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University students and professors' reflections on an averages-based approach to the fundamental theorem of calculus

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Abstract

Understanding the fundamental theorem of calculus (FTC) is complex for university students. In this paper, an alternative approach to the FTC that relies on the use of the calculation of average values of a continuous function for a certain interval is shown. Likewise, researchers present the reflections of twelve advanced university mathematics students on solving five tasks with this approach, as well as the reflections of two professors of calculus and mathematical analysis about the alternative approach and the resolutions given by the students. Students' reflections reveal that they found the approach to FTC generally appropriate, because it helped them mobilize their knowledge of calculus and supported their understanding of the theorem. The professors expressed in their reflections that the approach is pedagogically viable, and they seemed it a way of strengthening the students' conceptual knowledge in calculus.

Keywords: reflections, average, fundamental theorem of calculus, university students, professors

INTRODUCTION

In the usual teaching of the fundamental theorem of calculus (FTC), differentiation and integration are treated as inverse processes. That is, the definite integral provides an antiderivative, and antiderivative provides a means to evaluate a definite integral. Thus, the FTC is often presented in the classroom and in calculus textbooks in two parts, as shown below (Pedersen, 2015, p. 141-142):

1. FTC, part I. Suppose $f:[a,b]\to\mathbb{R}$ is integrable on $[a,b]$ and continuous at x_0. Let $g(x) := \int_a^x f$. Then g is differentiable at x_0 and $g'(x_0) = f(x_0)$.
2. FTC, part II. If $f:[a,b]\to\mathbb{R}$ is integrable and $F:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$, differentiable on $]a,b[$, and $F' = f$ on $]a,b[$, then $\int_a^b f = F(b) - F(a)$.

The intention of calculus textbook authors and teachers in presenting the FTC in two parts is precisely to show the inverse relationship between differentiation and integration, and to enable students to make more efficient use of the theorem when solving problems. However, the history of the conceptual development of calculus shows that the understanding of this theorem requires a variational view of the total change as an accumulation of small changes that are proportional to the instantaneous rate of change, very similar to the ideas of Barrow (1630-1677) and Newton (1643-1727). Earlier, the relationship between variation and accumulation is also found in the work of Oresme (1323-1382), who established that the distance travelled by a mobile moving with uniform acceleration is equal to the distance travelled at a constant average velocity. In this sense, the understanding of FTC cannot be situated in the merely operational, it is also necessary to connect the operational with the conceptual by following the variational principles and relations underlying the theorem.

For decades, research has reported that students have difficulties in articulating the two parts of FTC to achieve an adequate understanding. In the study conducted by Thompson (1994) with mathematics students, it was found that such difficulties are associated with a poor conceptualization of the rate of change, as well as underdeveloped images of this concept and a deficient understanding of covariation. In general, the literature on the teaching of FTC shows that, on the one hand, students find it difficult to articulate their procedural

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Contribution to the literature

- This paper presents an alternative approach to the FTC based on the calculation of average values of a continuous function. The approach relates the notions of variation and accumulation, and the arithmetic mean of ratios of change to the integral to support understanding of the FTC.
- The reflections of university professors and students on this approach and its scope in the mathematical comprehension of students are reported.
- The focus on reflections allows recognizing the potentialities and limitations of the approach to strengthen students' understanding of FTC.

and conceptual thinking to achieve an adequate understanding and use of FTC. On the other hand, teaching calculus with a traditionalist approach has had little impact on the generation of meaningful learning (Bressoud et al., 2016). In this regard, some researchers have proposed alternatives for teaching FTC by attempting to address the principles of calculus and the concepts involved, such as the concepts of function, limit, derivative and integral (e.g., Hardy, 2011; Kouropatov & Dreyfus, 2013; Robles et al., 2014), as well as the ideas of variation, accumulation and even the use of differentials (e.g., Carlson et al., 2003; Thompson & Dreyfus, 2016; Thompson & Silverman, 2008).

Recently, teaching and learning proposals have been generated to improve students' understanding of the FTC with an approach based on the ideas of variation and accumulation. For example, Sokolowski (2021) reports that by means of mathematical modelling and emulating scientific reasoning, high school students improved their understanding of the theorem. The proposal made by Zubieta and Meza (2022) consists of using GeoGebra software to give a dynamic treatment of the theorem based on Isaac Barrow's original ideas and version of the theorem. The use of GeoGebra as a mediator in the teaching of the FTC is also found in the work done by Acevedo-Rincón and Ramos-Rodríguez (2022) who use it to explore tasks with a certain level of cognitive demand associated with better learning of the theorem. In the attempt to make FTC more accessible to students, the active role that they should play in their learning is neglected, for example, analyzing their thoughts and actions, as well as generating logical explanations of these or of the possible conceptual relationships involved in the performance of one or more mathematical tasks.

In higher mathematics education, it is suggested to promote the implementation of pedagogical strategies in which students are more actively involved in their learning (e.g., Bennoun & Holm, 2021; Da, 2023; Laursen & Rasmussen, 2019; Mingus & Koelling, 2021; Stanberry, 2018). However, according to Alam (2020), teachers still face major challenges in promoting cognitive processes that enable students to connect the core concepts of calculus, improve their conceptual and procedural understanding, and articulate graphical and algebraic representations. One way to achieve this is to engage students in a process of reflection, for as Dewey (1938) mentions, learning is not a direct result of experiences, but of reflection on them. It has also been proven that reflection helps conceptual understanding of mathematical objects (Reinholz, 2016). Nevertheless, research has yet to be done on ways in which reflective thinking can be developed by mathematics students (Ariany et al., 2021) and what kind of treatment can be given to FTC so that it is the students' own reflections that contribute to their understanding.

With the intention of knowing more about how to help students achieve a better understanding of the FTC from their own thoughts and actions, in this study it was decided to explore the type of reflections that a group of students and university professors of an undergraduate degree in mathematics generate regarding an alternative approach to teaching the FTC, based on the calculation of average values of a continuous function in a certain interval. This would allow us to obtain information on the potential of such an approach to engage students in a reflective process and analysis of the existing relationship between the processes of variation and accumulation present in the theorem, as well as to know the professors' reflections in relation to

(1) the approach,

(2) the solutions given by the students to the tasks that make up the approach, and

(3) the students' reflections on the approach.

This approach emphasizes the importance of providing a more variational treatment of the calculus over those installed in the static or operational model (Larsen et al., 2017; López-Leyton et al., 2024; Tall, 2012; Thompson & Harel, 2021). Some instructors with extensive experience teaching derivatives at the undergraduate level agree that the study of the relationship between the concept of the average rate of change and the derivative, as well as the use of various representations, are useful for the conceptualization of the derivative (Mkhatshwa, 2024). In line with this idea, it was sought to establish relationships between the concept of average rate of change with that of slope of secant lines, as well as that of instantaneous rate of change with that of the slope of tangent lines, through coordinating numerical and graphical representations. In other words, the relationship between the average values of a function with that of the derivative.

Something similar happens in the case of the calculation of definite integrals when the functions are of the form: $f(x) = x^n$, $n \epsilon \mathbb{Z}^+$, an example of which is the mean value theorem for integrals, where the relationship between slopes and areas (equivalently, derivatives and integrals) is clearer.

Objective of the Study

Describe the type of reflections that a group of students and professors of an undergraduate degree in mathematics about an alternative approach to teaching the FTC, based on the calculation of average values of a continuous function.

CONCEPTUAL FRAMEWORK

Two important concepts to understand the perspective of this study are: reflection and average. The first refers to a process in education that is closely related to learning and the second is a transversal concept in the mathematics curriculum.

Reflection and Mathematical Learning

Reflection consists of an awareness of personal experiences (Dewey, 1938). Therefore, from an educational point of view, reflections account for the meaning that people attribute to their experiences or the type of conscious considerations when constructing objects (things with meaning). In this direction, Shön (1983) refers that reflection is the way in which people become aware of their own thoughts when they come into contact with objects. In other words, the action of the individual is the main object of reflection. This quality makes it particularly important and fundamental for learning mathematics since the latter will occur to the extent that students (also professors) reflect on their actions in doing mathematics.

For some authors such as Raelin (2007), reflection is largely interactive, so one way to promote it is through dialogue or conversation with others or with oneself. These dialogues or conversations when materialized in the form of written text allow the individual not only to organize his or her thoughts around what he or she considers most significant about an experience and thus become aware of it, but also to make them perceptible to others.

Average Value in Calculus

It was considered that averages provide an important conceptual basis for understanding various mathematical concepts in any area of mathematics and beyond. Their operational character allows estimating or calculating values of deterministic or non-deterministic situations, through ideas of proximity and weighting. For example, from a geometric perspective, the mean value theorem in differential calculus establishes a relationship between a secant line and a tangent line to the curve at a specific point. Such a relationship is equality between the slope of a secant and the slope of a tangent for some intermediate point.

The average values themselves represent, or constitute, values approximating a specific value of the same nature as the average values are refined. In relation to these ideas and for the case of classical mechanics, the passage from the concept of average velocity to that of instantaneous velocity is given as an example. An example of an average widely treated in mathematics education at any educational level is the arithmetic average and, in this case, some authors claim that it is an "object that is easy to calculate and provides a powerful and intuitively satisfying insight into many scientific ideas" (Grossman et al., 1983, p. 3). This is perhaps what explains the diversity of connotations of use and application of this mathematical object in different areas of mathematical knowledge, including Differential and Integral calculus (Larsen et al., 2017; Rondero, 2010; Thompson & Silverman, 2008). In relation to the above, the arithmetic average is also a complex and transversal epistemic object that serves as a basis for the study of other concepts and properties of calculus (Rondero & Font, 2015).

The arithmetic average of n numbers: $a_1, \cdots, a_n$ is defined as the number $(a_1 + \cdots + a_n)/n$ and it is from such a definition that in this work the concept of average is specifically assumed and used for the case of continuous functions in closed intervals, as follows: The arithmetic average of a continuous function $f(x)$ on a closed interval $[a, b]$ is determined by the value $\bar{f} = \frac{f(x_1) + \cdots + f(x_n)}{n-1}$, for a uniform partition $P = \{x_1, x_2, x_3, \cdots, x_n\}$, with $x_1 = a$; $x_n = b$ and $x_{k+1} - x_k = \frac{b-a}{n-1}$ for all $k \in \{1,2, \cdots, n-1\}$. Here the function to be averaged is the derivative of the original function f given.

Considering that the rate of change of f in the subinterval $[x_k, x_{k+1}]$ with $k \in \{1,2, \dots, n-1\}$ is $\left.\frac{\Delta f}{\Delta x}\right|_{x \in [x_k, x_{k+1}]} = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} = (n-1)\frac{f(x_{k+1}) - f(x_k)}{b-a}$, then taking the arithmetic average of these change ratios would give the following:

$$\frac{1}{n-1}\sum_{k=1}^{n-1}(n-1)\frac{f(x_{k+1}) - f(x_k)}{b-a} = \frac{f(b) - f(a)}{b-a}. \quad (1)$$

This is a telescoped sum, therefore, the rate of change of the entire interval is equal to the average of the rates of change in the subintervals. The f is the function whose ratios of change are averaged, not its derivative, i.e., it is the function that is integrated. This leads to the following equality:

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx. \quad (2)$$

The above definition of arithmetic average for continuous functions, although simple, illustrates how the arithmetic average is a limit of a convergent sequence

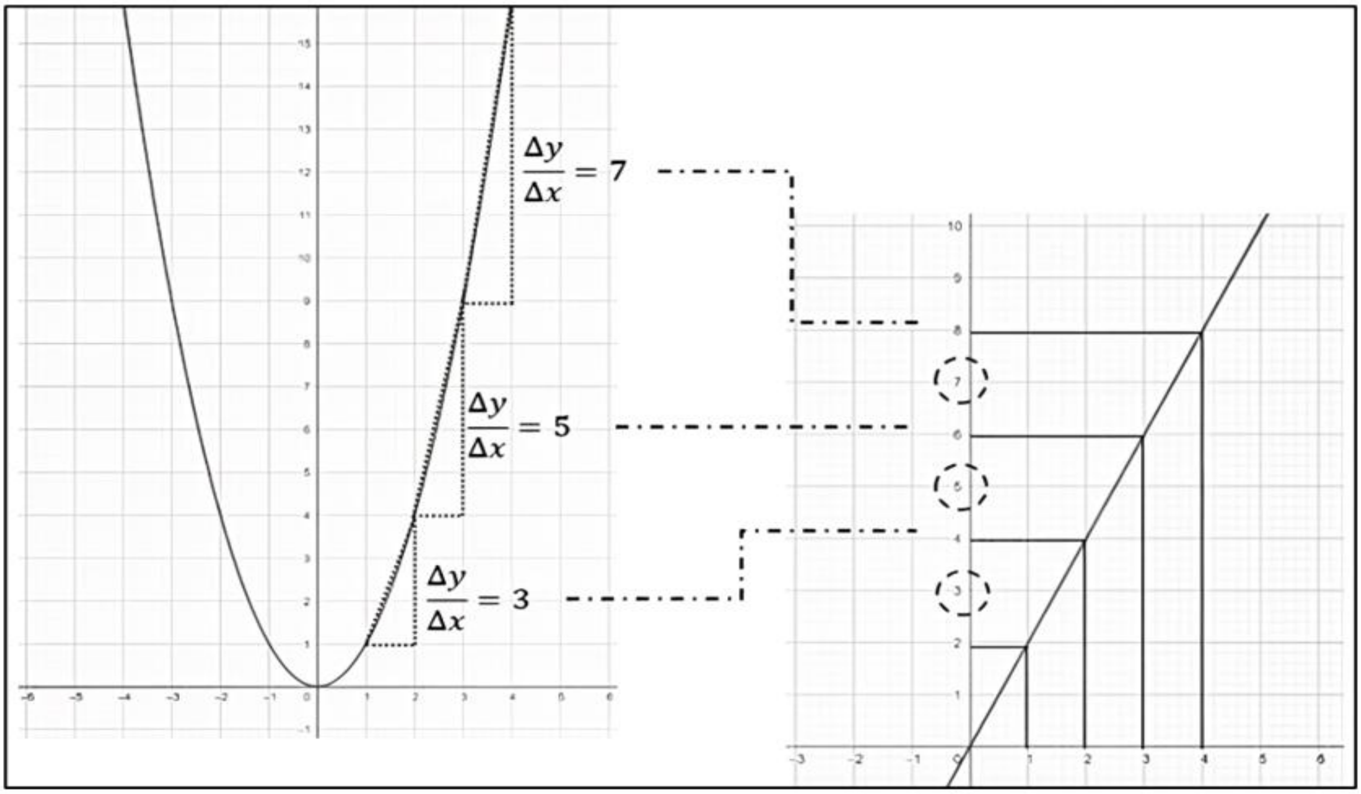

Figure 1. Graphical-numerical relationship between the average rates of change of $g(x) = x^2$ and the average heights of $f(x) = 2x$ in the interval $[1, 4]$ (Source: Authors' own elaboration)

and indicates a way in which the dual nature of two mathematical objects such as the derivative and integral, as well as the variation nature of both, are part of a process of conceptualizing FTC. This process is based on the idea of averaging and mean value approximations of a function in an interval, while allowing for a connection between various semiotic (as well as mental) representations of such objects in FTC. Specifically, it establishes a connection between the arithmetic, the graphical and the algebraic, as particularly illustrated below with the function $f(x) = 2x$ in the interval $[1, 4]$.

$$\bar{f} = \frac{2(1.5)+2(2.5)+2(3.5)}{3} = \frac{3+5+7}{3} = 5. \quad (3)$$

$$\bar{f} = \frac{1}{4-1}\int_1^4 2x dx = \frac{1}{3}[x^2]_1^4 = \frac{1}{3}[16-1] = 5. \quad (4)$$

That is, one can use the mean value theorem for derivatives to make the relationship between the average of the ratios of change (which is equal to the ratio of change over the entire interval) and the mean value of the derivative of the original function over the entire interval. **Figure 1** illustrates that the average height of the function $f(x) = 2x$ in the interval $[1, 2]$, is equal to the slope (average rate of change) of the function $g(x) = x^2$ in the same interval and so on for the intervals $[2, 3]$ and $[3, 4]$. These three intervals can be considered as a uniform partition of the interval $[1, 4]$. Therefore, and in this order of ideas, it can be said without loss of generality that the height of $f(x)$ is equal to the slope of $g(x) = x^2$. It should be noted that even any set of decimal values used in the average produces the same final average value of 5.

The previous result can also be stated, as follows: "the rate of change of an accumulation function is the value of the rate of change function from which it is generated", which means that any accumulated quantity $h(t)$ from some reference point t_0, would be determined by $h(t) = h(t_0) + \int_{t_0}^{t} j(x)dx$, for some rate of change $j(x)$. This opens the possibility of approaching the understanding of FTC by means of the analysis of variational relations using average values. It is in this sense that the notion of arithmetic average of continuous functions was used in this work to analyze the reflections made by advanced students of a degree in mathematics on the variational relationship between the arithmetic average of the average change ratios of a function and the cumulative value, as an alternative approach to FTC. Likewise, analyze the reflections of university professors of calculus on the approach and reflections given by students.

In this approach to FTC based on averages, tasks were used where the idea of variational approximation serves as a mechanism to support the establishment and understanding of the relationship between a process of differentiation and one of integration, while at the same time giving greater meaning to the methods, techniques, and concepts of calculus. Through these tasks it was explored to what extent this conceptual approach to FTC in a variational way is feasible for learning the theorem at higher education level, both from the perspective of advanced students and professors.

RESEARCH METHODS

The study followed a qualitative methodological orientation since it sought to explore and describe the type of reflections of students and professors when they experience a particular mathematical situation, in this case, an alternative approach to the one usually given in books and educational practices to FTC, without neglecting their interpretations and meanings associated with that situation (Creswell, 2013).

Participants and Data Collection

The participants were two types of populations: on the one hand, twelve university students who were in their final year of undergraduate studies in mathematics at a public university in Mexico. Therefore, they had mathematical knowledge and a broad command in the areas of algebra (symbolic, linear, and abstract), geometry (Euclidean, analytic, and topology), calculus (real and advanced calculus, differential equations, mathematical analysis) and probability and statistical inference, among other mathematical areas that require a higher level of theoretical abstraction. The other population consisted of two professors in mathematics, with doctoral studies and extensive experience teaching calculus and mathematical analysis at a higher level. The professors belonged to the same university as the students.

In the case of the students, data collection was done through the application of a printed and individual instrument. The instrument consisted of an activity made up of five tasks: three on the calculation of ratios of change (average) of the quadratic function $f(x) = x^2$ in specific intervals; one task about the relationship of equality between ratios of change and arithmetic average of the function $f(x) = x^2$; and one related to the calculation of the value of a continuous function, given certain conditions. At the end of the activity, students were asked to write down their reflections on the resolution of all the tasks. The five tasks and the related item are shown below.

1. Let $f: R \rightarrow R$, given by the following formula: $f(x) = x^2$, determine the rate of change of f in the interval $[1, 4]$.
2. Let $P = \{1, 2, 3, 4\}$ a partition of the interval $[1, 4]$ for the function f given in task 1. That is, the interval [1,4] is divided into three subintervals: $I = [1, 2]$, $J = [2, 3]$, $K = [3, 4]$. Determine the ratios of change of f for each subinterval.
3. Determine the arithmetic average of the change ratios of f obtained in sub-intervals I, J and K of task 2.
4. Use the fact that the results of task 1 and task 3 are the same for:
   a. Give an explanation as to why such an equal relationship between the results occurs.
   b. Mathematically generalize the underlying relationship between the two tasks.
   c. Give graphical or analytical arguments to support your generalization.
   d. Give a concrete example showing the validity of the generalization.
5. Use the results obtained from task 4 to do the following. If the value of a function f is known at some value a in its domain and that g, the derivative of f, is continuous and known at every number in the interval $[a, b]$, determine the value of $f(b)$.

Additionally, students were asked to write up their reflections on the tasks. In the case of the professors, data collection was done by invitation via email. Their participation consisted of sharing in writing their reflections on the alternative approach to FTC based on the review of the solutions given by the students to the tasks and their respective reflections. In addition, they were asked, as far as possible, to allude to mathematical and didactic aspects in their reflections, for example, an observation or comment on the mathematical knowledge and skills of the students, as well as their own assessment of the didactic scope or limitations of the approach.

Data Analysis

For the analysis of the data (reflections), the method of thematic analysis was used because it is possible to identify, organize and provide systematic information about the type of reflections made by both students and professors, which in turn allows the researcher to see and make sense of individual or collective reflections and experiences (Braun & Clarke, 2012). In other words, this method is a way of identifying what is common in the way a topic is reflected upon and making sense of those commonalities.

The first action for the analysis was to aggregate the reflections given by the students when solving the five tasks. The second was to apply the thematic analysis to the set of students' written reflections to classify them by themes and thus determine which were the most frequent considerations associated with their lived experience when completing the items. The third was to ask a mathematics education researcher to indicate, according to his/her opinion, whether in the set of reflections he/she identified (if applicable) indications of metacognition (how the students processed the experience of solving the tasks), specifically, how they made sense of what they knew or did not know when solving the tasks, and to point out the reflections in which he/she observed some indication of self-evaluation and thus strengthen the identification of the type of reflections made by the students. **Figure 2** shows the actions referred to.

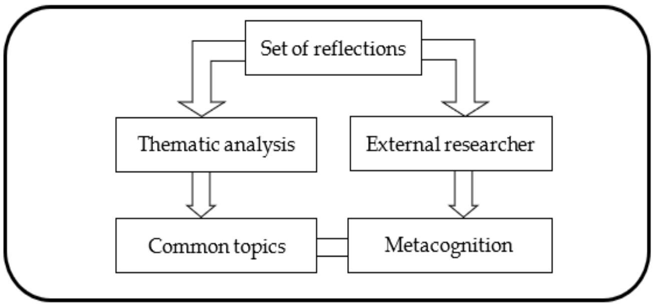

Figure 2. Process of analyzing the reflections given by the students on the tasks (Source: Authors' own elaboration)

For the analysis of the professors' reflections, their responses were combined in a single document and the thematic analysis was used to classify them by themes and to infer the type of reflections common to all of them.

The aim of this analysis was to identify general and specific reflections on the pedagogical scope of the alternative approach to FTC.

RESULTS

In accordance with the aim of this research, researchers identified the types of reflections of a group of students and professors of a mathematics degree about an alternative approach to teaching FTC based on the calculation of average values of a continuous function in a certain interval.

Types of Students' Reflections

From the thematic analysis applied to the reflections made by the students, it was found that most of them found the tasks interesting and important as they considered that they helped them, firstly, to remember basic concepts of calculus and, secondly, to reflect on another way of seeing some important relationships to understand FTC.

Table 1 presents the type of reflection expressed on the role of the tasks in the recall of some concepts.

Students also reflected on the role of the tasks in terms of their knowledge of calculus and approaching FTC in an alternative way to that usually employed in books or teaching practice. Some of these reflections are shown in **Table 2**.

About the results obtained by the external researcher concerning the identification of metacognitive processes in the students' reflections, the following was obtained. The researcher used Carlson and Bloom's (2005) characterization of metacognition, which they consider including control, monitoring, and associated behaviors.

Control comprises "the metacognitive behaviors and global decisions that influence the solution path (…) selection and implementation of resources and strategies, as well as behaviors that determine the efficiency with which facts, techniques, and strategies are exploited" (p. 50).

Table 1. Students' reflections about the role of tasks to remember calculus concepts

Student	Statement
S1	I found it fun to do the tasks, it's a good reminder of the things I've learned before.
S3	The tasks helped me to remember and strengthen my knowledge in calculus.
S4	It was interesting to solve basic numeracy tasks.
S9	I found it a simple activity that only requires basic knowledge and as an extra activity it is good to refresh some concepts.
S12	The tasks helped me to remember basic concepts of calculus.

Table 2. Students' reflections about the approach to the FTC

Student	Statement
S1	I believe that the transition from task 4 to task 5 should be more gradual if there is insufficient basis for the transition.
S2	It is a good idea to re-evaluate and rethink our current knowledge base.
S4	Finding and demonstrating the generalization of task 4 was intuitive, however, it was necessary to think through the details. Task 5 was a challenge; it reminds me of FTC.
S6	I saw that the purpose was not to solve 5 small tasks but to answer a larger task through 5 tasks, where 3 tasks were exploratory and 2 were central to the topic.
S7	The tasks are interesting for a clear understanding of some concepts, although the difficulty changes markedly in task 5, where more tools and knowledge of other things are required to solve it.
S8	It is a reflection on the particularities of the behavior of the Riemann integral. And its quality of being a "sum" implies that it deals well with another function such as the average function.
S10	I had never thought about this relationship before. I find this kind of task enriching, because although they are not very complex, they give you a new perspective on things.
S11	These are tasks to understand and realize the qualities of the derivative and why some equalities happen. I think there should be a task between 4 and 5, which allows us to take an intermediate step between the two concepts.

Table 3. Professors' reflections on the pedagogical scope of the approach to FTC

Professor	Statement
P1	I consider that the tasks represent an interesting approach to introduce the FTC from a different perspective and can contribute to reinforce the knowledge of mathematical analysis, including concepts of the ratio of change, the intermediate value theorem, and the definition of the Riemann integral. On the other hand, I think that task 5 could be presented in a more broken-down way to make it easier for students to solve, as its difficulty is considerably higher than the other tasks.
P2	I find the tasks interesting. What is new to me is how to arrive at the FTC via averages and average value of a function on an interval.

According to Schoenfeld (1992), the decisions to be made concern what knowledge to use, how and when to use it, and what to do in a problem. Monitoring (or self-assessment) refers to reflecting on whether the chosen strategy or resolution process is being effective in solving the problem or whether another process is chosen. Self-regulation is the actions taken as the progress of the problem-solving process is evaluated.

The researcher reported having identified signs of metacognition associated with control and self-assessment in the students' reflections, however, he also mentioned not having recognized forms of self-regulation. Below is the researcher's report where the actions associated with monitoring or self-assessment are indicated in italics.

It was found that only four of the twelve participants show signs of the metacognitive processes control and self-evaluation. For example, S4 manifests *control* actions when he refers *to being clear that the activity deals* with the FTC and *that it is required to determine* the value of a function in an interval $[a, b]$ or the function itself from information about its derivative. In fact, it seems that the latter was the idea that allowed him to associate and *make sense of the* FTC in the activity. He also shows to have had control in his *decisions of the mathematical resources to be used to solve it* by placing it in calculus and identifying the use of knowledge about the derivative of a function. In particular, he identifies the derivative as a useful concept in his solving process by interpreting it as a rate of change and he knows (he seems to know, except for what is evident in his answers) *how to carry out a process* of generalization of the average of the rates of change in partitions of an interval of the domain of a function *to obtain what is requested* in tasks 4 and 5. An extract in which S4 gives indications of control is, as follows:

> It was interesting to solve basic calculus tasks. Finding and proving the generalization for task 4 was quite intuitive, however, it was necessary to stop for a moment to think about the details. Task 5 was a challenge, as I wanted to find a way to give the exact value of $f(b)$. It reminds me of the fundamental theorem of calculus, as we want to find a function from knowing its derivative.

In addition, participant S4 identifies self-evaluation in his solving process when he mentions (in the above extract) having taken some time *to think about the details of his generalization* in task 4. There is also a confidence or positive appraisal *of having correctly demonstrated* what was asked for in that task, saying that it was "quite intuitive". He also self-assesses *the usefulness of the ideas* associated with the FTC in solving the tasks (especially task 4 and task 5), as can be seen in the following extract:

> It is possible that the ideas in this theorem [fundamental theorem of the calculus] may be useful in finding the solution, however, I am satisfied with the solution I have provided, even if I have not been able to give an exact solution.

Most students only evidenced control in solving the activity. For example, S8 seems *to be clear that the activity involves* knowledge of the Riemann integral and the process of obtaining an antiderivative through a sum of ratios of mean change *but does not indicate any reflection on the effectiveness or relevance of using this knowledge* to determine the function requested in task 5, nor its possible relationship to FTC. Such a form of control and lack of self-assessment was acknowledged in the following extract:

> It reflects particularities of the behavior of the Riemann integral and its quality of being a "sum" implies that it is well treated with another function such as the average function.

Types of Professors' Reflections

Both professors agreed that the tasks offer a different alternative to the usual way in which FTC is taught in higher level calculus courses and even to reinforce some concepts of the analysis. However, both commented that it is necessary to consider some considerations such as the uniformity of the partitions to verify the relationship of equality of the average of the ratios of change in the subintervals of the partition with the ratio of change in the whole interval and to present task 5 in a more disaggregated way to facilitate its resolution. **Table 3** contains part of the reflections regarding the pedagogical scope of the alternative approach to FTC. The label P1 and label P2 are used to refer to one and the other professor.

Professors also offered some reflections on the resolutions given by the students. In general, they agree that most students were able to solve the first three tasks

Table 4. Professors' reflections on the students' responses

Professor	Statement
P1	(a) Students had no problems solving tasks 1 to 3. In task 4, it is observed that many had difficulty in graphing the generalization of equality, even though they got the analytical argument right. This suggests that they are not able to relate the approximation to the curve by ratios of average change. This in turn reflects that the students do not recognize the rectification of the curve, for example, considering that the derivative (slope of the tangent line to the curve, is the best linear approximation to the curve: in this case, by the hypotenuse of several right triangles whose legs are the heights and increments respectively). (b) Task 5 is where the greatest difficulty was observed, as only two students managed to determine an exact expression for $f(b)$ using the intermediate value theorem.
P2	(a) Most students proved equality with an arbitrary function and with a uniform partition. (b) Some students, when in task 5 it is assumed that the derivative g of f exists and is continuous over the whole interval $[a, b]$, were not able to use this information to relate the previous tasks to the FTC. One of them achieved this relationship by using the average value of a function f on interval $[a, b]$ defined as the integral of this function evaluated from a to b divided by $b - a$, which is the length of the interval $[a, b]$. (c) To use the additional hypotheses of task 5, it was necessary to use the mean value theorem to use the hypothesis of the existence and continuity of g in [a,b] to relate the previous tasks to the FTC, or alternatively, to use the definition of the derivative as a limit of the rate of change when the increment of the independent variable tends to zero and to view the integral as the limit of a Riemann sum. Not all students were able to do this process correctly, involving the value of the integral from a to b of the function g. One of them used the FTC to obtain $f(b)$ using the value of $f(a)$ and the additional assumptions, but without relating it to the results of task 4. Another obtained the Riemann sum, but mentioned taking the limit when n grows, but did not link it to the Riemann integral of g from a to b. Another took the rate of change as an approximation of $g(a)$ and made a clearance but clarified that it is only a good approximation if a is very close to b, not involving the FTC.

without difficulty, however, they recognize conceptual and even procedural difficulties in tasks 4 and 5. For example, the professors consider that, while it is favorable that some students asked relevant conceptual questions, it is very unfortunate that they did not seek to provide answers to them. They were also struck by the fact that only two students were able to determine an exact expression for $f(b)$, and the fact that they had trouble relating the approximation to the curve by ratios of average change. **Table 4** shows this type of professors' reflections.

DISCUSSION

Reflections of both students and professors suggest that the presented approach can help with a better understanding of the FTC, since the tasks promote students to reflect on the variational relationship between the arithmetic average of the ratios of change and the integral when solving the first four tasks. However, there were also difficulties in recognizing or evoking some essential geometric ideas in calculus, for example, the rectification of curves.

Based on the reflections of the professors and the analysis of the external researcher, the need to strengthen a more variational treatment in calculus courses was identified. Even though the students participating in this study had already accredited their calculus and mathematical Analysis courses, several of them showed difficulties mobilizing their covariational reasoning (Thompson & Carlson, 2017; Thompson & Harel, 2021). In fact, very few gave the graphical representations requested to show the variational relationship between the arithmetic average of the ratios of change and the integral. In this respect, it should be said that not all graphical representations were adequate.

Finally, both students and professors expressed that if the aim is to use this kind of task so that students themselves, supported by their analysis and reflections, establish the variational relationship between differentiation and integration, it is necessary to consider an extra task that serves as a conceptual bridge to move from task 4 to task 5. In this sense, it is necessary to reflect on how to achieve a better balance of procedural and conceptual knowledge in this approach, and to promote the transition between different semiotic representations (Duval, 2017). According to the results, this was a weak aspect for most of the students, despite showing analytical skills. In other words, the answers given by the students indicate the need to improve their skills in articulating the graphic with the analytical approach. This would require the design and use of more tasks or activities based on various representations of the derivative and the integral (Hamdan, 2019).

CONCLUSION

This work showed the possibility of approximating the FTC to students without necessarily resorting to limit theory, using only the idea of average values. Moreover, from the reflections obtained, it makes sense to say that this approach can help students visualize that ordinal changes in linear elevation of an integral exactly equal the magnitude of area swept out by its derivative over that interval, an aspect that is often not fully seen or understood during a calculus course.

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REFERENCES

Acevedo-Rincón, J. P., & Ramos-Rodríguez, E. (2022). Fundamental theorem of calculus: Cognitive demands and learning limitations on mathematical tasks. *Acta Scientiae, 24*(7), 4-24. https://doi.org/10.17648/acta.scientiae.7099

Alam, A. (2020). Challenges and possibilities in teaching and learning of calculus: A case study of India. *Journal for the Education of Gifted Young Scientists, 8*(1), 407-433. https://doi.org/10.17478/jegys.660201

Ariany, R. L., Widiastuti, T. T., Jauhari, A. L. R., & Fardillah, F. (2021). Classification of student's mathematical reflective thinking in calculus class. *Journal of Physics: Conference Series, 1764*, Article 012117. https://doi.org/10.1088/17426596/1764/1/012117

Bennoun, S. & Holm, T. (2021). Establishing consistent active learning in a calculus I course. *PRIMUS, 31*(3-5), 565-577. https://doi.org/10.1080/10511970.2020.1746453

Braun, V., & Clarke, V. (2012). Thematic analysis. In H. Cooper, P. M. Camic, D. L. Long, A. T. Panter, D. Rindskopf, & K. J. Sher (Eds.), *APA handbook of research methods in psychology, vol. 2. Research designs: Quantitative, qualitative, neuropsychological, and biological* (pp. 57-71). American Psychological Association. https://doi.org/10.1037/13620-004

Bressoud, D., Ghedamsi, I., Martinez-Luaces, V., & Törner, G. (2016). *Teaching and learning of calculus: ICME-13 topical surveys*. Springer. https://doi.org/10.1007/978-3-319-32975-8

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45-75. https://doi.org/10.1007/s10649-005-0808-x

Carlson, M. P., Smith, N., & Persson, J. (2003). Developing and connecting calculus students' notions of rate-of change and accumulation: The fundamental theorem of calculus. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th International Group for the Psychology of Mathematics Education Conference* (vol. 2, pp. 165-172). Honolulu, HI.

Creswell, J. (2013). *Qualitative inquiry and research design: Choosing among five approaches* (3rd ed.). Sage.

Da, N. T. (2023). The effect of realistic mathematics education on the problem-solving competency of high school students through learning calculus topics. *Contemporary Mathematics and Science Education, 4*(1), Article ep23013. https://doi.org/10.30935/conmaths/13041

Dewey, J. (1938). *Education and experience*. Horace Liveright.

Duval, R. (2017). *Understanding the mathematical way of thinking–The registers of semiotic representations*. Springer. https://doi.org/10.1007/978-3-319-56910-9

Grossman, J., Grossman, M., & Katz, R. (1983). *Averages: A new approach*. The University of Michigan.

Hamdan, M. (2019). Filling an area discretely and continuously. *International Journal of Mathematical Education in Science and Technology, 50*(8), 1210-1222. https://doi.org/10.1080/0020739X.2019.1617440

Hardy, M. (2011). The basic principle of calculus? *PRIMUS, 21*(7), 659-668. https://doi.org/10.1080/10511970.2010.518550

Kouropatov, A., & Dreyfus, T. (2013). Constructing the fundamental theorem of calculus. In A. Heinze, & A. Lindmeier (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education Methodology and Methods in Mathematics Education* (vol. 3, pp. 201-208).

Larsen, S., Marrongelle, K., Bressoud, D. M., & Graham, K. (2017). Understanding the concepts of calculus: Frameworks and roadmaps emerging from educational research. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 526-550). National Council of Teachers of Mathematics.

Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education, 5*(1), 129-146. https://doi.org/10.1007/s40753-019-00085-6

López-Leyton, C., Aldana-Bermúdez, E., & Flórez-Laiseca, A.-M. (2024). A socio-epistemological approach articulated with problem-solving in higher education: Teaching of integral calculus. *Eurasia Journal of Mathematics, Science and Technology Education, 20*(12), Article em2548. https://doi.org/10.29333/ejmste/15704

Mingus, T., & Koelling, M. (2021). A collaborative approach to coordinating calculus 1 to Improve student outcomes, *PRIMUS, 31(*3-5), 393-412. https://doi.org/10.1080/10511970.2020.1772919

Mkhatshwa, T. P. (2024). Best practices for teaching the concept of the derivative: Lessons from experienced calculus instructors. *Eurasia Journal of Mathematics, Science and Technology Education, 20*(4), Article em2426. https://doi.org/10.29333/ejmste/14380

Pedersen, S. (2015). *From calculus to analysis.* Springer. https://doi.org/10.1007/978-3-319-13641-7

Raelin, J. A. (2007). Toward an epistemology of practice. *Academy of Management Learning and Education, 6*(4), 495-519. https://doi.org/10.5465/amle.2007.27694950

Reinholz, D. (2016). Developing mathematical practices through reflection cycles. *Mathematics Education Research Journal, 28,* 441-455. https://doi.org/10.1007/s13394-016-0175-1

Robles, M., Tellechea, E., & Font, V. (2014). Una propuesta de acercamiento alternativo al teorema fundamental del cálculo [A proposal for an alternative approach to the fundamental theorem of calculus]. *Educación Matemática, 26*(2), 69-109. https://doi.org/10.24844/EM2602.03

Rondero, C. (2010). Cálculo promedial. El caso de la media aritmética [Average calculation. The case of the arithmetic mean]. *Revista Latinoamericana de Investigación en Matemática Educativa, 13*(4), 387-408.

Rondero, C., & Font, V. (2015). Articulación de la complejidad matemática de la media aritmética [Articulation of the mathematical complexity of the arithmetic mean]. *Enseñanza de las ciencias: Revista de investigación y experiencias didácticas, 33*(2), 29-49. https://doi.org/10.5565/rev/ensciencias.1386

Schön, D. (1983). *The reflective practitioner: How professionals think in action*. Basic Books.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334-370). MacMillan.

Sokolowski, A. (2021). Modelling the fundamental theorem of calculus using scientific inquiry. In F. K. S. Leung, G. A. Stillman, G. Kaiser, & K. L. Wong (Eds.), *Mathematical modelling education in east and west, international perspectives on the teaching and learning of mathematical modelling* (pp. 433-442). Springer. https://doi.org/10.1007/978-3-030-66996-6_36

Stanberry, M. L. (2018). Active learning: A case study of student engagement in college calculus. *International Journal of Mathematical Education in Science and Technology, 49*(6), 959-969. https://doi.org/10.1080/0020739X.2018.1440328

Tall, D. (2012). A Sensible approach to the calculus. *El Cálculo y su Enseñanza, 3*(1), 81-128. https://doi.org/10.61174/recacym.v3i1.139

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2-3), 229-274. https://doi.org/10.1007/BF01273664

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). National Council of Teachers of Mathematics.

Thompson, P. W. & Dreyfus, T. (2016). A coherent approach to the fundamental theorem of calculus using differentials. In R. Göller, R. Biehler, & R. Hochsmuth (Eds.), *Proceedings of the Conference on Didactics of Mathematics in Higher Education as a Scientific Discipline* (pp. 355-359). Hannover, Germany: KHDM.

Thompson, P., & Harel, G. (2021). Ideas foundational to calculus learning and their links to students' difficulties. *ZDM-Mathematics Education, 53*, 507-519. https://doi.org/10.1007/s11858-021-01270-1

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson, & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43-52). Mathematical Association of America. https://doi.org/10.5948/UPO9780883859759.005

Zubieta, G., & Meza, R. (2022). El teorema fundamental del cálculo: Una versión contemporánea y la otra basada en las ideas de I. Barrow, utilizando GeoGebra [The fundamental theorem of calculus: A contemporary version and the other based on the ideas of I. Barrow, using GeoGebra]. *El Cálculo y su Enseñanza, 18*(1), 53-60. https://doi.org/10.61174/recacym.v18i1.188