017) hypothesized that teachers with a chunky image of slope or as an index of steepness were less likely to conceive slope and rate of change in a multiplicative way due to their weak meaning of quotient. This study also found that both pre-service teachers struggled to recognize the multiplicative property of rate of change. These findings were consistent with their responses in other tasks, where their notion of slope as a rate of change was absent and, meaning of rate of change that was misinterpreted. This was further supported by their interpretation of ratio, where they viewed it either as a mere comparison between quantities or as an additive comparison, but not as a multiplicative comparison.

In summary, the findings have revealed the pre-service teachers' lack of knowledge on rate of change, including the constant rate of change. The inconsistencies in their notion of rate of change across different tasks highlight their weakness to conceive the concept. This raises concerns about the foundation of knowledge these pre-service teachers are bringing into their classrooms. If their rate of change knowledge is lacking, it may affect their ability to develop understanding of limits and derivatives (Meylani & Teuscher, 2011; Nagle & Moore-Russo, 2013; Teuscher & Reys, 2012). The concept of derivatives as an instantaneous rate of change is intricately linked to the idea of a rate of change, which stems from the slope of a straight line. Nagle and Moore-Russo (2013) stated that conceptualizing slope as a functional property or specifically as a constant rate of change is a learning goal in eighth-grade instruction. Therefore, teachers should be equipped with knowledge on rate of change, including slope as constant rate of change before projecting their understanding onto concept of derivatives. Understanding derivatives would be challenging without a solid grasp of rate of change, especially the constant rate of change (Teuscher & Reys, 2012). The study's findings also showed the participants' limited knowledge of rate of change. They recognized that the derivatives as a rate of change operator but struggled to describe rate of change as a relationship between two changing quantities, including their constant change in various representations. Students, especially teachers must not only be able to compute slope and derivatives but also grasp their meanings within specific contexts to understand how rate of change is presented and used. This knowledge is essential for teachers in creating effective learning instructions and conveying the concept of rate of change accurately. The understanding of rate of change, with its significant impact on students' long-term learning, because it is a crucial concept in higher-level mathematics (Teuscher & Reys, 2012) due to its application in biology, chemistry, engineering, physics, and more (Kertil, 2021; Mkhatshwa & Doerr, 2018).

CONCLUSIONS & RECOMMENDATIONS

The knowledge of rate of change exhibited by the participating pre-service teachers, Zheng and Amitha, was lacking in several ways, impacting their ability to grasp the concept of a constant rate of change. Their inconsistent notion of rate of change, evident in most tasks, indicates a loose connection between the concept of rate of change and slope conception, including its multiplicative property. While they were able to compute and understand rate of change when presented in tabular form with repetitive values, they struggled with other different representations. Another task demonstrates their limited knowledge of derivatives as an operator to obtain rate of change and their struggle to describe the changes that occurred between the quantities. Both pre-service teachers also struggled to fully comprehend the common applications of velocity and speed. Therefore, this study indicates lack of comprehensive knowledge of participating pre-service mathematics teachers on rate of change. As this study focused on two case studies of pre-service mathematics teachers, the researchers refrained from making broad generalizations about all Malaysian pre-service mathematics teachers. This study offers valuable insights into the difficulties and weaknesses in the knowledge of these future teachers, which may assist in informing other Malaysian educators and researchers about the enhancement of calculus knowledge, specifically focusing on the concept of rate of change.

The results of this study indicate the need for teacher training programs to incorporate specific approaches or modules aimed at developing the knowledge of future teachers in the concept of rate of change. We recommend thoroughly scrutinizing modules related to the topic of rate of change, including its pre-requisite topics such as slope, ratio, and rate. This approach would greatly aid in developing student teachers' SMK on rate of change. For example, when revisiting or introducing the concept of slope, we recommend describing it as a representation of the constant rate of change between two quantities. However, to address the multiplicative comparison between the quantities, it is essential to introduce the concepts of ratio and rate early on. Hence, an early introduction to the importance of multiplicative comparison within rate of change concept should be taught to future teachers. Different representations, including graphical, tabular, and equation forms, should be used to enhance a robust understanding of rate of change. University instructors are strongly encouraged to foster students' understanding of slope as a rate of change through multiple representations, as slope is commonly associated with algebraic computation. These representations encompass verbal, symbolic, and visual formats, enabling a deeper understanding. Integrating information and communication technology can further enhance the learning environment by providing 9students with more efficient and effective ways to visualize graphs and images.

Incorporating diverse real-life applications alongside rigorous motion contexts helps students conceive how the idea of rate of change is used differently in various contexts and functions in the real world. Engaging in real-life applications can enhance students' understanding of rate of change, fostering a broader perspective beyond just algebraic manipulation. An approachable module, particularly for pre-service teachers, can aid in conceiving rate of change before delving into the concept of derivatives as an instantaneous rate of change. This is consistent with the idea proposed by Thompson (2008) in his paper on significant learning and coherence learning. He emphasized the term 'significant' to refer to ideas that are carried through the instructional sequence, serving as foundational concepts for learning other subsequent ideas.

The study aimed to investigate pre-service mathematics teachers' SMK. Future research should extend to in-service teachers and students at the secondary and tertiary levels. Exploring whether in-service teachers and students exhibit a similar pattern of knowledge about rate of change, particularly in slope conception, would be worthwhile. Future research could further investigate other aspects of rate of change, such as derivative, limit, or covariational reasoning since this study has inquired on slope concept. These scrutinization will aid educators in identifying problematic aspects of rate of change, allowing for the implementation of better approaches to prevent inaccurate and insufficient knowledge delivery. By gaining a comprehensive understanding of the existing problems in calculus, particularly rate of change, early prevention measures and improvements to teaching and learning can be addressed and implemented effectively.

Author contributions: All authors have sufficiently contributed to the study and agreed with the results and conclusions.

Funding: This study was supported by the Ministry of Higher Education Malaysia under *Skim Latihan Akademik Bumiputera* (SLAB) [*Bumiputera Academic Training Scheme*] sponsorship.

Acknowledgements: The authors would like to thank Universiti Utara Malaysia who granted SLAB sponsorship, therefore, allowing this study to be accomplished.

Ethical statement: The authors stated that the study was approved by the Universiti Malaya Research Ethics Committee (non-medical) on 9 February 2023 with approval code UM.TNC2/UMREC_2207. Written informed consents were obtained from the participants.

Declaration of interest: No conflict of interest is declared by authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Amit, M., & Vinner, S. (1990). Some misconceptions in calculus–Anecdotes or the tip of an iceberg? In *Proceedings of the Annual Conference of the International Group for the Psychology of Mathematics Education* (pp. 3-10).

Ayebo, A., Ukkelberg, S., & Assuah, C. (2017). Success in introductory calculus: The role of high school and pre-calculus preparation. *International Journal of Research in Education and Science, 3*(1), 11-19. https://doi.org/10.21890/ijres.267359

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal, 90*(4), 449-466. https://doi.org/10.1086/461626

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407. https://doi.org/10.1177/0022487108324554

Bezuidenhout, J. (1998). First-year university students' understanding of rate of change. *International Journal of Mathematical Education in Science and Technology, 29*(3), 389-399. https://doi.org/10.1080/0020739980290309

Brijlall, D., & Ndlovu, Z. (2013). High school learners' mental construction during solving optimization problems in calculus: A South African case study. *South African Journal of Education, 33*(2), 2-18. https://doi.org/10.15700/saje.v33n2a679

Byerley, C. (2019). Calculus students' fraction and measure schemes and implications for teaching rate of change functions conceptually. *Journal of Mathematical Behavior, 55*, 100694. https://doi.org/10.1016/j.jmathb.2019.03.001

Byerley, C., & Thompson, P. W. (2014). Secondary teachers' relative size schemes. In *Proceedings of the 38th Meeting of the International Group for the Psychology of Mathematics Education* (pp. 217-224).

Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *Journal of Mathematical Behavior, 48*, 168-193. https://doi.org/10.1016/j.jmathb.2017.09.003

Carlson, M. P., Madison, B., & West, R. D. (2015). A study of students' readiness to learn calculus. *International Journal of Research in Undergraduate Mathematics Education, 1*(2), 209-233. https://doi.org/10.1007/s40753-015-0013-y

Cetin, N. (2009). The ability of students to comprehend the function-derivative relationship with regard to problems from their real life. *Primus, 19*(3), 232-244. https://doi.org/10.1080/10511970701686987

Creswell, J. W. (2013). *Qualitative inquiry and research design: Choosing among five approaches*. SAGE.

Dane, A., Cetin, O. F., Bas, F., & Ozturan Sagirli, M. (2016). A conceptual and procedural research on the hierarchical structure of mathematics emerging in the minds of university students: An example of limit-continuity-integral-derivative. *International Journal of Higher Education, 5*(2), 82-91. https://doi.org/10.5430/ijhe.v5n2p82

Desfitri, R. (2016). In-service teachers' understanding on the concept of limits and derivatives and the way they deliver the concepts to their high school students. *Journal of Physics: Conference Series, 693*, 012016. https://doi.org/10.1088/1742-6596/693/1/012016

Estonanto, A. J. J., & Dio, R. V. (2019). Factors causing mathematics anxiety of senior high school students in calculus. *Asian Journal of Education and E-Learning, 7*(1), 37-47. https://doi.org/10.24203/ajeel.v7i1.5701

Fothergill, L. (2011). Aspects of calculus for pre-service teachers. *Mathematics Educator, 21*(1), 23-31.

Fuadah, U. S., Sukma, Y., & Yanda, F. (2022). Book review: A review of the book by Liping Ma (2020), Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States. *Educational Studies in Mathematics, 110*(2), 379-391. https://doi.org/10.1007/s10649-021-10116-2

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education*. Lawrence Erlbaum Associates.

Gudmundsdottir, S., & Shulman, L. (1987). Pedagogical content knowledge in social studies. *Scandinavian Journal of Educational Research, 31*(2), 59-70. https://doi.org/10.1080/0031383870310201

Hashemi, N., Abu, M. S., Kashefi, H., & Rahimi, K. (2014). Undergraduate students' difficulties in conceptual understanding of derivation. *Procedia-Social and Behavioral Sciences, 143*, 358-366. https://doi.org/10.1016/j.sbspro.2014.07.495

Herbert, S., & Pierce, R. (2012). Revealing educationally critical aspects of rate. *Educational Studies in Mathematics, 81*(1), 85-101. https://doi.org/10.1007/s10649-011-9368-4

Jones, S. R. (2017). An exploratory study on student understandings of derivatives in real-world, non-kinematics contexts. *Journal of Mathematical Behavior, 45*, 95-110. https://doi.org/10.1016/j.jmathb.2016.11.002

Kertil, M. (2014). *Pre-service elementary mathematics teachers' understanding of derivative through a model development unit* [Doctoral dissertation, Middle East Technical University].

Kertil, M. (2021). Conceptual analysis of derivative as a rate of change and analysis of the mathematics textbooks. *Sakarya University Journal of Education, 11*, 545-568. https://doi.org/10.19126/suje.977200

Kertil, M., & Dede, H. G. (2022). Promoting prospective mathematics teachers' understanding of derivative across different real-life contexts. *International Journal for Mathematics Teaching and Learning, 23*(1), 1-24. https://doi.org/10.4256/ijmtl.v23i1.361

Lam, T. T. (2009). On in-service mathematics teachers' content knowledge of calculus and related concepts. *The Mathematics Educator, 12*(1), 69-86.

Leong, K. E., Meng, C. C., & Abdul Rahim, S. S. (2015). Understanding Malaysian pre-service teachers mathematical content knowledge and pedagogical content knowledge. *EURASIA Journal of Mathematics, Science and Technology Education, 11*(2), 363-370. https://doi.org/10.12973/eurasia.2015.1346a

Li, V. L., Julaihi, N. H., & Eng, T. H. (2017). Misconceptions and errors in learning integral calculus. *Asian Journal of University Education, 13*(2), 17-39.

Maher, C. A., & Sigley, R. (2014). Task-based interviews in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 580-582). https://doi.org/10.1007/978-94-007-4978-8

Makonye, J. P., & Luneta, K. (2014). Mathematical errors in differential calculus tasks in the senior school certificate examinations in South Africa. *Education as Change, 18*(1), 119-136. https://doi.org/10.1080/16823206.2013.847014

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. Jossey-Bass Publishers.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation*. Jossey-Bass.

Meylani, R., & Teuscher, D. (2011). Calculus readiness: Comparing student outcomes from traditional precalculus and AP calculus AB with a novel precalculus program. In *Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

Mkhatshwa, T. P. (2019). Calculus students' quantitative reasoning in the context of solving related rates of change problems. *Mathematical Thinking and Learning, 22*(2), 139-161. https://doi.org/10.1080/10986065.2019.1658055

Mkhatshwa, T. P., & Doerr, H. M. (2018). Undergraduate students' quantitative reasoning in economic contexts undergraduate students' quantitative reasoning in economic. *Mathematical Thinking and Learning, 20*(2), 142-161. https://doi.org/10.1080/10986065.2018.1442642

Nagle, C., & Moore-Russo, D. (2013). The concept of slope: Comparing teachers' concept images and instructional content. *Investigations in Mathematics Learning, 6*(2), 1-18. https://doi.org/10.1080/24727466.2013.11790330

Nagle, C., Martínez-Planell, R., & Moore-Russo, D. (2019). Using APOS theory as a framework for considering slope understanding. *Journal of Mathematical Behavior, 54*, 100684. https://doi.org/10.1016/j.jmathb.2018.12.003

Nasir, N. M., Hashim, Y., Zabidi, S. F. H. A., Awang, R. J., & Zaihidee, E. M. (2013). Preliminary study of student performance on algebraic concepts and differentiation. *World Applied Sciences Journal, 21*, 162-167.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics, 14*(3), 235-250. https://doi.org/10.1007/BF00410540

Orton, A. (1984). Understanding rate of change. *Mathematics in School, 13*(5), 23-26.

Piaget, J. (1929). *The child's conception of the world*. CreateSpace Independent Publishing Platform.

Pitt, D. G. (2015). On the scaling of NSW HSC marks in mathematics and encouraging higher participation in calculus-based courses. *Australian Journal of Education, 59*(1), 65-81. https://doi.org/10.1177/0004944115571943

Shamsuddin, M., Mahlan, S. B., Umar, N., & Alias, F. A. (2015). Mathematical errors in advanced calculus: A survey among engineering students. *Esteem Academic Journal, 11*(2), 37-44.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14. https://doi.org/10.3102/0013189X015002004

Stump, S. (1999). Secondary mathematics teachers' knowledge of slope. *Mathematics Education Research Journal, 11*, 124-144. https://doi.org/10.1007/BF03217065

Stump, S. L. (2001). Developing pre-service teachers' pedagogical content knowledge of slope. *Journal of Mathematical Behavior, 20*(2), 207-227. https://doi.org/10.1016/S0732-3123(01)00071-2

Tall, D. (1993). Students' difficulties in calculus. In *Proceedings of Working Group 3 on Students' Difficulties in Calculus* (pp. 13-28).

Tall, D. (2009). Dynamic mathematics and the blending of knowledge structures in the calculus. *ZDM-International Journal on Mathematics Education, 41*(4), 481-492. https://doi.org/10.1007/s11858-009-0192-6

Tall, D. (2019). *The evolution of calculus: A personal experience 1956-2019* [Opening plenary]. The Conference on Calculus in Upper Secondary and Beginning University Mathematics.

Teuscher, D., & Reys, R. E. (2012). Rate of change: AP Calculus students' understandings and misconceptions after completing different curricular paths. *School Science and Mathematics, 112*(6), 359-376. https://doi.org/10.1111/j.1949-8594.2012.00150.x

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education. In *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (pp. 31-49).

Thompson, P. W., & Thompson, A. G. (1992). Images of rate. In *Proceedings of the Annual Meeting of the American Educational Research Association*.

Tyne, J. G. (2016). *Calculus students' reasoning about slope and derivative as rates of change* [Master's thesis, University of North Carolina].

Ubuz, B. (2007). Interpreting a graph and constructing its derivative graph: Stability and change in students' conceptions. *International Journal of Mathematical Education in Science and Technology, 38*(5), 609-637. https://doi.org/10.1080/00207390701359313

Verzosa, D. (2015). Reading the world with calculus. *Primus, 25*(4), 349-368. https://doi.org/10.1080/10511970.2014.978983

Weber, E., & Dorko, A. (2014). Students' and experts' schemes for rate of change and its representations. *Journal of Mathematical Behavior, 34*, 14-32. https://doi.org/10.1016/j.jmathb.2014.01.002

Weber, E., & Thompson, P. W. (2014). Students' images of two-variable functions and their graphs. *Educational Studies in Mathematics, 87*, 67-85. https://doi.org/10.1007/s10649-014-9548-0

Weber, E., Tallman, M., Byerley, C., & Thompson, P. W. (2012). Introducing the derivative via calculus triangles. *Mathematics Teacher, 104*(4), 274-278. https://doi.org/10.5951/mathteacher.106.4.0274

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education, 8*, 103-127. https://doi.org/10.1090/cbmath/008/06

APPENDIX: INTERVIEW TASKS

1. What is meant by rate of change?
2. Suppose the number of oranges Sofia had is k, while Daniel had l oranges, thus, ratio number of oranges of Sofia to Daniel will be $k{:}l$, where $k > l$. How many oranges does Sofia has compared to Daniel? Choose the correct answer below and explain your choice.
   a. Sofia has $\frac{k}{l}$ oranges as many as Daniel had
   b. Sofia has $k.l$ oranges as many as Daniel had
   c. Sofia has more $\frac{k}{l}$ oranges than Daniel had
   d. Sofia has more $k - l$ oranges than Daniel had
   e. I do not know
3. Adam draws a new car model for his car company. The drawing measurement of car's model is $\frac{m}{n}$ millimetre of a real car dimension.
   a. Find the real car dimension height if the model's height were 150 millimeters?
   b. Explain your calculation.
4. A slope for a line is calculated by differences in y divided by differences in x using two points on the line. For instance, 9.5 divided by 2, getting 4.75 as a slope. Explain what does 4.75 means?
5. Given $f(x) = 5.8x + 2.5$. Let say $f(x)$ represent production cost in dollar to produce x gallon of cooking oil.
   a. Find rate of change for the function.
   b. Describe the meaning of your answer in *a* based on the given context.
6. An experiment is conducted on a shark movement as it swimming horizontally back and forth. The motion of the shark is given in **Figure A1**. Answer the following questions below and explain your answers.
   a. What is the instantaneous velocity at $t = 3$?
   b. What is the instantaneous speed at $t = 3$?

Figure A1. Motion of shark (Source: Authors' own work)

7. **Table A1** illustrated a situation, where the water is flowing into a container at a constant rate. **Table A1** recorded the water depth within the first seven hour and there was an increase of five unit at each hour. What is the rate of increase in the water depth when $t = 2\frac{1}{2}$? When $t = 5.34$? Explain your answer.

Table A1. Water depth vs. time

Time, t (hour)	Depth of water (unit)
0	0
1	5
2	10
3	15
4	20
5	25
6	30
7	35