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Undergraduate students' conceptualization of elementary row operations in solving systems of linear equations

Benjamin Tatira [1*]

[1] Faculty of Educational Sciences, Walter Sisulu University, Mthatha, SOUTH AFRICA

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Abstract

The concept of systems of linear equations (SLEs) is fundamental and core in linear algebra, a subject, which has many applications in a number of disciplines. Gaussian elimination is a versatile method, which can be used to solve almost all types of SLEs by using row-reductions. This study focused on exploring undergraduate students' conceptualizations of elementary row operations (EROs) as a means to solve SLEs. The purpose of this study was to explore undergraduate students' conceptualizations of row reductions and their applications to the solutions of systems of equations. The perspectives of the action-process-object-schema theoretical framework were used in analyzing data and discussing the findings. To explore the students' conceptualization of EROs, a descriptive research approach was followed. I considered a case study of 131 students registered for a mathematics for educators course, where linear algebra was one of the topics. The findings revealed that students attained the action conception of reducing a system with unique solutions but had challenges reducing and interpreting solutions to a system with non-unique solutions. The latter row-reduction implored process and object conceptions especially when variable elements in the augmented matrix were involved. As students find the learning of linear algebra difficult, this study contributes to the debate in literature on how to improve its teaching and make suggestions on the ways make more effective the learning of linear algebra.

Keywords: Gaussian elimination, elementary row operations, augmented matrices, APOS theory, system of linear equations, linear algebra, unique solution, infinitely many solutions

INTRODUCTION

South African students first encounter linear algebra at tertiary level. In secondary school, they are limited to basic methods of solving simultaneous equations using the elimination, graphical or substitution methods. The national curriculum and assessment policy statement states that starting in the tenth grade, students should be taught to "solve simultaneous linear equations in two unknowns" (Department of Education, 2011, p. 22). The solving of systems of linear equations (SLEs) in two variables forms the foundation for linear algebra, which starts in earnest in tertiary education. Linear algebra is concerned with manipulating SLEs and their representations in the vector spaces using matrices (Mutambara & Bansilal, 2018) and one of the integral topics in undergraduate and postgraduate mathematics. It is the study of planes and straight lines, mappings and vector spaces that are required for linear transformations. The concept of linear systems is a core and fundamental part of linear algebra (Karunakaran & Higgins, 2021) and a basis for studying further concepts of linear algebra such as linear transformations, characteristic values, rank and vectors (Dewi et al., 2021).

Systems of equations with three or more unknowns are the domain for undergraduate education in South Africa. In undergraduate mathematics, students study linear algebra as a means to solve SLEs using the matrix method. This study is confined to this aspect of linear algebra. This creates a transition from secondary school concepts and the more advanced and abstract concepts of linear algebra. The introduction of the matrix method commences with students exposed to the meaning and operations of matrices. Thereafter, properties of determinants, matrix inverses and elementary row

✉ btatira@wsu.ac.za **(*Correspondence)**

Contribution to the literature

- This study revealed a greater need for more problems, which provide students with the opportunities to scrutinize SLEs geometrically and algebraically for all types existing of solutions for all orders.
- The concept of EROs is often reduced to rules without meaning yet it plays a vital role in computing inverses and solving SLE using GE.
- The learning and teaching of SLE by GE should be problem-based as a means to help students develop the necessary mental structures of both the concepts of EROs and solving SLE.

operations (EROs) are learnt and all building up to the solution of SLEs.

The term system of equations refers to conditions, where two or more unknown variables are related to each other through an equal or unequal number of equations. Therefore, an SLE is an array of linear equations of the same unknowns, that is, a system composed of m linear equations $L_1; L_2; L_3; L_4; \ldots; L_m$ in n unknowns $x_1; x_2; x_3; x_4; \ldots; x_n$, which can expressed, as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \quad (1)$$

The terms a_{ij} and b_i are constants and the a_{ij} is the coefficient of the unknown x_j in, the equation L_i. The term b_i is the constant of the equation L_i. A solution of the system in Eq. (1) are values for the unknowns, which are solutions to each of the equations in the system. The secondary school methods of substitution and elimination become cumbersome, inefficient, unsustainable and unreliable to solve general system in Eq. (1). The challenge of solving Eq. (1) is the genesis of linear algebra, which provides a systematic way to effectively solve general SLEs of any number of unknowns. The system in Eq. (1) can be transformed into matrices composed of coefficient, variable and constant matrices. The system in Eq. (1) with m equations in n unknowns is associated with the following matrices: the coefficient matrix of the system $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$, the variable matrix $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and the constant matrix $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$. The augmented matrix of the system is given by $\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$, which is composed of the coefficient and the constant matrices separated by a vertical line. One of the ways to solve an SLE is by working with its augmented matrix rather than the equations themselves nor the individual coefficient, variable and constant matrices. Any ERO on the augmented matrix is equivalent to applying the corresponding operation on the original system.

The common techniques used to solve SLEs using the matrix method in an introductory linear algebra course are the Cramer's rule, the inverse method and the Gaussian elimination (GE). The first two methods are hinged on the concept of determinant and yield a unique solution when the determinant is non-zero. If the determinant is zero, the two methods are inconclusive. Moreover, Cramer and inverse methods cannot be used when the solution to SLE is infinitely many or does not exist. The inapplicability is based on the limitations of evaluating the determinant of a non-square coefficient matrix when there are more equations than unknowns, giving rise to free variables.

GE method of solving SLEs is versatile in the sense that it can determine the solutions to an SLE whether they are unique, are infinitely many or do not exist. GE method does not rely on the determinant but by its nature seeks to eliminate all known variables except one. This is achieved by reducing the coefficient matrix to row echelon by executing EROs. In linear algebra there are three rules for reducing a matrix to echelon, and these are

(a) swap any two rows,

(b) divide or multiply a row by a non-zero constant, and

(c) subtract or add a multiple of one row to another (Rindu, 2017).

EROs can also be used as an alternative method to evaluate the inverse of square matrices. Thus, EROs are taught firstly to reduce matrices to echelon form and secondly as an application to the solution of SLEs.

Statement of Problem

Linear algebra is one of the first abstract mathematics courses that students encounter at university, which has many applications in sciences, technology, engineering and economics (Harel, 2017; John et al., 2016; Plaxco & Wawro, 2015; Tai, 2020). Many problems in fields of fields mentioned above can be modelled by linear algebra (Arnawa et al., 2019; Liu, 2015). In its basic state, linear algebra is used to solve SLEs in systematic way. In understanding SLEs, the concepts of EROs, inverse, determinant and echelon matrices are essential concepts, which need to be emphasized. Thus, lack of adequate understanding of the essential concepts in SLEs negatively impact students' achievement (Arnawa et al., 2019). Hence, literature reports that students grapple to come to terms with the key concept of linear algebra courses (Harel, 2017; Kazunga & Bansilal, 2017; Mutambara & Bansilal, 2018, 2022; Stewart & Thomas, 2010; Tai, 2020). Tai (2020) posits that part of the difficulties may be attributed to the transition from elementary to advanced mathematics. This study intends to give an insight into the undergraduate students' hierarchical development of knowledge as they learn solving SLEs using the matrix method. This study focused on exploring undergraduate students' conceptualization of EROs as a means to solve SLEs by GE method. Such a study has not been done before and literature review in the next section elaborates related studies to this. The purpose of this study was to explore undergraduate students' conceptualization of EROs as a way to solve SLEs. The research questions for this study are, "What are undergraduate students' conceptualization of EROs in solving SLEs?" and "What are the undergraduate students' knowledge levels when they learn EROs?"

LITERATURE AND THEORY

In mathematics some concepts are learnt not as end in themselves but as a means to the end. In this regard, Kazunga and Bansilal (2020) looked into in-service teachers' understanding of the pre-requisite concepts of determinants and inverses as a means to solve SLEs using the Cramer and inverse methods. Their findings revealed that students could not solve SLE properly because they struggled with the pre-requisite concepts of determinants and inverses. Conversely, where students have sufficiently mastered the pre-requisite concepts, they can use the knowledge in further transformations. As a result, Kazunga and Bansilal (2020) explored students' understanding of determinants and inverses initially, followed by the students' understanding of the application of these concepts in solving SLEs using the obvious inverse method. In the same vein, students' lack of foundational knowledge might trifle full understanding of a given concept, similar to lack of pre-requisite concepts. On a closer inspection of the misconceptions and errors committed by 73 in-service teachers registered for mathematics teaching degree in the concept of linear independence, Mutambara and Bansilal (2022) discovered that students' errors were mostly foundational. The students erred in processes of row reductions and interpretation of solutions to SLEs. Consequently, the foundational errors had the negative effect on the conceptual understanding of linear independence. The authors suggested granting more teaching time to these in-service teachers since they operated under constrained time frames during their study periods. In another study, Mutambara and Bansilal (2018) revealed that in-service teachers struggled to understand vector subspaces due to non-encapsulation of pre-requisite concepts. The students missed concepts of sets and binary operations, as well as the role of counter-examples in proving that a set is not a subspace. Full schema development of a mathematics concept depends on the coherent collection of actions, processes and objects and other related pre-schemas, that is, according to the action-process-object-schema (APOS) theory (Arnon et al., 2014). Furthermore, Sepideh and Michael (2010) revealed that emphasis on the pre-requisite concepts on their own did not help second-year undergraduate students when the teaching approach procedural. In that study, students who were taught how to find the basis for a vector subspace using the matrix manipulation method still struggled to understand the construct of basis, which made further progress even more difficult. Detached emphasis on matrix processes on their own did not help students understand the concept of basis. The stern construction of the embodied and visual ideas based on Tall's (2004) three worlds was suggested as a possible remedy.

If students' hierarchical development of knowledge in linear algebra remains basic, they struggle to make sense of further linear algebra concepts. This is posited by Ndlovu and Brijlall (2015) who found that undergraduate students operated at the action or process stages of APOS theory. Moreover, they observed that the schema for basic algebra and real numbers were not robust, which militated familiarity with terminologies and notations. To counter this, the authors suggested revising the theoretical decomposition of matrix operations so that students were be able to make the required mental constructions of linear algebra concepts. On the other hand, Ndlovu and Brijlall (2016) investigated students' mental constructions as they learn the concept of determinants but without the application thereof. They discovered that most students conceptualized determinants procedurally and operated on the action or process stages of APOS theory. In both studies by Ndlovu and Brijlall above, students failed to attain the object mental structures, which inadvertently is the goal of all mathematics teaching (Voskoglou, 2015). The literature sources cited above were framed on APOS theory alone or jointly with the Three Worlds by Tall (2004). APOS is the same theoretical framework engaged in this study and it is described in the next paragraph.

APOS theoretical framework is a pedagogical approach and evaluative tool that is used to study the learning of undergraduate mathematics concepts (Asiala et al., 1996). According to Dubinsky and MacDonald (2001) students understand a mathematics concept through construction of knowledge called reflective abstraction and interaction with peers. The pedagogical approach for APOS is the activities-class discussion-exercises teaching cycles whereby a concept is developed through repetitive group-based activities before and after class, with a whole class discussion in between. This is achieved by instructor-researcher proposing a genetic decomposition (GD) of the concept to guide the activities. A GD describes sequence of tasks that demand mental constructions that students ought to attain as they learn a mathematics concept (Altieri & Schirmer, 2019). In this study, APOS theory was used as to evaluate undergraduate students' mental constructions of EROs as a means to solve SLEs after traditional teaching of the concept, as discussed by Arnon et al. (2014). The mental constructions are categorized as hierarchical growth of knowledge starting at actions, processes, objects and schemata levels.

The action conception of a mathematics occurs when students can perform step-by-step transformation and each step is externally connected to the next. Students cannot skip a step and each step is externally motivated. Students ordinarily start learning a concept at the action level. When students can reflect on an action and can do the same transformation mentally, they would have interiorized the action into a process. At this stage students can predict outcomes and work in reverse. When a student can view a transformation in totality whereby they realize that actions and processes can be applied to this totality, they would have encapsulated the mathematics concept into an object. With an objection conception, students can determine the applicability of an algorithm to a problem situation. Finally, a schema is a coherent collection of actions, processes, objects and other previously held schemata. New schemas are created as new knowledge is constructed through assimilation and accommodation.

EROs categories of the growth knowledge are given in the next paragraph, which is the formulated GD. The two categories are given separately as conceptualizing EROs and the application of EROs in solving SLEs.

Pre-Requisite Stratum: EROs

Pre-action: Individuals can transform an SLE into matrix format and vice-versa.

Action: The individual executes EROs to reduce a given matrix of any order to echelon or reduced row echelon. The steps are calculated sequentially an individual is bound to the steps related to the procedure. Ideally, all matrices of any order are reducible to echelon.

Process: Each step carried out implicitly and an individual can imagine carrying out the processes mentally. The individual can predict EROs that results in an upper triangle matrix without having to go through each step explicitly.

Object: An individual can reduce a matrix of order $m \times n$ by conceiving EROs as a totality upon which further actions and processes can be carried out. The properties of EROs can be explained and established because they are recognized entities, which can be manipulated and transformed. For example, an individual can reduce a matrix with variable elements.

Application Stratum: Solution of SLE Using Gaussian Elimination

Action: An individual performs EROs of an augmented matrix to solve an SLE.

Process: An individual can perform EROs mentally and predict the nature of solutions of an SLE.

Object: An individual can apply ERO to solve an SLE for any type of solution including those with variable elements. They use EROs to find the type of solution for a particular SLE. When presented with an SLE, where there are more unknowns than equations, an individual automatically perform EROs to find solutions involving free variables. For unique solution to an SLE, students realize they can also use Cramer or the inverse method.

METHODOLOGY

To explore the students' conceptualization of EROs, a descriptive research approach was followed. The usage of a descriptive research approach is "best suited to examining and trying to make sense of a situation or event *as it currently exists* in the world" (Leedy & Ormrod, 2021, p. 174). By focusing phenomena on their natural settings, descriptive research does not involve modifying a situation under investigation, nor does it intend to determine cause-and-effect relationships. I adopted the qualitative case study research design, and the main assumption is that phenomena is studied as a closed system (Creswell & Creswell, 2018). The author taught linear algebra to third-year secondary Bachelor of Education degree students majoring in mathematics and a science subject. The participants consisted of 131 students doing a mathematics for educators course, where linear algebra was one of the topics. This was the students' very first encounter with linear algebra. Of these, 80 were male and 51 were female. Data was collected through a formal test written individually by all students. The selection of the class was based on the fact that the researcher was the instructor, also a norm in classroom-based research. Only two tasks were selected for this study from the test. The two items were:

1. Use Gauss-Jordan elimination to solve following systems of equations:

$$\begin{aligned} -3y &= -6 \\ x - 2y - 2z &= -14 \\ 4y - x - 3z &= 5 \end{aligned} \tag{2}$$

2. Given the system

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned} \tag{3}$$

Find the value of a does the system have

(a) one solution

(b) infinitely many solutions, and

(c) no solution.

The first item was explicit on the Gauss-Jordan elimination but EROs played a key role to reduce the augmented matrix to row canonical form. However, the second item was silent on the method so that as part of schema development, students could figure out on their own that only GE method is applicable when all the three types of solutions are under consideration. The items were meticulously selected in order to create opportunities for students to construct the required mental constructions as indicated in GD (Ndlovu & Brijlall, 2015). Data analysis involves paying particular attention to students' reasoning reflected in the written responses (Radu & Weller, 2011). After the students wrote the test, a further eight students were purposively selected from the initial 131 for the semi-structured interviews. The purpose of the interviews were to examine students' conceptions and application of EROs more closely (Oktac et al., 2019) and to corroborate the initial analyses of the written responses. The interview transcriptions formed the second data set for this study. Normal ethical consideration as to any research were adhered to. For confidentiality, the participants' responses' were labelled X1, X2, up to X131. With confidentiality, only the researcher knew the individuals who provided specific responses. The participants' informed consent was sought at the start of the study, and they were told they could withdraw at any point of the research process if they felt threatened and unsafe.

The author marked the written responses and analyzed them. The responses were initially coded as *blank (B)*, *wrong (W)*, *partially-correct (P)*, and *correct (C)* and the frequencies were recorded. After the ordinal categories, inductive content analysis of written responses was done to assist in identifying different components of APOS theory in all of the correct, incorrect and partially-correct solution strategies. In addition, the interview transcriptions were analyzed thematically so that emerging patterns in the data were identified and discussed. After analyzing both the written and interview responses, meaning was assigned to them and the level whereat students operated was described in terms of APOS theory. The researcher compared the failure or success of students on the two tasks with specific mental structures they may/may not have attained and tried to explain the differences (Dubinsky & McDonald, 2001).

RESULTS

The frequencies for both items are given in **Table 1**.

From **Table 1**, similar experiences for both items were observed for blank and wrong categories but immense differences emerged in partial and correct. The explanations for the performances for each item are given separately in the following sections.

Table 1. Frequencies of ordinal data of written responses

Response	Item 1 frequencies	Item 2 frequencies
B	9	12
W	11	15
P	58	102
C	53	2
Total	131	131

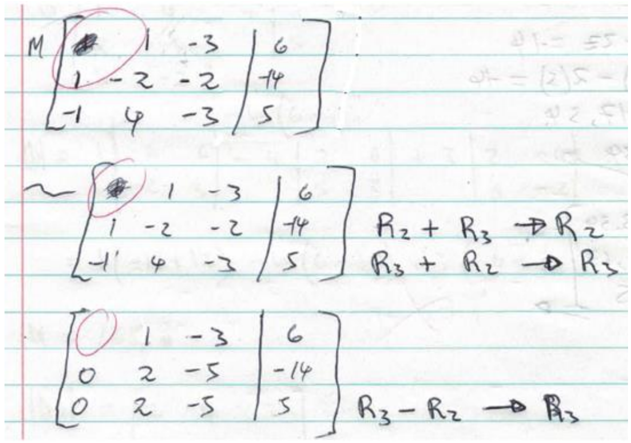

Figure 1. Flawed transformation to an augmented matrix by X14 (Source: Author's own illustration)

Item 1 Analysis

In this item, students were required to perform about eight EROs in order to reduce the augmented matric to row canonical and then determine the values of the unknowns.

Application of three rules of EROs

Firstly, nine students skipped this question entirely without rendering an attempt. Six students indeed attempted to transform SLE to matrices in order to perform EROs but were not precise in doing so. They did not realize that the order of the variables were jumbled, hence they took the coefficients in the order given as illustrated in **Figure 1**.

This denotes that some students operated at the pre-action conception. X14 could not append a zero a_{11}as expected. He also proceeded with EROs but with a null on the first pivot. The interview with X128 revealed also that students were not observant on the order of the variables:

Researcher: Can you see there is no coefficient of x?

X128: Zero is the coefficient of x.

R: Where did you write it?

X128: But I do not know why it is not there.

R: Who should know? Again, what's wrong here?

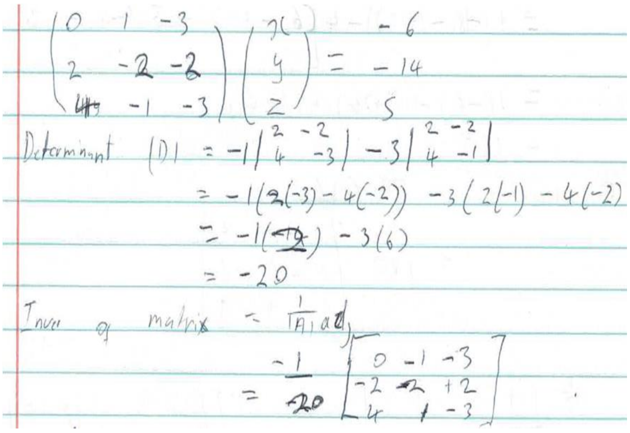

Figure 2. Multiple errors made in a wrong method by X118 (Source: Author's own illustration)

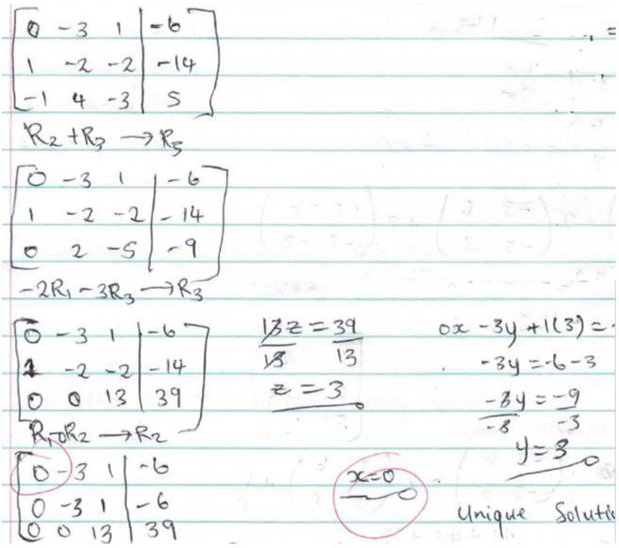

Figure 3. Failure to reduce to a Gauss-Jordan format by X28 (Source: Author's own illustration)

X128: The coefficient of x should be here on column 1.

R: Can you see I jumbled the variables?

X128: Yes and I missed that.

R: So it's all wrong.

X128: Yes.

Student X118 attempted to use the inverse method but failed to transform SLE into matrices and again failed to find the adjoint (as shown in **Figure 2**). If X118 was well conversant with EROs, she could have used them to find the inverse of the matrix.

Three more students transformed SLE correctly into matrices form but did not adhere to the format of Gauss-Jordan. By failing to do a row-swap, they performed row-reduction with a leading zero on the pivot, albeit with correct EROs. The response by X28 in **Figure 3** illustrates this anomaly. By failing to have a leading one in the pivot, X28 lost the solution of x, albeit the $y-$ and $z-$ solutions being correct. X104 and X127 made the misconception of getting the trivial first column. Moreover, the reduced matrix was not in row canonical form. Thus some students' action conception was not robust.

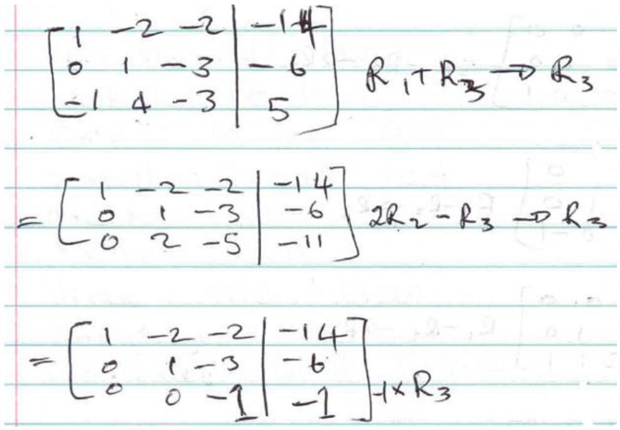

Figure 4. Computational error by X18, which spoiled rest of solution (Source: Author's own illustration)

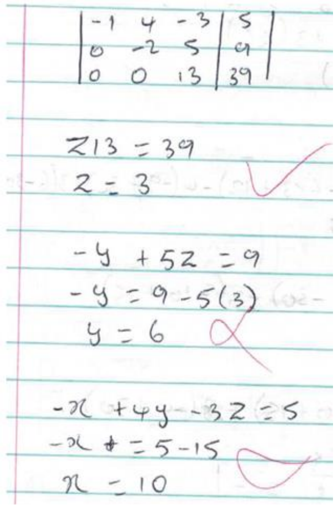

Figure 5. Non-systematic errors in row & back-substitution (Source: Author's own illustration)

The highest number of students in this item were in the category for partially-correct responses, with a 44%. The majority of these students encountered computation difficulties to get correct EROs. For example, X18 made a mistake of saying $(-14) + 5$ gives -11 as shown in **Figure 4**.

X18 duly went on to find the reduced row echelon but the final answers lacked precision. Some students like X15 were lost in simplifying fractions that arose in the process of row-reductions.

Besides, X104 who presented an incomplete solution, the rest made non-systematic errors, hence eight of them only managed to get $z = 3$ but with wrong $x-$ and $y-$values. A further four students coped with EROs to obtain the correct z and y values but incorrect x. This implied that the process of back-substitution was flawed. X41 had both row-reduction and back-substitution errors in his solution as shown in **Figure 5**.

Type of errors

Another common error observable in **Figure 5** is that of not reducing to row canonical form. Students thought since they could find the required unknown values after reducing to echelon form, there was no need to proceed to reduced row echelon form. This was in struck contrast to the instruction in the question. The dialogue with X130 illustrates this scenario:

R: The question was saying Gauss-Jordan. Do you know it?

X130: Yes.

R: Do you know the difference between Gauss and Gauss-Jordan elimination? Perhaps up to now you do not know!

X130: Both lower and upper triangles must be zeros for Gauss-Jordan.

R: Is that what you did there?

X130: No.

R: Why?

X130: I did not take that serious because the answers came out with upper triangle only.

Finally, about 40% of the students displayed evidence of conceptualization at the object level. X55 had the ingenuity of avoiding working with fractions by expressing row-reduction rules like $13r_2 - r_3 \rightarrow r_3$ instead of reducing all the pivots to one. The dialogue with X44 below confirms students' uneasiness to work with fractions. Together with those who went ahead to work flawlessly with fractions, these students understood concept and application of EROs in solving an SLE using Gauss-Jordan elimination. The fully-correct solution for X55 is shown in **Figure 6**.

Even though the pivot is non-unitary, X55 managed to reduce SLE to reduced row-echelon and obtained the correct solutions by dividing by the figure in the pivot. To show interiorization, X55 performed row-swapping without actually showing it. The dialogue with 44 alludes to this:

R: Why did you swap the rows?

X44: Because there is a zero on the pivot.

R: Is it a problem if there is a zero there?

X44: Yes because we must reduce to Gauss-Jordan.

R: But here; why did not you divide by three since it is Gauss-Jordan. We must have a 1 there, must not we?

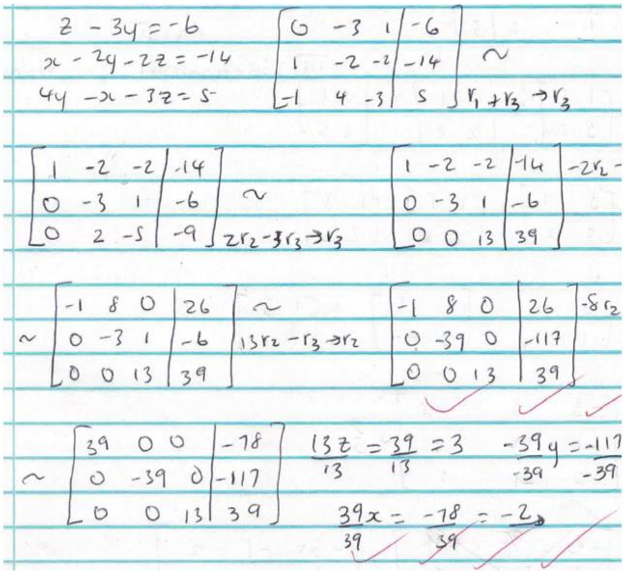

Figure 6. A fully correct solution, which avoided fractions (Source: Author's own illustration)

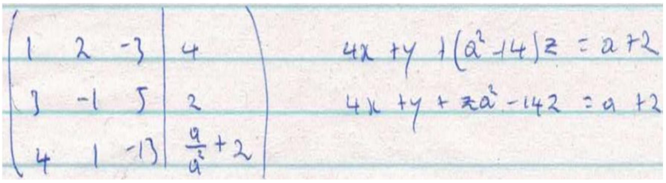

Figure 7. Incorrect augmented matrix as only written work submitted (Source: Author's own illustration)

X44: I was afraid of working with fractions.

R: With a 1 it's going to be easy, and you get 0 in the lower part.

X44: But those nasty fractions Sir!

Correct solution indicates that 53 students learned concept of EROs and its application in solving SLEs.

Item 2 Analysis

This item called for students to realize the usage of GE or Gauss-Jordan as the only method to determine no solution or infinitely many solutions to the given SLE with variable coefficients.

Method-Type of solving SLEs

Results showed that 12 students did not attempt the question and five of those just wrote the augmented matrix only and stopped. X22 failed to get the correct matrix form of SLE as shown in **Figure 7**.

According to GD, the evidence above represent an un-attained pre-action to the solution of SLE. On top of that, 16 students made attempts to solve SLE, but their efforts were in vain. X125 tried to use the Cramer's rule and went ahead to find the determinants required in the Cramer's as shown in **Figure 8**.

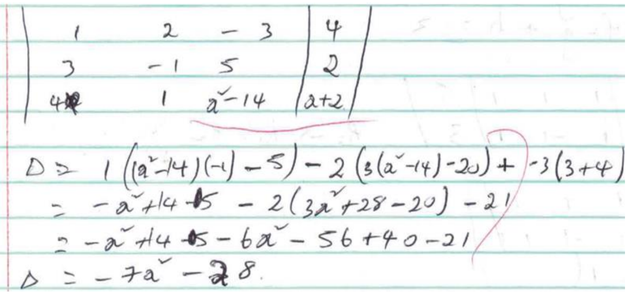

Figure 8. Inappropriate use of Cramer's rule by X125 (Source: Author's own illustration)

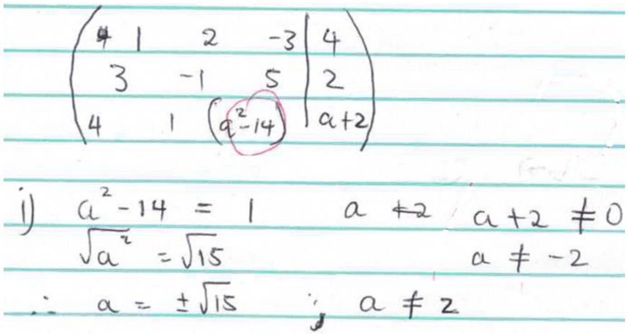

Figure 9. No evidence of use of EROs to solve problem by X16 (Source: Author's own illustration)

This student did not realize that only GE is applicable to this question and with it there is no need for the determinant. Coupled with this, six students did not perform row reduction to the augmented matrix as shown in **Figure 9**. Instead, X125 attempts to find the determinant. Without doing row reductions, then the students were not cognizant of their role in solving problems of this nature or they were not confident to reduce matrices with unknown coefficients.

X16 went straight to find the values of a for the three cases without reducing the augmented matrix first. It was not a case interiorization because the equations used to evaluate a were incorrect. The interview with X42 revealed the same weakness:

R: I can see you changed the equations to matrices. But where are the row operations?

X42: I was trying to solve it Sir but hey …

R: Solve it how?

X42: I was trying to find the value of a?

R: Without reducing to Gaussian?

X42: I never thought I would …

R: Do you know we look at the last row to find the unique, no or infinitely many solution?

X42: Yes yes, I know.

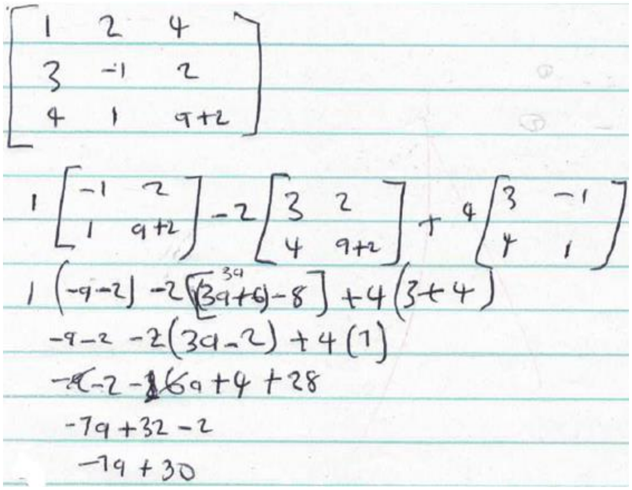

Figure 10. Finding determinant by expanding along row one by X21 (Source: Author's own illustration)

R: For example if it is zero zero, zero, and zero; what is the nature of solutions?

X42: They will be many.

R: And if it's zero, zero, a number and another number?

42: One solution.

R: Yes. And if the last row is zero, zero, zero and a number?

X42: No solution.

R: So where are the row operation to reduce to upper triangle?

X42: I never did that Sir.

Furthermore, six students got the wrong solution by computing the determinant of the 3×3 coefficient matrix as shown in **Figure 10**. However, after getting the expression of determinant, they hit a dead end. Among them, X127 attempted to calculate the inverse of the coefficient matrix using the adjoint method, but to no avail. Students who were not comfortable performing EROs with variable elements decided to find ways to replace a with a constant value. For example, X106 used $a = 1$ to obtain matrix $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 1 & -1 & 5 & 2 \\ 4 & 1 & -13 & 3 \end{array}\right)$ and then did EROs. When probed on the origin of $a = 1$, she said:

R: But there is a problem here. Where did you get this: $a = 1$?

X106: I said $a = 1$ and after I was going to substitute here.

R: Who told you $a = 1$?

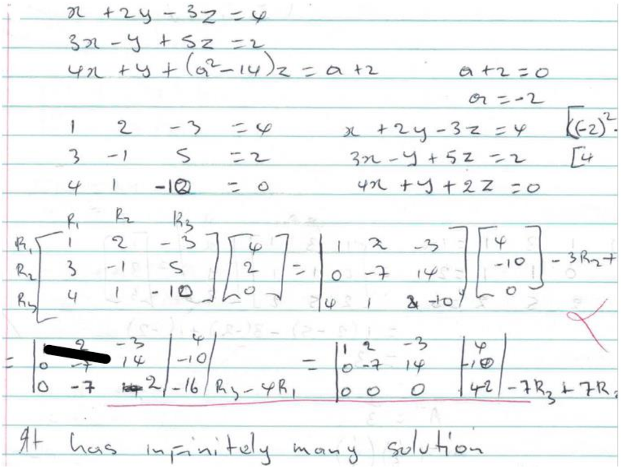

Figure 11. Simplifying augmented matrix with an unknown substitution (Source: Author's own illustration)

X106: No-one. So that's the problem because I did not attend the class.

X116 changed $a^2 - 14$ to $2a - 14$ to which he then performed row-reductions. X41 also used the substitution $a = 2$ to simplify the augmented matrix and then proceeded to perform EROs as shown in **Figure 11**.

Moreover, the row operation ($R_3 - 4R_1 \rightarrow R_3$) used to get 0 in the element a_{32} is not proper as it would destabilize the first column in the process. Instead, the pivot on a_{22} is the one to be used to eliminate the element on a_{32}.

The analysis of the partial responses of the 101 students revealed challenges with doing row-reductions and interpreting the solutions of the row-reduced augmented matrix. Student X128 started the row-reductions well but gave up before completing up to echelon form. In contrast, 79 students made conceited efforts to reduce the augmented matrix to echelon but failed to get the last row, which is decisive in interpreting the three types of solutions. The majority were not precise in the manipulations of the row-reduction as was the case with X30 shown in **Figure 12**.

It is very alarming to see such a huge number of students erring this way as this had the effect of spoiling the subsequent stages of interpreting the solutions. In GE, the behavior of the last row is critical to determine the nature of the solutions, and in this particular item, to evaluate the value(s) of the constant a. X79 had more complications with some of the row operations and subsequently failed to reduce the augmented matrix to echelon (shown in **Figure 13**).

Students like X79 still find executing row operations a bit of a challenge as they fail to reduce to echelon. The interview with X79 alludes to this:

R: Why is there no zero here? Did I ever talked about the upper triangle?

X79: Yes sir.

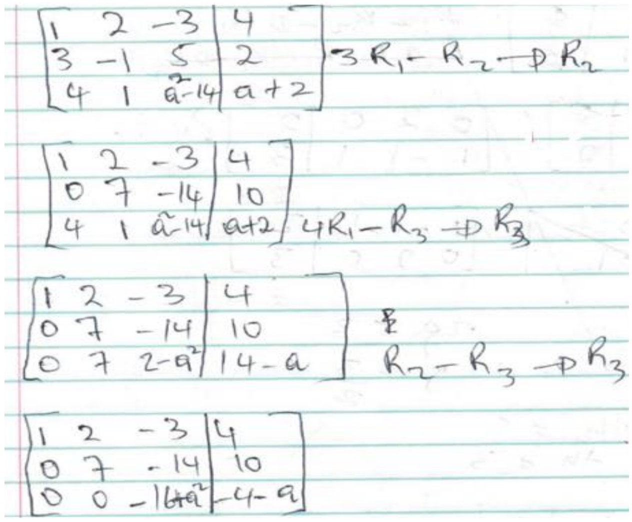

Figure 12. Lack of precision in computations (Source: Author's own illustration)

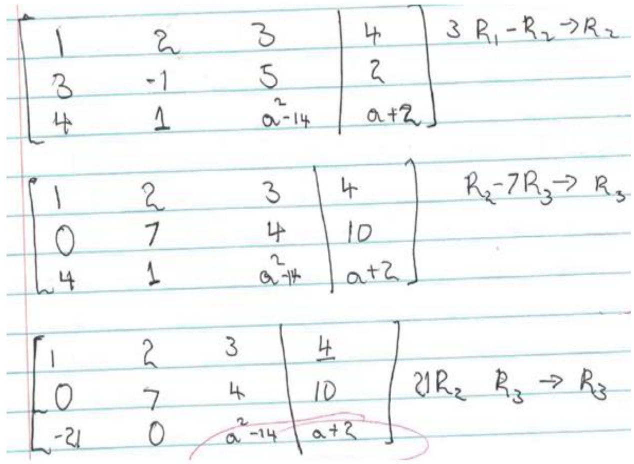

Figure 13. Complications in row-reductions (Source: Author's own illustration)

R: What is this four doing here?

X79: I struggled to do this question Sir.

R: Then you skipped this four and reduced to zero the next element. Would it work if you skipped this one?

X79: We put the zeros from left to the right I know. We cannot start on the middle column before there are zeros here.

R: Why did not you do it that way? How do you go to the interpretation with this number four here?

X79: I practiced but …

Nonetheless, EROs are learnt as a means to solve SLE problems using GE; this result revealed that 20 students completed all EROs but has challenges with the interpretation of the solutions.

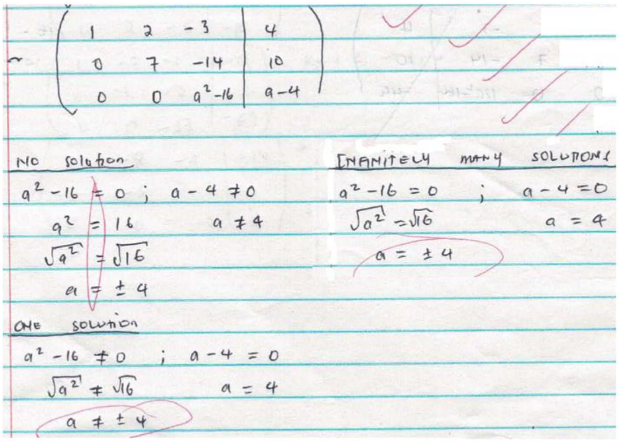

Figure 14. Correct EROs but incorrect interpretation of last row (Source: Author's own illustration)

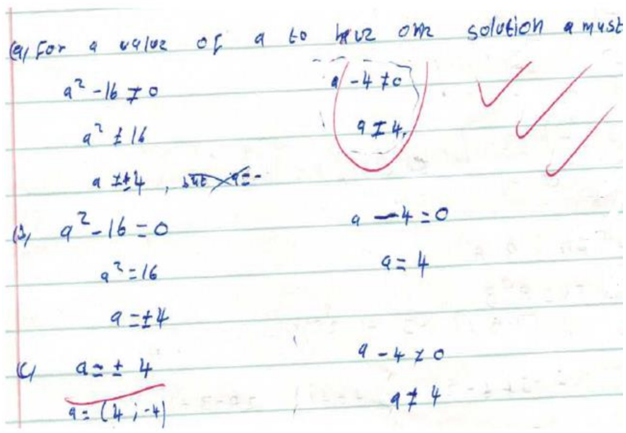

Figure 15. Incomplete solution lacking only intersection by X6 (Source: Author's own illustration)

Types and Interpretation of Solutions

Nine failed to interpret the connotation of the last row of the row-reduced matrix to the type of solutions associated with it. All of them managed to get the last matrix correct as shown in **Figure 14** but it was the interpretation thereof which was problematic. In the interview, X2 indicated that:

> R: Tell me; when you get at this stage, what is your interpretation of
>
> (a) unique,
>
> (b) infinitely many, and
>
> (c) no solutions?
>
> X2: the problem is that I did not understand those things of nature of solutions. That's why I got it all wrong.
>
> R: But we did an example on this one in class, did not we?
>
> X2: Yes. According to me, it was not enough. I would have wanted more and more examples.

X122's response is depicted in the vignette in **Figure 14**, and X110 also presented the same response. X102 did not even attempt to find the values of a that correspond to each type of solution whilst X123 managed to get the answer to *no solution* only. Even in the answer for the *no solution*, X123 did not find the intersection of the in-equations $a^2 - 16 \neq 0$ and $a - 4 \neq 0$ as is the case.

In connection to intersection solution, eleven students solved the equations for the three cases almost in the expected way but failed to find the common solution. **Figure 15** depicts an otherwise correct solution, which only lacked the intersection of the two parts.

X11 also missed the intersection for (a), did not complete (b) and skipped (c) entirely as described in the following dialogue.

> R: There is supposed to be one solution. It cannot be both. We are saying AND.
>
> X11: Eeeeh. This side we get these two solutions. This side we get this one.
>
> R: We want the number, which is common to both solutions.
>
> X11: I was supposed to … I can see.
>
> R: Where is the intersection? This four is common in both.
>
> X11: Oh I can see. Next time I will make sure I will not repeat.

Hence lack of robust understanding of the behavior of the last row to the three types of solutions was the hindrance.

Finally, only two students successfully performed EROs and interpreted the solution types to SLE. These were X2 and X75. The execution of EROs, the interpretation of the solution and the intersection of solutions were effortlessly done in **Figure 16** by X75 (part of the solution was cropped off to reduce the image size). However, it is a very small proportion of students who fully developed the schema.

The above sub-sections in this section can be coalesced to give a good picture of undergraduate students' conceptualization of EROs as they solve given problems. Initially, students are expected to determine and apply the three rules for EROs. This is followed by the need for them to figure out the appropriate method to solve given SLEs based on the limitations of each. APOS theory provided the level to determine the degree to which students' understanding developed in the process of learning linear algebra. Thirdly, lack of full

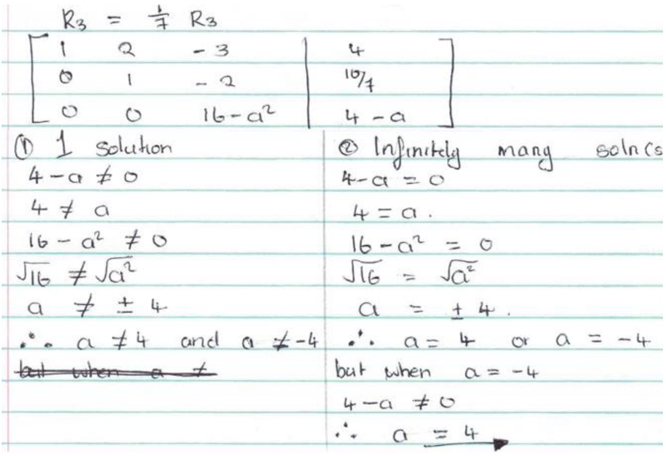

Figure 16. Solution, which depicts encapsulation of EROs in solving SLE (Source: Author's own illustration)

understanding of EROs in solving SLEs lead to many types of cognitive and procedural errors.

Eventually, this culminated in students displaying different levels of interpretation of results, which they obtained after solving an SLE. Unique solutions were rather straightforward to determine, but no solution and infinitely many solutions required higher-levels of mental constructions. With this mind map of the journey that students take to conceptualize EROs, I now discuss how this mind map of results support or refute current literature in the next section.

DISCUSSION

Results give a snapshot of nature of undergraduate students' mental constructions when learning EROs concept and its application in solving SLE.

APOS Theory in Research

As a theory of learning, APOS can only make external observations of students' learning through what they write down and speak. This necessitates taking a snapshot of students' understanding at a particular point in time through the guidelines of the researcher's proposed decomposition of the concept (Martin et al., 2010). This seemingly transcends the dynamic nature of the development of mathematical schemata. Hence, our analyses of students' responses may not be the exact thoughts processes of the students before, during and after data collection (Kazunga & Bansilal, 2017). Their written and verbal responses acts as proxy of how they reason and think about solutions of SLEs using GE. Nevertheless, the use of APOS theory is insightful in that instruction of mathematics concepts are designed around the pre-determined steps in mental constructions called GD. The theoretical analysis of the concepts is designed by the researcher who also happens to be the instructor in such a way that students are actively involved in the learning process (Chagwiza et al., 2021). As a data analysis tool, researchers use APOS theory to compare the students' failure or success on mathematics tasks against specific mental constructions called for by GD (Dubinsky & McDonald, 2001). The researcher then tries to explain the possible differences between the projected and actual students' performances by pointing to the actions, processes and objects mental constructions.

Gaussian Elimination

The second item was silent on the method, which students could use to solve the problem and the choice of GE was supposed to be spontaneous. GE is usually used to solve SLEs (Srinivasan et al., 2017). The majority of students managed to do exactly that except a few. This lack of spontaneity was also observed by Harel (2017) in a study involving SLEs with a large number of unknowns. In such a situation, GE offers a system approach that uses EROs to reduce the matrix to echelon form, but students in that study resorted to the manual elimination of variables technique. That showed a lack of object conception skills. The results indicated that the concept of EROs was essential to solving SLEs when the solution is not unique, thus it must be emphasized in the learning of SLEs. The lack of understanding of essential concepts negatively impacts students' achievement in SLE (Arnawa et al., 2019).

With regard to item 1, many students made use of the action conception whereby they used GE algorithm to calculate the values of x, y and z. Algorithms are step-by-step instructions, which enable accomplishment of a given task (Kontorovich, 2020). In APOS theory, the algorithm is explicit and externally driven. This concurs with research conducted in linear algebra that reports that students cope well with procedural aspects but have conceptual challenges of linear algebra concepts (Kazunga & Bansilal, 2017). However, not all students who solved SLE interiorized because they did not realize that even though SLE can be solved by GE, they were instead required to use the Gauss-Jordan elimination. These students did not show understanding that indeed the two were different methods to solve SLE but only one of them was called for. The interviews revealed that students did not bother further reducing to reduced row echelon on the fact that they had obtained the required solution already. This meant doing a few more row operations to reduce the augmented matrix to both lower and upper triangle, which was only done by 53 students. However, doing these reductions to Gauss-Jordan may involve painful and tedious operations and fractions therein are inevitable (Rindu, 2017). Naturally students prefer GE as it is faster than the Gauss-Jordan method (Gharib et al., 2015). To students, the selection of a method to use depends on accuracy and speed in solving large systems of equations because the volume of computations involved are bulky (Mandal et al., 2021). Nevertheless, all students except 20 understood EROs as a tool for solving SLE using the Gauss/Gauss-Jordan methods (Martin et al., 2010), which demonstrated an object conception. The 20 either skipped the question altogether or evaded EROs in their solution processes.

Procedures for Elementary Row Operations

Siahaan et al. (2023) posit that college students do not comprehend the procedures and concept of EROs, evidenced by students' inability to execute requisite EROs for each step of row-reduction. This weakness was more pronounced in the second item, which had variable elements. In that case, two students tried to simplify the row operations by eliminating the unknown element with a numeric. Generally, many students are not proficient in solving mathematics problems involving fractions (Smith & Powell, 2011). Student X55 tried to circumvent fractions by working with non-one pivots. By so doing he did not divide by the coefficient of the pivot, rather he multiplied the other row by the value of the coefficient. This resulted in dealing with big numbers in addition to flouting the rules of Gauss-Jordan for the main diagonal. Unfortunately, students who attempted to avoid fractions made more mistakes in the process so that they got partially-correct responses. The best way was to take the fractions head-on, which many students managed to do so.

The frequency of students who managed to row-reduce the augmented matrix to Gauss-Jordan in item 1 was more than for item 2. Item 1 required use of action conception to reduce the augmented matrix to row canonical form, leading to the unique solution for the given SLE. Students have an intuition regarding SLEs with unique solutions and this is represented geometrically by the intersection of all the linear graphs making up the system (Karunakaran & Higgins, 2021). Instruction also "foregrounds solving methods that focus first on unique solution cases, before moving to the much more common cases of no solution or infinitely many solutions" (Karunakaran & Higgins, 2021, p. 308). However, item 2 involved row-reducing a matrix with variable elements and this required students' process conception of linear algebra. Fewer students consequently managed to execute the correct EROs. Among those who got EROs correct, one of their greatest challenge to interpret the last row of the reduced matrix so that they could find the value of a for which the system had unique, no or infinitely many solutions. Object conception of linear algebra was called upon to do the correct interpretation. Oktac (2018) comment that students unsuccessfully relied on procedural knowledge and memorization to interpret $0 = 0$ or $0 = 9$ to decide which one has no solution infinitely many solutions. In this study, their decisions were complicated in that the last row had quadratic expression $a^2 - 16$ on the pivot and $a - 4$ after the vertical line. Students were supposed to think at object conception to solve and find the intersection of $a^2 - 16 = 0$ and $a - 4 = 0$, giving the final answer $a = 4$ for infinitely many solutions. Only X2 and X75 encapsulated the concept of solving SLEs in item 2. This is disquieting as the goal of instruction is help students attain the object conception of the mathematics topics they learn. In addition, GE is an efficient and structured method of solving SLEs and conceptual understanding is needed at every step of the solution process (Dewi et al., 2021). A conceptual understanding is regarded as the coherent knowledge that enable students to learn new ideas by linking them to what they already know (Kilpatrick, 2001). The lack of encapsulation also saw some students opting to use the Cramer's rule and the inverse method in item 2. But after computing the determinant, inverse and co-factor, the reached a dead-end.

Types of Errors

The results revealed many types of errors made by students as they learn linear algebra (Mutambara & Bansilal, 2022). Some errors were foundational in nature, meaning manipulation and baseline knowledge errors. I also identified errors that were procedural, meaning errors related to the step-by-step processes in the solution. For example, students failed to transform SLE into matrices. According to Possani et al. (2009), transforming an equation into an equivalent one is action-level conception. On the contrary, the preliminary GD indicated that transforming equations into matrices is pre-action. However, whether classified as pre-action or action, students reflect on this knowledge and incorporate it into algorithms or procedures. In APOS, this is called interiorization of actions into processes. Lastly, there are conceptual errors, which are associated with deep-seated misunderstandings, which Makonye (2012) described as misconceptions. If students thought of SLEs as a whole, compare and predict their types of solutions, they would have constructed an object conception (Possani et al., 2009). But some students failed to perform EROs to solve SLEs with no solutions or infinitely many solutions and failed to interpret the types of solutions. Students were supposed to realize that for unique solutions, other methods like the Cramer or inverse method can be used (Kazunga & Bansilal, 2020) but only GE is applicable for no solution or infinitely many solutions. This happens when students do not encapsulate the actions and processes of solving SLEs. GE method makes use of EROs to reduce a given augmented matrix to echelon, hence non-encapsulation of EROs resulted in students struggling determining solutions to SLEs (Mutambara & Bansilal, 2018). The most common type of error was procedural, where 79 students started well doing EROs but failed to get the correct last row. In GE, the last row is critical to figure out the type of solutions associated with a given SLE. When reduced to upper triangle, it is the last row that has the largest number of noughts.

CONCLUSIONS AND IMPLICATIONS

The schema development of EROs was moderately fared by many students. Executing the three rules required to reduce an augmented matrix to echelon form was understood by some students using procedural proficiency while developing conceptual development as well. However, the majority of students who had encapsulated the pre-requisite concept could not encapsulate the application thereof to solve SLE using GE. In this study, quite a number of students had flawless EROs but struggled with the interpretation of the solving SLEs. Students' thinking was more intuitive to interpret for unique solutions to SLEs relative to systems with infinitely many solutions or no solution (Karunakaran & Higgins, 2021; Oktac, 2018).

Similarly, those students who struggled with EROs obviously could not engage SLE properly. This study contextualized EROs in solving SLEs, a concept that is known to challenging to undergraduate students. Despite the fact the majority of elementary linear algebra are regarded as abstract and lack connection to what students already know (Dorier et al., 2000), solution of SLEs are done in secondary school using the technique of elimination of variables. As this technique is cumbersome for multiple equations with multiple unknowns and SLEs with non-unique solutions, a connection was established to undergraduate mathematics, where a systematic elimination technique called GE through EROs replaces the elimination method. This study revealed a greater need for more problems, which provide students with the opportunities to scrutinize SLEs geometrically and algebraically for all types of solutions for all orders. In this study, students demonstrated inroads into EROs as a concept but struggled to apply it in solving SLEs.

The use of APOS theory has intension to instructional strategies as understanding the mental constructions that students make when learning a mathematical concept leads to improved practice (Ndlovu & Brijlall, 2016). The mental constructions made by the students concurred with the preliminary concept decompositions for both the concept of EROs and their applications. It has also come to the fore in this study that some concepts are not learnt as an end to themselves but as means to the end.

The concept of EROs is often reduced to rules without meaning yet it plays a vital role in solving SLEs using GE. The learning and teaching of SLEs by GE should be problem-based as a means to help students develop the necessary mental structures of both the concepts of EROs and solving SLEs. The types of problems should be selected to invoke each of the action, process or object conceptions (Tatira, 2023). I suggest designing GD-informed instruction, as well as steering instruction towards the achievement of the mental constructions spelled out by GD. As students find the learning of linear algebra difficult (Altieri & Schirmer, 2019; Possani et al., 2009; Salgado & Trigueros, 2015) this study contributes to the debate in literature on how to improve the teaching of linear algebra.

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