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The structural and operational complementarity: Grade nine learners' pitfalls and gains of simplifying algebraic expressions

Zwelithini Bongani Dhlamini [1*]

[1] Department of Mathematics Science and Technology Education, University of Limpopo, Limpopo, SOUTH AFRICA

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Abstract
This study focused on the investigation of the structural and operational complementarity in the simplification of algebraic expressions. The dual nature of mathematical conceptions is the theoretical framework that underpin this study. Qualitative content analysis was employed as a method and research design to investigate the complementarity between the operational and structural learners' conceptions of the simplification. The findings revealed that learners' undeveloped and fragmented structures of algebra caused the inability to simplify expressions. Ironically, for the very few learners who managed to simplify expressions, used skills and knowledge of the previous grade. Irrespective of this, the complementarity of learners' conceptions of systems and components is undoubtfully key to their success during the simplification of expressions. This study recommends teaching and learning of algebra should address the complementarity of the operational and structural notions.

Keywords: indeterminate, object, operational, structural

INTRODUCTION

Globally, studies on the simplification of algebraic expressions ponders discursive computational conceptual algebraic pitfalls and gains (Faramarzpoor & Fadaee, 2020; Novotná & Hoch, 2008; Tirosh et al., 1998). The computational pitfalls and gains are reported in diverse focus, such as, insights and analysis of errors and mistakes (Pournara et al., 2016; Seng, 2010), exploration of misconceptions and errors (AL-Rababaha et al., 2020; Baidoo, 2019; Faramarzpoor & Fadaee, 2020), awareness of teachers and related teaching approaches (Chalouh & Herscovics, 1988; Tirosh et al., 1998). A coherent understanding of the pitfalls and gains of mathematical conceptions requires theoretical shift that is coherent. I use the words, 'concepts' and 'conceptions' interchangeably, the former, also called 'notions' referring to theoretical aspects mathematical ideas while the latter posits the internalized human knowing. Mathematical conceptions hinge on two complementary notions, both structural and operational (Sfard & Linchevski, 1995). Literature (Baidoo, 2019; Faramarzpoor & Fadaee, 2020; Seng, 2010; Tirosh et al., 1998) classifies the concepts into both operational and structural. The operational (process-oriented), are routines and computations, which (Sfard & Linchevski, 1995) argues that it emerges first during interiorization. The routines refer to specific steps and procedures followed in the simplification (Faramarzpoor & Fadaee, 2020), while computations refer to the ability of handling operations related to components, structures, and routines (Seng, 2010). In contrast, the structural (object-oriented), refers to components and systems (Linchevski & Livneh, 1999), which develop afterwards through the process of reification (Sfard, 1995). On one hand, components posit the distinct parts of algebraic expressions such as variables, numbers, brackets, operations and notation (Kieran, 1992, 2007). On the other hand, systems are algorithms and concepts that are specific for algebra concepts (Novotná & Hoch, 2008).

Algebra consists of symbols, numbers, and variables (Kieran, 2007). Correspondingly, numbers and variables are among various representations of explicit and implicit quantities (Booth, 1988). Kieran (1992) and

This article is based on available data from the dissertation of the author conducted in the institution of affiliation.

✉ zwelithini.dhlamini@ul.ac.za **(*Correspondence)**

Contribution to the literature

- This study re-emphasizes benefits of the complementarity of mathematical conceptions, especially algebra.
- It highlights that current literature in the discourse of algebra is skewed towards the operational disregarding the importance of systems and components of algebra.
- The study revealed that conceptions of the structures and systems of algebra conceptions are determinant on learners' recontextualization.

Mason (1996) refer to these quantities as either the particular or abstract entities. The description of the schema, properties, and relationships of these abstract entities, which Novotná and Hoch (2008) refer to as structural. In contrast, algorithms and computations that guide the utility of the abstract entities Sfard and Linchevski (1995) refers to as operational. Structural, algebraic expressions emanate from the generality of numeric expressions using variables to represent unknown quantities (Moodliar & Abdulhamid, 2021; Warren, 2003). In fact, they are indeterminate (Sfard, 1995), such as, '$4x + 3$'. It is configured using alphanumeric, the generic ($4x$), the sign ($+$), and a numeral (3) hence it posits an indeterminate quantity (Tirosh et al., 1998). Contrary, operational this expression depicts, 'add four times x and three', in particular 'x' represents the general.

Noticeable, during algebra instruction, the simplification of expressions in earlier grades of schooling evolves from numeric to alphanumeric (Baidoo, 2019; Carraher et al., 2007). Structural, the simplification of algebraic expressions relies on the mastery of other algebra concepts such as equations and exponents (Baidoo, 2019; Warren, 2003). Such mastery is most likely to posit gains and contrary a deficit of the algebra concepts results in pitfalls. On this notion, the indeterminate nature of algebraic expressions is a recipe for operational conceptions (Sfard, 1995). Conceptions based on features, schema and relationships are gains for the simplification of algebraic expressions (Carraher & Schliemann, 2007). Hence, algebra instruction should equip learners with structural sound concepts relevant for the operational simplification of algebraic expressions.

I define the simplification of algebraic expressions as the process of their decomposition into simpler versions. It posits a gain by enabling learners to distinguish both the relevant processes and the objects for this domain from other related algebra concepts such as factorization, operations, and exponential laws (AL-Rababaha et al., 2020; Tirosh et al., 1998). Specifically, Pournara et al. (2016) and Seng (2010) ascertain the existence of a wide range of concepts required for simplifying expressions, such as, grouping like terms, expanding brackets, handling basic operations and attachment of notation and signs. In addition, the grasp of the concept of numbers and fraction remain critical aspects of mastering algebra (Bansilal & Ubah, 2020; Booth et al., 2014).

Learners often experience discursive pitfalls that are associated with both the structural and operational knowledge when simplifying algebraic expressions. Literature (AL-Rababaha et al., 2020; Bansilal & Ubah, 2020; Faramarzpoor & Fadaee, 2020; Herscovics & Linchevski, 1994; Linchevski & Herscovics; 1996; Pournara at al., 2016; Seng, 2010), points to various challenges. First, is the misuse of algebra rules, such as the distributive law and BODMAS, in the earlier grades of high school. Second, confusing arithmetic concepts with those of algebra by introducing the equal sign. Third, misconceiving algebraic expressions as determinate through assigning numerical values. Fourth, oversimplification and incorrect cancellation. Fifth, misconceiving structures of algebraic expressions by conjoining unlike terms during simplification. Amid all these vast learners' challenges reported in the literature, the complementarity of the structural and operational simplification of algebraic expressions remains unclear, which pose a knowledge gap. Hence, the purpose of this study is to investigate the structural and operational complementarity in the simplification of algebraic expressions. Learners' pitfalls when simplifying algebraic expressions were vital in achieving the purpose. The following research question was:

- How do learners operational and structural conceptions complement when simplifying algebraic expressions?

The current study taps into available data (Dhlamini, 2018) and uses the theory of the dual nature of mathematical conceptions (DN) (Sfard, 1991) to succinctly explain the simplifying of algebraic expressions.

THEORETICAL UNDERPINNINGS

DN (Sfard, 1991) posits an interplay between two complementary tenets, the operational as a process and the structural as an object of the simplification of algebraic expressions (**Figure 1**). Sfard (1995) explained that in most mathematics notions, the operational process precedes the structural object, except for geometry, where it is vice versa. Otte (1990) argued for the complementarity of mathematical objects and tools in the sense they simultaneously differ in structure,

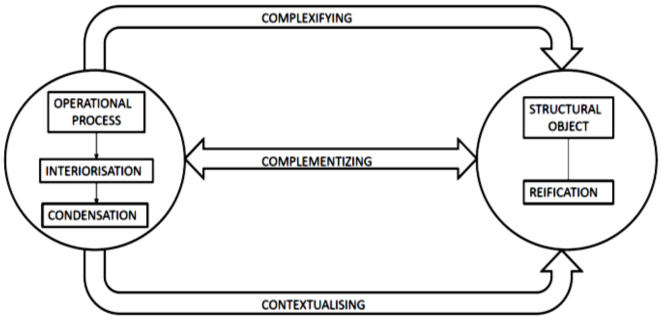

Figure 1. Conceptual framework for DN (Source: Author's own elaboration)

nonetheless they relate operational. Similarly, Vithal (2008) perceives complementarity as extraordinary means of explicating the co-existence of considerably distinct, even contrary object, which explains the same process. For an example, Seng (2010) outlined a learner who simplified the expression '$3a - 6a$' to '$3a$'. When interviewed, the learner said, 'I take $6a$ and minus $3a$ so I got $3a$' (p. 150). This learner incorrectly used concepts of arithmetic, i.e., a large value minus small value. Instinctively, '$3a - 6a$' as an object, consist of distinct indeterminate components, '$3a$', positive alpha numeric indeterminate term of the expression, '$-$', a minus sign (Jiang et al., 2014), '$-6a$', negative numeric indeterminate term of the expression. Contrary, as a process $3a - 6a$' depicts subtracting a negative from a positive. Yet, the minus here is an operation depicting a subtraction, positing complementarity. The indeterminate terms, postulate algebra components, like terms, which can be subtracted.

On the one hand, the operational process is characterized by two tenets, *interiorization* and *condensation*. During simplification of algebraic expressions, learners manipulate components and systems using properties, rules, concepts and algorithms to develop new mental conceptions, and this is referred to as *interiorization* (Scheiner, 2016). Subsequently, learners grasp the compact bigger picture of concepts in a process called *condensation* (Chimoni & Pitta-Pantazi, 2017; Thompson & Sfard, 1994). For an example, in the study conducted by Pournara (2020), a learner simplified as follows: '$-3x + x$' to '$-4x$'. Such manipulation signifies that integers addition rule is misapplied in algebra and depicts the learner not grasping *condensation*. In contrast, Irwati and Ali, (2018) discovered a learner who added $4 + 3x^2$ to get $7x^2$. This illustrates further the learners' disregard of the indeterminate nature of components of algebraic expressions (Moodliar & Abdulhamid, 2021). In contrast, Ndemo and Ndemo (2018) studied the simplification of $x(a + b) \div x(a + b)$, the learner rewrites and expands brackets as $\frac{ax+bx}{x+xd}$, cancels the 'x_s' and the answer was $\frac{a+b}{d}$. This learner posits misapplication of division in an algebraic expression. In the three studies, learners indicate having imbalance between *interiorization* and *condensation*.

On the other hand, during the structural object, learners' master the indeterminate algebra structures and components of algebraic expressions through balancing *interiorization* and *condensation* to postulate *reification* (Chimoni & Pitta-Pantazi, 2017; Linchevski & Livneh, 1999). To master the structural object, learners first engage with three tenets during the operational process that are both contrasting and complementary (Zeljić, 2015). These are fundamental in balancing *interiorization,* and *condensation*, they are, contextualizing, complementizing and complexifying (Schneider & Pinto 2019). For *contextualization*, learners extract meaning of an object dependent on, how it is represented, the context, where it is used, the mathematics domains, where it resides and how is recontextualized (van Oers, 1998). In contrast, during complementizing, learners should co-ordinate various contexts to create coherent conceptual structures (Scheiner, 2016). The other notion, complexifying, prescribes that learners should be enabled to navigate from simple conceptual structures to complex structures through recontextualizing, and coherently co-ordination conceptual structures (Schneider & Pinto 2019). However, in the simplification of algebraic expressions complexification is recontextualized in the reverse, from complex to simple (Moodliar & Abdulhamid, 2021). All these is the key balancing factors of *interiorization and condensation*, which is explicated using empirical literature below.

I reflect on a learners' responses extrapolated from three empirical studies on the simplification of algebraic expressions to clarify the structural object. First, Faramarzpoor and Fadaee (2020) reported a learner who simplified $9b^2 + 4b^2$, and the answer was $13b^4$. The components of the algebraic expression, '$9b^2$', and '$4b^2$', are alphanumeric like terms (AL-Rababaha et al., 2020), they can be simplified further only by addition. When confronted with the exponents, the learner recontextualizes to the domain of multiplying exponents. Similarly, in Zulfa et al. (2020) a learner simplified $\frac{5}{a} + \frac{20}{a}$ to get $\frac{25}{a^2}$. This simplification posits misconceived recontextualizing addition to multiplication of exponents, causing imbalance to *interiorization and condensation*. Hence, such posits in undeveloped reification and the structural object of the simplification of algebraic expressions.

METHODS

This study employed the qualitative content analysis as method and research design (Elo & Kyngäs, 2008) to elucidate learners' simplification of algebraic expressions. This approach is flexible on the procedures a study can undertake as guided by the research problem (Harwood & Garry, 2003). The study purposively

Table 1. Codes for learners' algebraic simplification

Coding algebraic simplification		
AS3.1A: *Process*, used inappropriate ingredients for simplification neither expansion nor factoring common factor, resulting in operational irrelevant simplification. *Object*, failure to grasp components, unlike terms and exponents negatively affected knowledge of systems.	AS3.1B: *Process*, expansion inside brackets, as ingredients for simplification; however, operational irrelevant simplification of expression due to various irrelevant computations such as, failure to manage minus sign, misapplication of exponential rules, & brackets, conjoining unlike terms & mishandling of BODMAS rule. *Object*, although learners realized systems such as expansion, which could not filter into components, unlike terms & exponents.	AS3.1C: *Process*, expansion inside brackets, grouping like terms, appropriate handling of minus sign and brackets, resulting in relevant simplification. *Object*, led to grasp of the indeterminate nature of components of the expression, and the realization of systems, like terms, to reach the simpler version of the expression.
AS3.2A: *Process*, used inappropriate ingredients for simplification neither add like terms & divide nor use common denominator, resulting in operational irrelevant simplification. *Object*, failure to realize systems, division, led to mishandling of components, exponents.	AS3.2B: *Process*, add, divide by $'8x^2y^3'$ as ingredients for the simplification, however operational irrelevant simplification due to misapplication of exponential rules. *Object*, realization of systems, division, could not transfer to the components, exponents.	AS3.2C: *Process*, add and divide by $'8x^2y^3'$, resulting in relevant simplification. *Object*, realization of systems, the divisor, which enabled the handling of components, like terms, to reach the simpler version.
AS3.3A: *For process,* used unfit ingredients for simplification, misapplication of exponential rules, conjoining unlike terms, irrelevant cancelling resulting in operational irrelevant simplification. *Object*, failure to grasp systems of division, negatively affected grasp of components, divisors.	AS3.3B: *Process*, factors, & divisor $(x-4)$ as ingredients for simplification is appropriate; however, operational irrelevant simplification due to various incorrect computations such as factorization & cancelling. *Object,* learners' recognition of systems could not assist in managing components, factorization, & division.	AS3.3C: *Process*, factors, and divisor $(x-4)$ as ingredients for the simplification, resulting in operational relevant simplification. *Object*, consciousness of components, the divisor, which allowed handling of systems factorization and division to reach the simpler version.
AS3.4A: *Process*, could not use common denominator as an ingredient for simplification, causing the mishandling of exponential rules which resulted in operational irrelevant simplification. *Object*, failure to grasp systems, divisor, led to mishandling of components.	AS3.4B: *Process,* like terms, common denominator as ingredients for the simplification, however operational irrelevant simplification due to mishandling computations involving the common denominator. *Object*, grasp of systems, divisor, could not filter into components, divisors.	AS3.4C: *Process*, like terms, common denominator as ingredients for simplification, resulting in operational relevant simplification. *Object*, realization of common denominator in components of non-equivalent fractions, which permitted handling of systems involving operations (addition & subtraction) to reach simpler version of expression.

sampled ninety (n=90) learners' scripts from available data (Dhlamini, 2018).

Structural and operational learners' simplification of algebraic expressions required explanations and descriptions as per guidance from the literature (Schreier, 2012). Hence, two experienced mathematics educators coded the learners' responses. The research processes were in the following sequence. First, *preparation phase*, involved sourcing available data on algebraic expressions, and learners' simplification. The COVID-19 regulation has made it difficult and almost impossible to access research sites such as schools. In the mist of this pandemic, critical issues remain unclear and even missing in the literature and require urgent attention. Hence, available data from previous bigger study (Dhlamini, 2018) elucidate silent issues in the simplification of algebraic expressions. Making sense of the data started with the review of the *process* and *object* as posed by algebraic expressions. Then, familiarization of the learners' simplification of algebraic expressions with reference to literature in the discourse. Second, *the organizing phase* followed three processes, coding, creating categories, and abstraction (Schreier, 2012).

When conducting open coding (**Table 1**), the learners' algebra simplification procedures generated categories. To reach abstraction categories were both collapsed and refined. Third, *data analysis and interpretation*: inductive analysis was conducted, from open coding (**Table 1**) of the learners' response's themes emerged (Braun & Clarke, 2006). Concurrently, the data analysis was guided by the theoretical basis (**Figure 1**). During interpretation, the theoretical prescripts of DN

Table 2. Documenting learners' simplification of algebraic expressions

Learners' responses to the simplification of algebraic expressions (F [%])				
Questions	Irrelevant process & object	Fragmented process & object	Relevant process & object	Total frequency
Question 3.1	76 (84.45)	13 (14.44)	1 (1.11)	90 (100)
Question 3.2	81 (90.00)	1 (1.11)	8 (8.89)	90 (100)
Question 3.3	84 (93.33)	5 (5.56)	1 (1.11)	90 (100)
Question 3.4	80 (88.89)	3 (3.33)	7 (7.78)	90 (100)

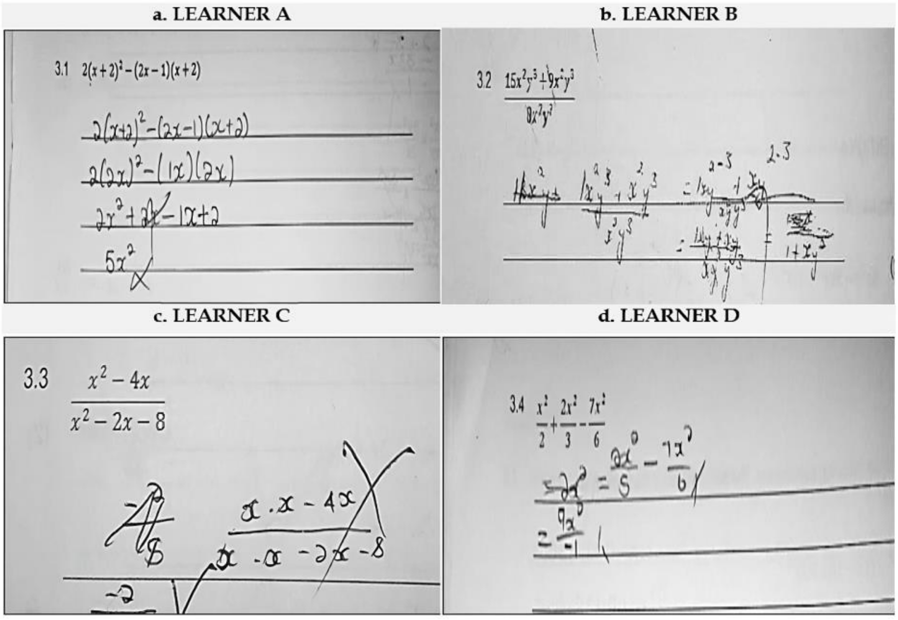

Figure 2. Learners' structural & operational irrelevant simplifications (Dhlamini, 2018)

theory (Sfard, 1991) and literature on simplification of algebraic expression interrogated the results.

RESULTS

The overall results posit that learners' simplification algebraic expressions had three categories. First, a bulk of results were learners who exhibited both irrelevant *process* and *object* for the simplification of algebraic expressions. This reveals the absolute lack of the indeterminate conceptions for structures of algebraic expressions and the corresponding computations of its systems. Second, a few results were learners with both fragmented *process* and *object* for the simplification. This exposes imbalances in the grasp of the structures and systems. Third, were very few results for learners relevant *operational* and *structural*. This posits the application of relevant systems, which enabled the handling of components of algebraic expressions.

How Learners Simplified Algebraic Expressions

The data in **Table 2** is a synopsis that indicates the trend on learners how learners simplified the algebraic expressions. The data is skewed towards irrelevant, *process* and *object*, positing vast challenges related to the simplification of algebraic expressions. A small quantity of the data was in *process* and *object* fragmented revealing limited skills of the simplification. A very small portion of the data was in *process* and *object* relevant, signifying the existence of learners with knowledge and skills of simplifying algebraic expressions.

Theme 1: Process & Object Irrelevant Simplification

In **Figure 2**, are vignettes (part a, part b, part c, & part d), for a sample of learners' responses in this theme. First, learner A's simplification of question 3.1, one of 76 out of 90 coded as AS3.1A. For the *process*, this learner conjoined unlike terms, $5x^2$, $(x+2)^2$to $2x^2$, $(2x-1)$ to $1x$, $(x+2)$ to $2x$, misapplied exponential laws and BODMAS rule, $2(2x)^2-(1x)(2x)$ to $2x^2+2x-1x+2$. Further in the *process*, the learner conjoined unlike terms, misapplied exponential laws $2x^2+2x+1x$ and eliminated the number 'two,' to reach a simpler version $5x^2$ (part a in **Figure 2**). This implies that, for the *object*, the learner failed to grasp systems required for simplification of components, such as unlike terms and exponents. Second, learner B's simplification of question 3.2 coded AS3.2A and one of 81 out of 90 classified in this theme. The *process,* the learner incorrectly cancelled the numbers in both the numerator and denominator, $\frac{15x^2y^3+9x^2y^3}{8x^2y^3}$ to $\frac{1x^2y^3+x^2y^3}{x^2y^3}$ misapplied exponential law, mishandled the minus signs in the exponents, $\frac{1xy^{2-3}+xy^{2-3}}{x^2y^3}$to get$\frac{1xy+xy}{x^2y^3}$. Finally, the learner incorrectly cancelled, $\frac{1xy+xy}{x^2y^3}$ to reach an irrelevant simpler version,

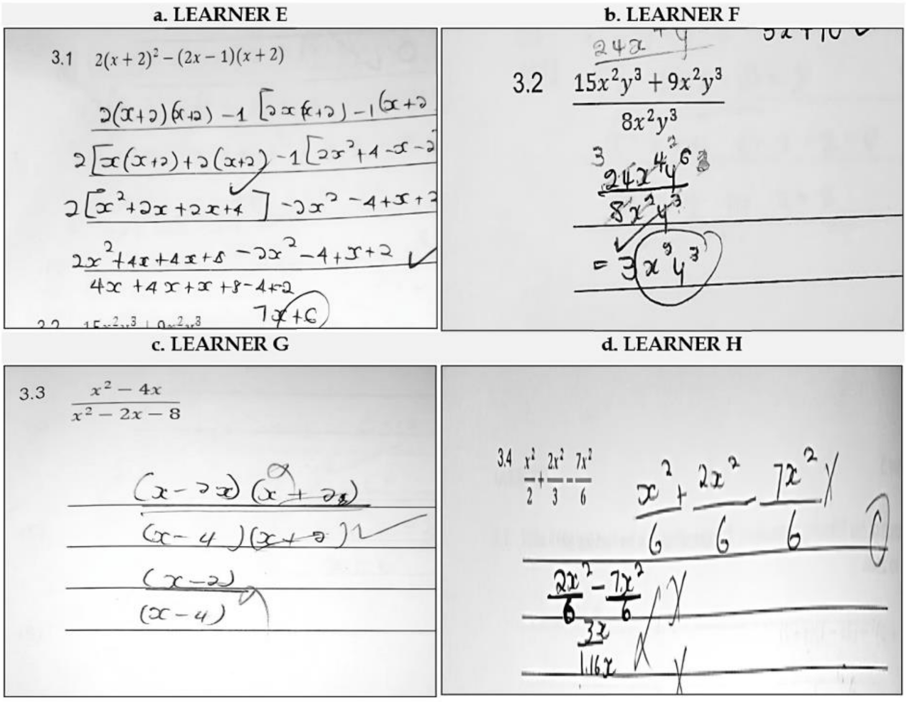

Figure 3. Learners' structural appropriate & operational irrelevant simplifications (Dhlamini, 2018)

$1 + xy$, (part b in **Figure 2**). The coding posits that this learner neither realized systems of the components, the addition of like terms, division nor common denominator as the *object*. Third, learner C's simplification of question 3.3 was coded AS3.3A, which was one of 84 out of 90 classified in this theme. In the *process*, the learner correctly applied law of exponents, x^2 to '$x.x$' in the denominator and numerator. Learners unbundle exponents when preparation for factorization. Instead, the learner incorrectly cancelled $\frac{x.x}{x.x}$, divided $\frac{-4x}{-2x}$ detached '-8, mishandled the minus signs in the division to reach an incorrect simpler version, -2 (part c in **Figure 2**). During coding, the *object*, the learner neither used systems for factorization nor components, the divisor $(x-4)$ as ingredients for the simplification. Forth, learner D's simplification of question 3.4 was coded AS3.4A one of 80 out of 90 classified in this theme. For the *process*, the learner misapplied the addition and subtraction of fractions. The learner observed BODMAS rule (addition first) and did not use the common denominator when adding $\frac{x}{2} + \frac{2x}{3}$ to $\frac{2x^2}{5}$. Afterward, subtraction, again not obeying the common denominator, $\frac{2x^2}{5} - \frac{7x^2}{6}$ to $\frac{9x^2}{-1}$, (part d in **Figure 2**), which was an incorrect simpler version. For *object*, the learner could not use correct systems of components, common denominator as an ingredient for the simplification.

Theme 2: Process & Object Fragmented Simplification

Figure 3 consist of vignettes (part a, part b, part c, & part d), for sampled learners' responses in theme 2. First, learner E's simplification of question 3.1 was coded AS3.1B, one of 13 out of 90 classified in this theme. During the *process*, the learner handled properly the distributive law and minus sign when applying the BODMAS rule (part a in **Figure 3**). However, the learner detached the symbol '*X*' from the number '4' in the second set of brackets during the expansion, $[2x(x+2) - 1(x+2)] to [2x^2 + 4 - x - 2]$. Further there was an incorrect addition, $4x + 4x + x$ to $7x$. Consequently, the simpler version was $7x + 6$, (part a in **Figure 3**), not the correct answer. For the *object*, the learner recognized the systems for simplification and could not reach the required simpler version due to mishandling of components. Second, learner F's simplification of question 3.2 was coded AS3.2B, one out of 90 classified in this theme. During the *process*, the learner failed to realize like terms and misapplied exponential laws by using multiplication law of exponents instead of addition, $15x^2y^3 + 9x^2y^3$ to $24x^4y^6$. The further simplification was correct $\frac{24x^4y^6}{8x^2y^3}$ to $3x^2y^3$ (part b in **Figure 3**), although incorrect simpler version due to the earlier misapplication of exponential rules. In the *object*, by not realizing components, like terms, the learner misapplied exponential rules. Third, learner G's simplification of question 3.3 was coded AS3.3B, one of five out of 90 classified in this theme. During the *process*, the learner incorrectly factorized in the numerator, $x^2 - 4x$ to $(x-2x)(x+2x)$. Here the learner computed the difference of two squares instead of factoring 'x. Factorization in the denominator was appropriate. Further the learner detached 'X' from $(x-2x)(x+2x)$ to divide by $(x+2)$ to reach an incorrect simpler version $\frac{x-2}{x-4}$, (part c in **Figure 3**). For the *object*, the learner misconceived the systems of factorization.

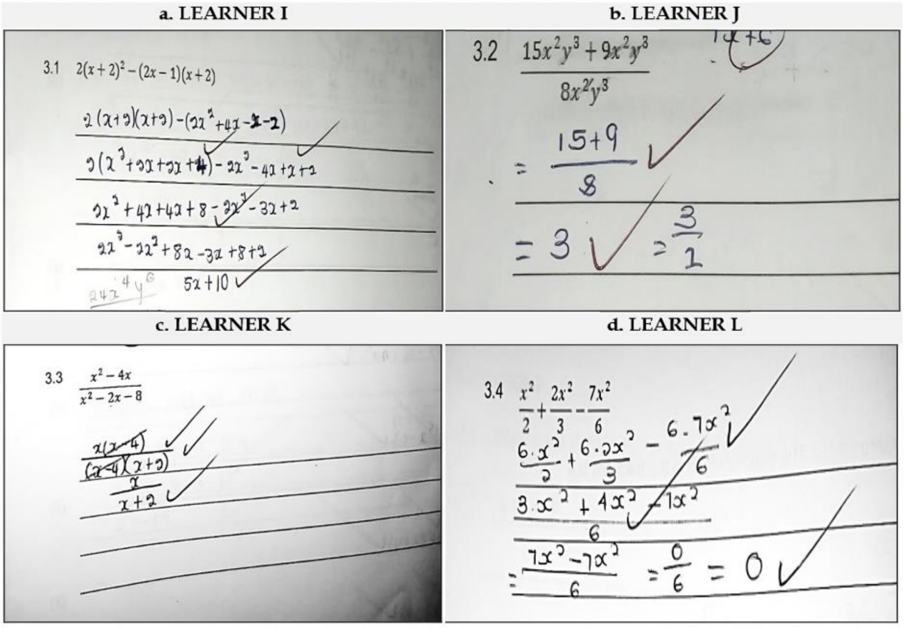

Figure 4. Learners' structural & operational relevant simplifications (Dhlamini, 2018)

Forth, learner H's simplification of question 3.4 was coded AS3.4B, one of three out of 90 classified in this theme. For the *process*, the learner misapplied rules of converting equivalent fractions, as follows:

1. $\frac{x^2}{2}$ to $\frac{x^2}{6}$ and
2. $\frac{2x^2}{3}$ to $\frac{2x^2}{6}$.

During the simplification, the learner incorrectly performed the addition of fractions like terms with same denominator, $\frac{x^2}{6}+\frac{2x^2}{6}$ to $\frac{2x^2}{6}$. Further, the learner incorrectly subtracted like terms, and attached 'x' in the denominator as follows: $\frac{2x^2}{6}-\frac{7x^2}{6}$ to $\frac{3x}{1.16x}$, (part d in **Figure 3**) a simpler version, not the required one due to the fragmented simplification. The learner used the common denominator, In the *object*, the learner misapplied the systems of division, which filtered to the mishandling of the components.

Theme 3: Process & Object Relevant Simplification

Figure 4 are vignettes (part a, part b, part c, & part d), for sampled learners' responses that classified in this theme. First, learner L's simplification of question 3.1 was coded AS3.1C, one out of 90 classified in this theme. During the simplification, the *process* is the accuracy of the expansion of brackets $2(x^2+2x+2x+4)-(2x^2+4x-x-2)$. In addition, the learner efficiently managed the distributive law and the minus sign $2x^2+4x+4x+8-2x^2-3x+2$. Further, the learner appropriately grouped like terms $2x^2-2x^2+8x-3x+8+2$, correctly applied BODMAS rule to reach the required simpler version $5x+10$, (part a in **Figure 4**). The *object* is the proper handling of the systems of BODMAS, which hinges on the indeterminate nature of the components, like terms and exponents. Second, learner J's simplification of question 3.2 was coded AS3.2C, one of eight out of 90 classified in this theme. For the *process*, the learner divided the two terms of the numerator by the denominator, $\frac{15x^3y^3+9x^2y^3}{8x^2y^3}$ to $\frac{15+9}{8}$. Then simplified the fraction to get the simpler version '3' (part b in **Figure 4**). The *object* is the by realization of systems of division, which enabled the handling of components, the divisor and like terms Third, learner K's simplification of question 3.3 was coded AS3.3C, one out of 90 classified in this theme. In the *process*, the learner factorized both the numerator and the denominator, then divide by $x-4$ to reach the simpler version $\frac{x}{x+2}$, (part c in **Figure 4**). The *object* assisted the learner to be conscious of relevant components, the divisor, which allowed the simultaneous handling of systems factorization and division. Forth, learner L's simplification of question 3.4 was coded AS3.4C, one of seven out of 90 classified in this theme. During the *process*, the learner converted the expression to a common denominator '6' $\frac{3x^2+4x^2-7x^2}{6}$. Then the learner appropriately applied BODMAS rule for addition and subtraction in the numerator reach the simpler version '0 (part d in **Figure 4**). For the *object*, the learner realized the common denominator in components of non-equivalent fractions, which permitted the handling of systems involving basic operations.

DISCUSSION

The purpose of this study is to investigate the structural and operational complementarity in the simplification of algebraic expressions. To achieve this, DN theory (Sfard, 1991) clarified learners' simplification of algebraic expression in various ways as underpinned and anchored by the two tenets the object and process.

On one hand, the process posed two categories, interiorization and condensation. Interiorization was sought from learners' simplification using various properties, rules, and algorithms from four algebraic expressions. To reach condensation, the evaluation focused on how learners interiorized these aspects to understand the indeterminate nature of algebraic expressions. On the other hand, the object was sought from contextualizing, complexifying and complementizing to deduce balance between interiorization and condensation, a process known as reification (Schneider & Pinto 2019). Furthermore, the theoretical interrogation of learners' simplification of algebraic expressions hinges on the learners' context, recontextualization, from complex to simple (Faramarzpoor & Fadaee, 2020).

The most worrying results were the bulk of learners who could not handle the systems and components of the simplification resulting in pitfalls related to irrelevant process and object. The pitfalls are associated with the following concepts, misapplication of exponential and BODMAS rules, mishandling of the minus sign, incorrect cancelling in algebraic fractions, conjoining of unlike terms. The misapplication of the exponential rules and incorrect cancelling suggests irrelevant recontextualization, which misappropriates the interiorization of concepts and procedures of simplifying algebraic expressions (Schneider, 2016; Baidoo, 2019). Consequently, condensation is based on inappropriate concepts and systems of simplifying algebraic expressions causing inappropriate reification (Chimoni & Pitta-Pantazi, 2017). These results are consistent with those reported by both Baidoo et al. (2020) and Ndemo and Ndemo (2018). The major difference is that in these previous studies, the results were skewed towards reporting only the operational process. In contrast, there are various inconsistencies in relation to learners' mishandling of concepts systems and structure of algebraic expressions. Such are, conjoining of unlike terms (Moodliar & Abdulhamid, 2021), also showing (part a in **Figure 2**) and mishandling of the minus sign (part d in **Figure 2**) as reflected by Pournara (2020). On the one hand, the conjoining of unlike terms is evidence of irrelevant recontextualization (Schneider, 2016), of algebra using rules, concepts and algorithms of integers (Baidoo et al., 2020). On the other hand, the inappropriate application of the minus sign, signals learners' imbalance towards the complementizing of the dual nature, as a sign and as an operator (Jiang et al., 2014; Zeljić, 2015). The bulk of learners categorized in the theme irrelevant process and object, point to inappropriate balance of interiorization and condensation in all aspects of DN (**Figure 1**) including complexifying (complex to simpler) the simplification of algebraic expressions (Moodliar & Abdulhamid, 2021).

A few results indicated that learners posited various gains and pitfalls during simplification due to fragmented process and object. The positives in this category are various gains of, proper handling of the distributive law, minus sign (part a in **Figure 3**), relevant factorization, realization of the LCD (part b in **Figure 3**).

The gains posit some traits of relevant learners' manipulation of algebra concepts and systems at the level of interiorization (Schneider, 2016). These results are similar to those reported in other studies in this domain (Irwati & Ali, 2018; Ndemo & Ndemo, 2018; Zulfa et al., 2020). Simultaneously the results are worrisome due to numerous pitfalls that resulted from, detaching negative symbols in terms (part d in **Figure 3**), misapplication of laws of exponents (part b in **Figure 3**), and misapplied factorization (part c in **Figure 3**).

These are attributed to both improper recontextualizing of laws of integers in exponents (Pournara, 2020; Schneider & Pinto 2019) and misuse in irrelevant domain (Schneider, 2016). Consequently, learners could not grasp a coherent conceptual structure (condensation) of the simplification, positing fragmented complexification. Hence the simplification is characterized by imbalances between interiorization and condensation and not reaching reification in this domain (Baidoo et al., 2020; Zeljić, 2015).

Moreover, it was most disturbing to discover from the results that very few learners correctly simplified algebraic expressions. These results were ascribed to appropriate interiorization of manipulated concepts, systems and procedures of simplifying algebraic expressions (Schneider, 2016). The results pointed the consistency of handling algebra rules, concepts, of signs, division of fractions. exponential laws consistent with those reported in other studies (Irwati & Ali, 2018; Pournara, 2020). As a consequence, learners grasped complexification of the simplification of algebraic expressions (Schneider & Pinto, 2019). Hence there is balance between interiorization and condensation a signal for reification.

However, there results were worrisome, the curriculum (Department of Basic Education [DBE], 2011) specifies that grade nine learners should accomplish simplification using factorization. In contrast, simplification using BODMAS rule (expansion) is an outcome that should be achieved in the previous grades, eight and seven. This signifies that these grade nine were exhibiting grade eight skills of simplification.

CONCLUSIONS

This study contributes to the complexification of algebraic expressions from recontextualization and complementizing. DN permitted the study to harvest silent issues in the literature, especially the role of the *object* in algebra conceptions. The structural *object* has the potential of empowering learners with complexities of conceptions required for the simplification of algebraic expressions. Hence the importance of complementarity in the manipulation of properties, skills, algorithms, operational *process* and their corresponding abstract conceptions, structural *object* proved as remedies for the simplification of algebraic expressions.

The current study responded to the following research question: *How do learners operational and structural conceptions complement when simplifying algebraic expressions?* Answers to this question emanated from three main findings, which were sought from the three themes. First, majority of learners inappropriately recontextualize algebra systems and algorithms to those of integers resulting the irrelevant co-ordination of irrelevant simplification. Second, fragmented simplification was attributed by a blend of relevant and irrelevant contextualization of the simplification, resulting in distorted conceptual complexification of the expressions. Last, very few learners managed to appropriately complexify algebraic expressions through relevant recontextualization and coherent structures. However, when simplifying certain expressions, they applied only knowledge from the previous grade, which raised concerns on their conceptual progression. Hence, it implies that teaching and learning should capacitate learners first with algebra concepts, rules and systems preceded by skills of recontextualizing them to the domain of algebra during simplification. In conclusion, learners' operational and structural conceptions of simplifying algebraic expressions were not complementary due to underdeveloped structures and systems algebra. This is attributed to the misuse of structures and systems from other discourses such as integers during recontextualization and complexifying of the expressions.

The limitations are attributed to the focus only on available data, the learners' responses to the simplification of algebraic expressions. Although DN is a useful tool to evaluate the results, it could have been worthy to probe learners to justify and clarify simplifications. Nonetheless, this study has reignited the dual conception of mathematical notions, which is key in understanding learners' cognition. Hence, I suggest further empirical studies in this discourse should use various methods such as interviews to harvest some of the silent issues in the literature.

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