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Towards a Theory of Identity and Agency in Coming to Learn Mathematics

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In writing this paper we draw considerably on the work of Jo Boaler and Leone Burton. Boaler's studies of classrooms have been particularly poignant in alerting the mathematics education community to a number of key features of successful classrooms, and how such features can turn around the successes for students who traditionally perform poorly in school mathematics. This is supplemented by the recent work of Leone Burton who worked extensively with research mathematicians in order to understand their communities and ways of working. Collectively these two seminal works provide valuable insights into potential ways to move the field of school mathematics forward. In times when there is international recognition of the plight of school mathematics, there is a need for new teaching practices that overcome the hiatus of contemporary school mathematics.

Keywords: Learning Theory, Identity, Agency, Working As A Mathematician.

INTRODUCTION

For a long time now we have known that there have been serious problems with mathematics participation and engagement. The situation facing mathematics has been highlighted recently in Australia by two significant reviews into the mathematical sciences:

Statistics at Australian universities (Statistical Society of Australia, 2005)

Mathematics and statistics: Critical skills for Australia's future (Australian Academy of Science, 2006).

Although these reviews were conducted in Australia, a similar story has emerged around the world. In these reviews, particular attention has rightly been given to school mathematics and the problems of non-engagement with an increasing number of students in higher level courses of mathematical study. That said, it has been known for a long time, through the many descriptive studies that have been undertaken since the 1970s, that mathematics has been unpopular and disliked, and yet the problems appear to grow unabated and little progress has been made to arrest the decline.

In this paper we draw on the work of two researchers - Jo Boaler and Leone Burton - who collectively create a new space for theorizing a way out of the potential teacher blame. Drawing on these works, we seek to illustrate the power of agency in working mathematically. For too long, the pedagogy of school mathematics has focused on procedural knowledge rather than depth of understanding. Combining the work of Boaler and Burton, we draw on an illustrative example of teachers working to solve a task. They draw on many of the concepts identified in the works of Boaler and Burton. Our contention is that the combining of Boaler's 'dance of agency' with Burton's 'working as a mathematician' enables a rich way forward in the teaching of school mathematics.

Mathematics and Teaching: A Link?

Recently there have been reviews of the preparation, qualities and qualifications of mathematics teachers (e.g., "The Preparation of Mathematics Teachers in Australia" (Harris & Jensz, 2006) for the Australian Council of Deans of Science). There have been reports highlighting

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the poor mathematical content knowledge of teachers, particularly primary teachers and non-specialist teachers who are placed in front of secondary classes. Primary preservice teachers often are not confident with the study of mathematics and generally have low levels of understanding of many mathematical concepts (Kanes & Nisbet, 1996). Many preservice teachers enter their teacher education courses with low levels of mathematics knowledge as well as considerable anxiety towards the subject (Brown, McNamara, Hanley, & Jones, 1999; Cooney, Shealy, & Arvold, 1998; Grootenboer, 2003). In many cases, preservice primary teachers have opted for studies in areas other than mathematics so when they enter their courses they have low levels of mathematics content knowledge and frequently have an anxiety towards involvement in the discipline (Goulding, Rowland, & Barber, 2002). The development of a strong content knowledge is central to the development of quality mathematics teachers. For example, Mandeville and Lui (1997) concluded that the level of teacher knowledge impacted significantly on the learning of the students, whereby teachers with high levels of mathematical understandings provided higher quality learning opportunities for their students than did their peers with limited understandings of mathematics. Thus, the role of teacher education is to scaffold teachers into confident and competent developers and users of mathematics so that they are better able to teach mathematics.

Simon (1993) has raised concerns about primary preservice teachers' weak conceptual knowledge and Cooney et al. (1998) have noted similar difficulties with secondary teachers' content knowledge. In their study of preservice teachers in the UK, Goulding et al. (2002) found that there was a significant link between "poor subject knowledge [being] associated with weaknesses in planning and teaching primary mathematics" (p. 699). Recognizing that such a correlation does not imply causation, the authors elaborated further that the positive links were potentially due to the connection that preservice teachers were making between content knowledge and pedagogic knowledge. Goulding, et al contended that the link was due to both cognitive and affective dimensions of the students. Being strong in content knowledge offered a sense of confidence, which in turn was realized through teacher actions. Offering a strengthened program in content knowledge gave students resources upon which they could draw as they planned their teaching. The authors concluded that where students had secure mathematical foundations, they had greater confidence in their own knowledge as a teacher.

Preservice teachers often enter their initial training courses with self doubt about their capacity to learn mathematics (Cooney et al., 1998; Philippou & Christou, 1998). These conceptions come to frame how they will organize learning environments once they begin to plan for teaching (Sánchez & García, 2008). This extends to practicing teachers: Bibby (2002) showed that the belief that mathematics is about 'right answers' brings about feelings of shame amongst practicing teachers if they do not know the answers. This produces teaching practices that are governed by teachers ensuring they have correct answers, thereby offering a restricted repertoire of learning experiences for students. Ball (1990) argued strongly that the focus in teacher preparation needed to be one that encouraged students to relearn the content knowledge in order to develop new understandings of pedagogic knowledge. In attempting to break the distinction between content knowledge and how it is taught, Ball (1990) argued that preservice teachers needed to develop connections between mathematical knowledge and teaching knowledge. Strength in content knowledge can be transferred to pedagogical knowledge. This possibility was made evident by Mandeville and Lui (1997), who reported that teachers with a strong knowledge were "[able to provide] greater depth in dealing with concepts, better equipped to lead students to use their knowledge and use more higher-order content than teachers less knowledgeable about the content" (p. 406).

At this critical point we want to suggest that it is time to move on from studies that repetitively show that mathematics is suffering from poor teacher knowledge and attitudes towards mathematics—either with teachers or students—and to try and look forward by offering some positive directions. To advance this agenda we need more than good ideas that seemed to have worked in a particular context; we need to begin developing a theoretical, robust framework that will address these concerns in a coherent and holistic fashion. In this paper we have drawn on the seminal works of Burton and Boaler to consider mathematical learning from both the discipline knowledge and the mathematical activity perspectives. After reviewing Burton's findings from her study with research mathematicians we briefly highlight some relevant points from Boaler's classroom studies. After presenting an example from teacher education we finish by employing the metaphor of a 'dance of agency' (Pickering, 1995) to discuss mathematics learning, particularly in the light of the current crisis.

The Practice of Mathematicians

The two recent reviews of mathematical sciences in Australia mentioned previously both made significant comment on and recommendations for school mathematics education. Interestingly, the authors of these reports were mathematical scientists and there appeared to be little input from mathematics educators and mathematics teachers. Although this is problematic, it does perhaps highlight the gap that seems to exist between mathematicians and statisticians, and teachers and teacher educators. This is unhealthy and if the current decline in participation and interest in mathematics is to be arrested these groups need to engage in dialogue and mutual projects. To this end, the work of Burton (1999a, 1999b, 2001, 2002) is helpful because her research explored the practices of research mathematicians and their implications for the learning of mathematics.

In 1997 Burton studied the practices of 70 research mathematicians in Great Britain and one of the key features she identified was the collaborative nature of their practice. The benefits for collaborating included practical (e.g., sharing the work), quality (e.g., greater range of ideas on problems), educational (e.g., learning from one another) and emotional (e.g., feeling less isolated) reasons. Clearly, working together with other mathematicians was seen as important, but there appeared to be a distinction between the public perception of mathematics as a lonely enterprise and the reality of mathematicians' practice, in which collaboration is highly valued.

Perhaps another anomaly from public perception was Burton's finding that mathematicians have emotional, aesthetic and personal responses to mathematics.

> … although knowing when you know is extremely important, you have to live with uncertainty. You gain pleasure and satisfaction from the feelings that are associated with knowing. These feelings are exceptionally important since, often despite being unsure about the best path to take to reach your objective, because of your feelings you remain convinced that a path is there. … This is particularly poignant in the light of the picture painted of mathematics as being emotion-free … (Burton, 1999a, p. 134)

The mathematicians in her study highlighted the power of the "aha!" moment and the joy of mathematical discovery, revealing the clear link between mathematics and those who produce it. Allied to their emotional responses to their mathematical practice were aesthetic reactions. They described mathematics in terms such as "wonder", "beauty" and "delight" and these personal responses provided motivation for continued engagement and fuelled a passion for the discipline of mathematics. Davis and Hersh (1998, p. 169) lamented that "blindness to the aesthetic element in mathematics is widespread and can account for the feeling that mathematics is dry as dust, as exciting as a telephone book …".

Another feature of research mathematicians' practice was the importance of intuition or insight. While the mathematicians were less than clear in describing what intuition and/or insight were, they were unambiguous in highlighting the importance of these factors in their mathematical practice. The suggestion was that intuition can be developed through the application of knowledge and experience in mathematical discovery and reflection upon such investigations.

Burton highlighted other features of the practice of mathematicians including the desire to seek and see rich connections between the various branches of mathematics and between mathematics and other disciplines, but her other main agenda was to highlight the pedagogical implications of her findings. Throughout her reports Burton highlighted the distinction that is evident between the work and learning practices of research mathematicians, and the learning experiences of mathematics students at almost all other levels from preschool to undergraduate degree programs. This led her to assert that "we have a responsibility to make the learning of mathematics more akin to how mathematicians learn and to be less obsessed with the necessity to teach 'the basics' in the absence of any student's need to know" (Burton, 2001, p. 598). Even at a very general level, this would require mathematical pedagogy to be characterized by collaboration and group work with attention paid to the emotional, aesthetic and intuitive dimensions of the discipline. This encompasses the 'doing' of mathematics that has been under-emphasized in education as it has focused on the 'knowing' of mathematics. Indeed, perhaps an issue with the educational recommendations in the Australian review of mathematical sciences was the emphasis on mathematical content knowledge that can be taught largely through a transmission model. On this point Boaler (2003) commented:

> There is a widespread public perception that good teachers simply need to know a lot. But teaching is not a knowledge base, it is an action, and teacher knowledge is only useful to the extent that it interacts productively with all the different variables in teaching. Knowledge of subject, curriculum, or even teaching methods, need to combine with teachers' own thoughts and ideas as they too engage in something of a conceptual dance. (p. 12)

In her seminal work in England and the United States, Boaler (1997, 2008) explored the mathematical practices of teachers and students in two different sorts of mathematics classrooms. In one group of classes, the mathematical pedagogy was 'traditional' and the students learned standard algorithms through worked examples and textbook exercises. The other classrooms were characterized by open-ended projects, group work and discussion (Boaler & Staples, 2008). Not surprisingly, Boaler found that generally students learned a form of mathematics that was consistent with the mathematical epistemology and pedagogy of their classroom experiences. In addition the students in the 'non-traditional' classes performed better in a range of assessment tasks and overall they developed more positive attitudes towards the subject and a stronger sense of their own mathematical identity. While the detail is light here, it seemed in short that the experiences of the students in the non-traditional classrooms were akin to the mathematical practices of research mathematicians outlined above.

The studies undertaken by Boaler and Burton were substantially different in terms of their participants - school students and research mathematicians - but offer poignant insights into the ways in which working as a mathematician enhances the potential for learning school mathematics. The characteristics identified by Burton as being the ways of working as a research mathematician encouraged her to plead for schools to adopt such practices in school mathematics classrooms. She hoped that this would improve the learning outcomes for students. Boaler's study indicates that when teachers embrace the characteristics mentioned above there is enhanced performance of students. But, as we noted at the outset of this paper, there is a strong sense that many of the teachers entering school mathematics classrooms may have weak content knowledge, weak pedagogic content knowledge and/or a fear of mathematics. If school mathematics is to be reformed, then we propose that there should be some sense of agency among teachers that will enable them to move forward with their existing knowledge. To this end, we draw on Boaler's notion of agency which she expands from the work of Pickering (1995).

The Dance of Agency

The claims of Burton regarding the working practices of mathematicians and the classroom evidence of Boaler (2003) together seem to make a strong case for considering the learning of mathematics to be like 'working as a mathematician'. Conceptually, this requires engaging in what Pickering (1995) calls a 'dance of agency'. In studying the practices of research scientists and mathematicians he noted that they choreographed a complex routine by which, at times, they drew on their own agency as scientists or mathematicians, and yet at other times they would concede authority to the agency of their discipline and associated community of practice. This is like the interplay between the activity of mathematics and the content knowledge of mathematics that was highlighted above. Rather than seeing the practice or knowledge-base being supreme, it reveals a dialectic interdependence where the mathematician (at any level) requires both to meaningfully and successfully engage in the mathematical enterprise. Likewise, teachers also need to engage in a dance of agency where they appraise and decide when to encourage and support the students' own agency as mathematicians and when to defer to the authority of the discipline (e.g., the requirement to follow a standard procedure or form of presentation). It is worth noting that mathematicians do defer to the agency of the discipline in their practice and it is this authority that is credible in a mathematics classroom. However, in traditional mathematics classrooms the authority usually resides with the textbook and the teacher, both of which are temporal aspects of students' mathematical development and they do not endure as the discipline itself does.

Boaler's (2003) use of the dance of agency in her work illustrates the importance of the learner having a robust and empowering identity in relation to mathematics. Knowing how and when to draw on mathematical ideas to solve problems is a critical part of the dance of agency. Boaler used examples of learners who could not solve tasks but drew on a range of skills, knowledge and collective wisdom in order to solve those problems. This process is akin to that identified in Burton's work with research mathematicians. The practices offered by Boaler and Burton may offer a way forward and out of the quagmire of contemporary school mathematics that is being identified by many external forces.

In the remainder of this paper, we draw on an example taken from a professional development activity that one of us undertook with a group of primary school teachers. We argue that the level of the learners is not the feature of the analysis as we contend this example can be used across all sectors of learning—primary, secondary and preservice/inservice education. Rather, the analysis focuses on the ways of working, which is the significant aspect of the example. These provide an illustration of how learners, in this case teachers, can draw on previous knowledge to work collectively to achieve a common goal. That is, they drew on their sense of agency around particular mathematical ideas and their collective wisdom as a group to solve the task. Collectively the goal is attained but not without considerable input from the learners. The input varies in form and timing, and helps to illustrate the powerful learning made possible when working in ways similar to mathematicians but also having a sense of agency that allows for the legitimate use of learners' understandings that enable the building of deeper understandings. However, as Boaler's work has highlighted, such success is dependent on the learners' sense of identity with mathematics and their sense of agency through which they can 'dance' between the known and the unknown in order to build deeper understandings. It is for this reason we have used this example. After describing and illustrating the mathematical practices of these teachers we will draw on their example to discuss the features of mathematical classrooms that promote the development of robust mathematical identities through an authentic 'dance of agency'. We use this illustrative example to show how the mathematical identity of learners may be constituted through particular practices of mathematics.

The data provided in the following example are drawn from field notes from the professional development activity. The quotes and drawings are those written by the observer and are representative of the discussion made by the participants as no formal recording tools (tape recorders) were used. The data were triangulated with participants so that they are an accurate summation of the interactions in the workshop.

Sum of the Interior Angles of an Octagon: A Working Example

A group of primary school teachers had been working on mathematical problems as part of a professional development day. The participating teachers were engaged in the mathematical tasks in order to better understand their own teaching of mathematics. A standard geometry task was provided that required the teachers to work out the sum of the interior angles of an octagon. There was some discussion as to what an octagon is, and how many sides it has. Once this was clarified, the teachers worked in small groups. In the example we have used here, there were four teachers in the group and they were all relatively experienced teachers.

I have no idea on how to work this out.

Well if you look at it you can divide it into triangles. [T divides octagon into 8 triangles; see Figure 1]. See, there are 8 triangles. Each triangle has got 180º so to work out what the angles are on the bottom of the triangle, you have to work out how many degrees are in the top angle there [draws an arrow to the centre].

Ah, so that is 360° divided by 8

Huh? [unsure of where the figures are coming from]

Well you know that there are 360° in a circle [draws a circle around the centre where the apexes of the triangles meet] and you can see there are 8 triangles making up that circle.

So, 360 ÷ 8 is [some talk on how to work this out, two teachers use pencil and paper for the division] … 45.

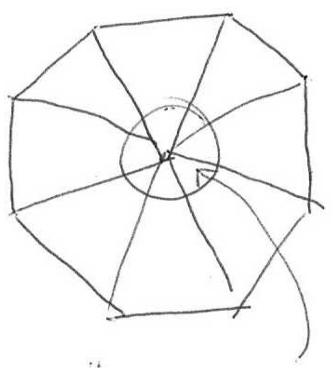

Figure 1. Participant's diagram

OK now what we have to do is work out how big the other angles are. They are the same size so you take 45 from 180 and then divide by 2.

Why?

Well there are two angles [points to the two angles at the bottom of one triangle] and we need to see how big one is.

The discussion continues so that the group identifies the size of one of the interior angles of the constructed triangles as being 67.5º. There is some discussion that it cannot be correct. One teacher commented that she thought it must be incorrect as the leader would not have given them an angle with a half in it. Calculations are checked and the answer is seen to be correct. Someone then suggested that they have to multiply it by 8 so it will not be a "half number" any more.

To this point, the teachers have been drawing on their shared knowledge of the properties of a triangle, in particular, the internal angles of the triangle. There has been considerable sharing of intellectual resources that have enabled the group to move forward. At this point, one of the teachers noticed incongruence between what had been calculated and her knowledge of angle types.

Another teacher in the group comments that it cannot be right as the number they have calculated is less than 90° which would make for a less than 'straight angle' [assumed to mean a 'right angle']. There is some discussion and movement of the shape and then agreement that they have done something wrong.

I know what it is… that is only half of the angle. See look, we have worked out half of the angle; the other part is in the triangle next door.

You're right, so the size of one angle is really double what we found so that makes it 135. And that is bigger than 90 so we must be right now.

Ok, then we multiply by 8 and find out what the total size is.

Someone in the group then multiplied 135 by 8 using pencil-and-paper to come to an answer of 1080.

At this point the group has successfully completed the task of finding the internal angles of an octagon. This is often the end of a mathematics exercise. However, such an approach leads to shallow thinking and not what we would see as working as a mathematician. Much like the task of the scientist, the task of the mathematician is to find generalizations and to prove their results. In this case, all that has been achieved is the answer to a routine problem. To facilitate the moving into the 'working as a mathematician' the leader of the session then asked the group to find patterns.

Once the group has finished, the leader then asks them to find out what it might be for a hexagon and some other shapes. The group goes through a similar process, this time drawing the hexagon, finding the magnitude of the central angle and then the size of each interior base angle. This is then doubled and multiplied by 6. At this point, a woman who had not contributed too much of the discussion interrupts and poses the following:

You know what we are doing… making more work for ourselves. Look at this. You divided the 120 by 2 and got the size of the angle inside the triangle and then you doubled it. We halved and then doubled so we have just done the same thing twice.

The teachers then go on to do two more shapes of their own choosing. The leader then posed the problem to see if they could make a prediction for any shape and how would they do it. The response was that this means they needed to make a formula for the problem.

For us, it is this step that makes this activity more akin to Burton's proposition that schools adopt the practices of working as mathematicians. By seeking ways to make generalizations, the teachers were being asked to think as mathematicians do. As the following section of the field notes suggest, the two groups of teachers being observed used different strategies, one of which was more effective in resolving the generalization task.

Group one made a table for their results. Aside from the triangle which they knew had 180º, they had only made shapes with even numbers of sides so that the table looked like that shown in Figure 2.

Hey, look at that you can see a pattern there. Each time we go up by 2 sides, it gets bigger by 360. That is a square so if we only increased by one side it would be getter bigger by 180º – that is a triangle.

However, this group was unable to move beyond this observation to make a more generalizable statement.

Group two used a similar method and when it came to the discussion at the end of the session during which groups shared their findings, this group explained that they found that the pattern was "increasing by 180º each time a side was added to a shape" but you could not go below one triangle as this was the lowest point. One teacher explained their generalization as follows:

We found that what the pattern is- is that each shape is the number of sides take away 2 and then you multiply by 180º. So if you use a hexagon as the example, you can see that it has 6 sides but if you takeaway 2, you have 4 and then if you multiply it by 180 you get the sum of the interior angles. We thought you could say it like (number of sides minus 2) and then multiply by 180 so that is (n-2) x 180. We checked it out with the others and it worked. So if you use the triangle. It has 3 sides, so that is 3-1 and then times 180 so that is 180 and that is right.

sides	0°	△s.
3	180	1
4	360	2
5		
6	720	4
7		
8	1080	6
9		
10	1440	8

Figure 2. Participant's table

This final part of the activity we see as critical in enabling participants to justify and explain their working processes. Again, as Burton's work indicated, this justification strategy is used by working mathematicians. It made the teachers use metacognitive processes to think through and then articulate their working and thinking strategies.

In the next section we draw on this example to theorize the aspects of 'working as a mathematician' from the combined works of Boaler and Burton. In this section, we identify three key elements to working as a mathematician which are evident in the example cited.

Coming to Understand "Working as a Mathematician"

In drawing on Burton's and Boaler's work, we propose that there are three elements to developing a sense of working as a mathematician. There are the cognitive aspects of knowing mathematics and thinking like a mathematician. Burton draws considerably on the cognitive features of working mathematically. Both Boaler and Burton recognize the importance of the social context within which learning occurs. The pedagogy employed at Railside was strongly influenced by Complex Instruction (Cohen & Latan, 1997; Cohen, Latan, Scarloss, & Arellano, 1999) in terms of organizing the learning environment. Burton draws more closely on the literature regarding communities of practice (Wenger, 1998) to theorize her position, and in doing so, sees that "knowledge and the knower are mutually constituted within these dialogic communities" (1999a, p.132). Collectively the two positions provide a more comprehensive picture of the potential for classroom practice. Finally, the focus of both authors, and this paper, is that of mathematics. This tripartite model – social/cultural, cognitive/affective and mathematics – is represented diagrammatically in Figure 3.

What can be seen in this example are a number of features about working as a mathematician. We use the example presented above to illustrate the notion of

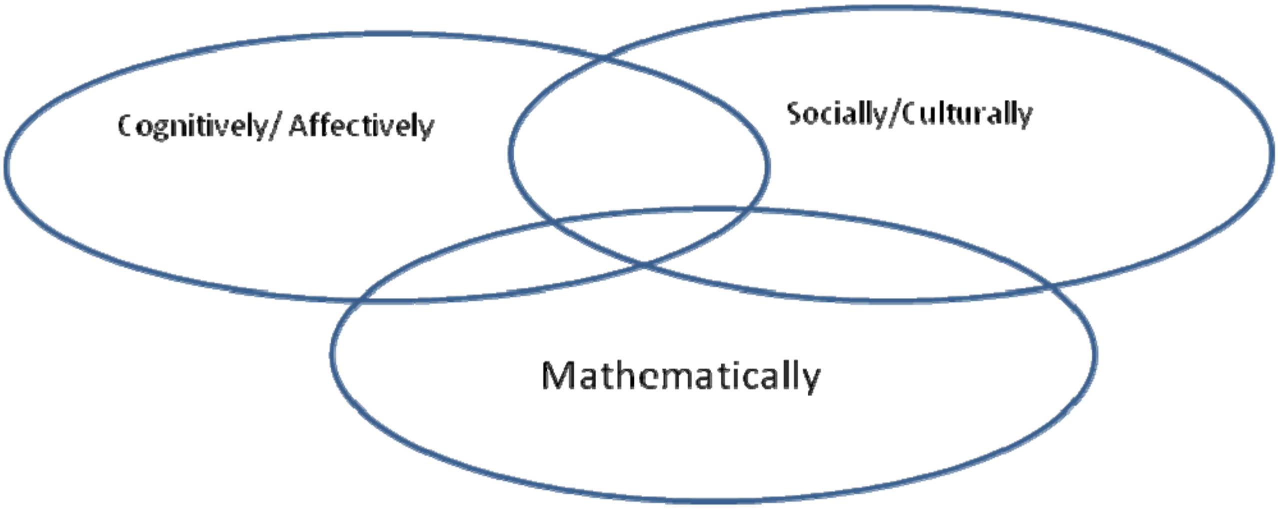

Figure 3. Aspects of working as a mathematician

working as a mathematician and the importance of agency in this process. In so doing, we link this to classroom practice as a means of moving forward the debates on mathematical thinking and learning.

Socially

We define the context within which learning and working is occurring as the social dimension. This includes the ways in which the learning environment is organized along with the social and cultural dispositions that learners bring to that environment. Indeed, the social context of mathematical learning has been widely discussed in the literature and many of the concepts that emerged (e.g., agency) have been dealt with in greater depth elsewhere, but here we want to particularly highlight aspects of the example through which we can see features that enabled the learners to work as mathematicians.

Group Work: Being part of a group and working as a collective enabled the teachers to share their knowledge which is often tacit and not well understood. The sharing and discussion of mathematical knowledge can also be generative and leads to more complete understandings. Furthermore, it can reduce the pressure that individual students can experience in mathematics to memorize and readily recall mathematical rules and formulae and hence, they can devote more attention to mathematical thinking and problem solving. In this example, the teachers did not know the formulae and so they relied on their collective wisdom, which enabled them to fill in gaps in each other's knowledge. Without the input from various members of the group it is unlikely that the collective would have advanced as far with their thinking as was evident in the observations. It is also important to note that this form of collaborative group work is consistent with the practices of mathematicians highlighted by Burton in the studies we reviewed earlier. It seems that when individuals are released from the pressure of having to carry the complete package of relevant knowledge to work on a particular mathematical problem, they are free to engage more fully in the generative mathematical thinking and conceptualizing, and more significant outcomes are possible. This sort of community activity is based on constructive discussion.

Collaborative Talk: The interactions between the participants were focused on the task, and thus enabled them to talk through their observations. When working alone the individual has to undertake the roles of worker and observer either simultaneously or by flipping between the two (or some combination), but in the group context there are opportunities for learners to negotiate, either overtly or tacitly, times of activity and times of reflection and observation. In the example above, having some participants working on the task and others observing enabled the observers to gain insights into the actions and base their discussion on shared and recent experiences. In this case, one of the teachers was able to 'see' that her colleagues were halving and then doubling. Being able to provide this input in a non-threatening way to colleagues enabled the group to move forward in a productive way.

Ethos: The environment established in this session was non-threatening and supportive so that learners could actively engage in the task at levels that met their current needs and understandings. Issues relating to student affect in learning mathematics have received a lot of attention in recent years, and it is clear that there have been real problems for many students when they are stressed and anxious about mathematical activity. The benefits of developing and sustaining a supportive ethos have been documented in Boaler's studies (Boaler, 2002a, 2002b) as enabling learners to participate without threat and hence open opportunities for participation and learning. While teachers have a significant role in developing this sort of learning environment, there is a sense of camaraderie and shared mission when the students can provide this mutual support through operating in a collaborative group.

Agency: In the example presented, the participants seemed to have a sense of agency because they were able to draw on their own understandings of the situation and use these to develop richer understandings that were strongly mathematical. Given their ages and teaching background, it would not have been unreasonable to expect that they recalled the formulae for internal angles of a polygon. However, none of them could remember this formula (which is what we had hoped in planning the activity). Instead, they drew on their preexisting knowledge in ways that enabled them to move forward with the problem, and to ultimately solve it—and to generate their own formula. Being able to draw on existing knowledge to solve the problem in non-traditional ways ensured task completion and also allowed the participants to gain a strong sense of achievement. Their sense of agency was not based on their knowledge, attitude, aptitude or ability to single-handedly complete the investigation, but rather on being able to contribute something to the shared dynamic that emerged as they engaged with the task collaboratively.

Task: A critical dimension to the successful mathematical work of the participants was the task. The design of the task may be seen as quite traditional but the leader deviated from those practices often found in classrooms where rote procedures are applied to a range of questions and little opportunity is provided to develop richer understandings. Extending the task to find the generalization enabled the teachers to develop ways of thinking mathematically and to construct their own formula/generalization. It is important to note that the task was inherently mathematical in both content and process and, as such, it was consistent with the practices of mathematicians (as highlighted by Burton and summarized previously). Moving away from tasks that can be solved through the application of formula that are applied in a rote, lock-step manner is critical in fostering learning environments that encourage deep learning.

Working as a Mathematician

This aspect of the learning environment is very different from the format of the traditional classroom where the learner is often situated as a 'consumer' or user of mathematics rather than a creator of mathematics. The practices of research mathematicians are creative in that their work is to 'create new knowledge' by drawing on their own sense of agency and by working with others in ways consistent with their discipline. While the knowledge developed by the participants in the example was not unknown, it was new to them and, as such, it involved a creative process. Furthermore, the participants drew on their collective mathematical knowledge and developed their thinking within a mathematical framework.

Mathematically

This aspect of working as a mathematician draws on features that can be considered as part of the mathematical content knowledge or the pedagogical content knowledge identified by Shulman (1986). These features are often distinctly mathematical and are what can be seen to differentiate mathematics from other curriculum areas. The features involve mathematical knowledge, but also mathematical practices. Unlike traditional classrooms where rote-and-drill learning, textbook-based exercises and strong teacher direction dominate, mathematicians employ practices that are quite different from school mathematics practices. Indeed, this difference does raise concerns about the 'mathematics-ness' of what occurs in classrooms as we would argue students' contemporary mathematical experiences are not necessarily based on mathematical behaviour. Below we highlight some of the key mathematical practices that were identified in the example.

Identifying Patterns: Creating the table enabled the participants to observe a pattern. Some participants were able to describe the pattern but not the generalization. For others, seeing the pattern through representing the information on the table enabled them to construct the generalization. It is important to note that although most students can be taught to draw a table (and they are in most classrooms), the drawing of the table was not an end in itself, but rather it was a technology to help the participants engage in the mathematical task of identifying a pattern. Of course, teaching mathematical learners to identify a pattern is a much more difficult task than to teaching them to simply draw a table or memorize a set algorithm to see prescribed patterns - it involves less tangible aspects of mathematicians' practices such as insight and perception. But it is these very aspects that make it a rich mathematical experience rather than the dehydrated pseudo-mathematical task that most students experience - the clear and easily defined mathematics that has been carefully programmed, pre-processed and homogenized so all can get the right answer. Perhaps in our attempts to make mathematical knowledge more accessible to students we have kept the knowledge but lost the mathematical behaviours, and in the process the mathematical experiences of the classroom can no longer be regarded as 'mathematical'.

Constructing Generalizations: Another integral part of working as a mathematician is about making the generalizable statement. The insight to see and construct the generalizable statement, and to be able to state it clearly, is an important mathematical practice. In this case, the development of a formula for the interior angles of a polygon was part of the task. Unlike traditional mathematics classrooms where the generalization (i.e., the rule) is often the starting point and learners are encouraged to practise on examples, this learning enabled the participants to generate their own generalization. At this point it is worth noting that the participants in our example did not perhaps take the final mathematical step of proving their result. Indeed, they were not far away from it and if they could have drawn the construction lines shown in Figure 4, they may have been able to complete their 'proof'.

We do acknowledge that it is also an important activity to apply and use known mathematical rules, but this is relatively straight-forward and simple, and perhaps inherently less mathematical, than the engaging and creative task of constructing generalizations. Boaler's research suggests that mathematical knowledge developed in this way is more robust and accessible for learners than prepackaged formulas that are memorized as preordained facts.

Using a Simple Example to Test the Hypothesis: Once a potential generalization had been developed, the participants applied this to a simple example (the triangle) to check its validity. In this case it worked so the generalization appeared valid to the participants. They also applied the generalization to the examples that they had worked out (and recorded in the table) to check that the generalization was valid in other examples. It seems that in a traditional classroom there is little scope for conjecturing and hypothesizing, as the route to the answer is known and the task of the student is to travel the prescribed and clearly structured route to the known answer. Dead-ends and time-wasting side tracks are thus avoided and the journey is quick and efficient. However, this is not consistent with mathematical practice (as outlined previously) and hence its place in the mathematics classroom deserves consideration.

Identifying Limits: Finally, part of working on a mathematical task is being able to determine parameters. Of course, it is not a necessary mathematical action if all the mathematical tasks faced are bounded and clearly defined. As noted by one group, the limit in this activity was that the shape had to have three or more sides if the generalization was to work. Again, this was a relatively innocuous observation, but an integral mathematical process that can easily be lost in the process of sanitizing authentic mathematical tasks for the classroom, thus diminishing the true mathematical thinking required by students.

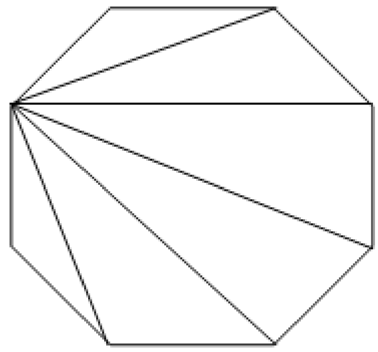

Figure 4. Octagon with construction lines

Cognitively

Drawn from Burton's work are aspects of cognition, affect and other constructs of the internal features of working as a mathematician. Rather than trying to delineate these various dimensions, we have accepted their inter-connectivity and tried to note them as they arose in the example. This approach is similar to that undertaken by Burton and means we do not apply a pre-determined theoretical framework. Historically, affect and cognition have largely been studied independently, or at least as separate concepts, but here we have not made that distinction. What we have done is identify particular features of cognition and dispositions that are part of the learners' ways of approaching the tasks, particularly as exemplified in the account above.

Thinking Styles: As shown in the example used, the learners engaged a range of thinking styles that included verbalization, drawing illustrations, and the use of tables to arrive at insights about the problem, the mathematics, and ways to solve the problem. These various thinking styles enabled the group to gain insight into the problem, and whether they would have been successful with a uni-dimensional approach is debatable. Indeed, drawing on a range of thinking styles– visual, analytic and conceptual - was identified by Burton (2001) in her study of mathematicians, and we can see how most of the participants in our example used a composite of these styles. The use of a variety of cognitive approaches is valued in the mathematics community because it is integral to, and enhances, the mathematical endeavour.

Insight/Intuition: Burton's (2001) mathematicians referred to the 'light being switched on' which enabled them to see what works and what does not work without being overtly aware of how they gained such insights. Barnes (2000) also studied 'aha moments' in school students' mathematical experiences. While these sorts of expressions seem to be common in general conversations about mathematics learning, they are relatively absent in the research literature, and yet both Burton and Barnes saw them as critical affective/cognitive components of doing mathematics. We cannot do justice to this topic in this example, but we do want to flag *insight* and *intuition* as aspects that require further research.

Making Connections: It can be seen from the example that various elements of mathematics have been linked together to form a coherent whole. Burton (2001) argues that it is akin to fitting the pieces of the jigsaw together. What can be seen in this example is how the teachers have drawn on various aspects of mathematical knowledge, in particular their knowledge of triangles, and have pooled this knowledge in order to come up with a deeper appreciation of mathematical understanding. This making of connections leads to a more robust and inter-connected mathematical knowledge by which mathematics is not seen as a collection of isolated procedures and concepts. In general, it seems that a more holistic and related mathematical understanding is not developed in mathematics classrooms because experiences are based around learning small, bite-sized conceptual chunks that are rarely stitched together into a broader conceptual framework. This is often exacerbated by the teach-test-and-forget program mentality that discourages applying a range of mathematical concepts to the solution of a problem. Thus, for mathematics learners to engage in the critical cognitive activity of making connections, they need problems and tasks that inherently demand more than one mathematical idea to solve.

In this section we have discussed the example presented earlier vis-à-vis the activity of working as a mathematician. To do this, we briefly explored the activities of the participants under quite a few themes. In the next section we look at the example at a more macro level, in particular noting the choreography of agency between their mathematical identities and the discipline of mathematics.

Identity and the Dance of Agency

What becomes possible to see through this example is that the learning situation draws considerably on those aspects of working as a mathematician as identified by Burton's work and on the aspects of classrooms and teaching identified by Boaler's work. Boaler's work has been particularly powerful in illustrating the importance of agency and identity. When we consider the activity identified in this paper, we recognize that the three features – social, mathematical and cognition – are critical variables in the provision of quality learning opportunities. If we are to emerge from the current demise in mathematics education identified at the start of this paper, then reforms are needed to enable change from the current, traditional practices to ones which are more empowering for learners. This requires a shift not only in pedagogy and curriculum but also in the dispositions of learners. As noted by Zevenbergen (2005) many of the current practices in school mathematics create particular mathematical habitus which are far from empowering for learners and indeed encourage disengagement with the discipline.

This example and our analysis of that practice highlight some of the features that foster the characteristics of working as a mathematician that have been identified through the combined work of Burton and Boaler. However, in this final section, we want to draw more constructively on Boaler's use of Pickering's (1995) notion of a 'dance of agency'. For Boaler this construct is critical, as it enables learners to draw on their mathematical understandings, to build on what they know, and to construct deeper understandings. This is one of the fundamental premises of much mathematical learning but it is improbable in many of mainstream classrooms due to the pedagogies being implemented. As shown in the Queensland School Longitudinal Reform Study (Education Queensland, 2001), the teaching of mathematics in schools is the most poorly taught area of school curriculum and dominated by shallow teaching approaches with little scope for students to engage substantially with ideas and deep learning. The example here provides some insights into the ways in which a commonly used activity can be adjusted to allow for depth of learning. However, as Boaler's work highlights, learners must feel some sense of agency to be confident to draw on other forms of knowing in order to solve problems.

In the example provided, we note that the first comment provided by a participant was "I have no idea of how to work this out". Such a comment is not a surprise for many mathematics educators and has been well documented as an outcome of the teaching of school mathematics. Yet, as the activity unravelled, the engagement and success of the participants illustrated the importance of a number of characteristics Burton identified among the practices of research mathematicians who strongly identify with mathematics. We suggest that the activity, including the way it was organized and presented to participants, enabled them to engage with the problem in order to solve it. It seems that allowing the participants/learners to engage in a collaborative group and to draw on pre-existing concepts, which they knew were robust, enabled them to engage successfully with the task. Further, it was critical to the dance of agency that the participants felt confident to draw on their existing knowledge to build deeper mathematical understandings. The participants appeared to be confident in their knowledge and they identified strongly with the concepts encountered in their teaching of primary mathematics, including the properties of triangles in particular, and polygons in general, along with the types of angles. They then drew on this knowledge to solve a more complex problem -

something that they did not encounter in their teaching in the primary school, and hence was unfamiliar to them.

We contend that traditional classrooms would have fostered learning activities around the application of a formula for calculating the sum of interior angles. In this example, the participants could not remember this formula (and it was not provided) so they needed to rely on their existing knowledge, the collective wisdom of the group and a sense that they could solve the problem. This sense of agency - where not only could they rely on their own knowledge in a legitimate sense, but also the collective knowledge across the group – enabled them to gain a sense of learning and achievement through the completion of the task. We contend that such practice is far more enabling and develops a strong sense of agency and identity with mathematics.

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