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USING DRAWINGS AND GENERATING INFORMATION IN MATHEMATICAL PROBLEM SOLVING PROCESSES

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ABSTRACT. The purpose of this paper is to investigate how drawings can contribute to generating new information when solvers use drawings in solving mathematical problems. For this purpose, two episodes, in which drawings enabled a solver to find ideas useful for his solutions, were qualitatively and closely analyzed, especially focusing on what roles drawings could play when the solver found new information about the problem situations. The analysis demonstrates the following aspects of use of drawings: (a) drawings can contribute to information generation by producing unexpected combinations or configurations of elements solvers identified in the problem situations previously, which led the solver to recognizing emergent patterns; (b) the use of drawings includes cyclic processes, through which the solver's understanding of the problem situations and his way of interacting with drawings varied each other. Implications for instruction will be also discussed on the basis of such aspects.

KEYWORDS. Mathematical Problem Solving, Mathematical Thinking, Heuristic Strategies, Use of Drawings, Problem Solving Processes.

INTRODUCTION

In discussing the use of drawings, one of the most well-known heuristic strategies for mathematical problem solving, the importance of self-generated drawings has attracted the attention (Gutstein & Romberg, 1995; Van Essen & Hamaker, 1990). Some of the recent researches have attended to the fact that drawings are usually modified or altered during problem solving processes (Bremigan, 2005; Nunokawa, 2000). For example, Gibson (1998) analyzed the problem solving processes of university students to investigate the effect of drawings in developing proofs. He observed that, in their search for proofs, "students added to, subtracted from, and redrew" their drawings and their alteration of drawings "allowed students to explore possible scenarios within the problem situation and, as a result, come up with ideas" (p. 296). In analyzing in detail the solution processes of high- and low-achieving secondary students, Lawson and Chinnappan (1994) considered the modification of drawings to be a kind of information generation and paid attention to the order in which each element of the drawings was drawn. Diezmann & English (2001) reported one student's solving process where the student's reflection on the adequacy of her diagram enabled the change in her understanding of the problem structure. Diezmann (2000) pointed out that diagrams were dynamic rather than static representation and it could be beneficial to produce more than one diagram.

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Such focus has led to the research on how drawings are actually used in mathematical problem solving. These works have shown the interrelationships between use of drawings and analytical thinking (Hershkowitz et al, 2001; Nunokawa, 1994, 2004; Stylianou, 2002; Zazkis et al, 1996). They have showed the interwoven relationship between visualizing and analyzing modes in the use of drawings, and highlighted solvers' analytical operation on drawings. For example, Stylianou (2002) pointed out the importance of systematic and thorough explorations of drawings. There is, however, little research that explores how drawings per se can contribute to generating new information in such processes.

Larkin & Simon (1987) seemed to present a kind of contribution of drawings in problem solving. They mentioned as one of the merits of use of drawings the fact that drawings can group together all information that is used together and enable a solver to avoid large amount of search for the elements needed to make a problem-solving inference (p. 98). Their analysis, however, utilized the input data structured for their cognitive model and the completed diagrammatic representation of that data. In order to understand how drawings can help students solve problems, this feature of drawings should be reexamined through the analysis of authentic problem solving processes in the context of the recent research trend to emphasize the processes of using drawings, because students who face a difficulty in solving a problem cannot necessarily construct perfect drawings immediately.

The purpose of this paper is to investigate contributions of drawings by analyzing actual problem solving processes and to attempt to deepen our understanding of how drawings can help students solve mathematical problems. For this purpose, two episodes, in which a student tackled rather tough mathematical problems and drawings enabled him to find ideas useful for solutions, were analyzed in detail. In these analyses, the focus was laid especially upon what roles drawings could play when the solver found new information about the problem situations.

METHOD

The subject was the same graduate student as the subject of Nunokawa (2004), who studied mathematics education and worked as a part-time teacher of mathematics at a senior high school. Thus, as far as high-school-level mathematics is concerned, he could be considered an expert problem solver. The problems from Klamkin (1988) were used to provide genuine problem solving settings for this subject. He participated in 9 problem solving sessions.

All sessions were recorded by audio and video tape recorders. Those records were transformed into protocols of the solution processes. In transcribing, steps of making each drawing were incorporated into the protocols, as well as utterances of the solver and the written sentences and expressions. Based on such protocols and the video-taped records, the relationships between the steps of drawing and the solver's utterances or thought were analyzed to explore how the solver obtained new information and how drawings contributed to generating new information in the process of using drawings.

THE SOLVER'S USE OF DRAWINGS IN SOLVING MATHEMATICAL PROBLEMS

Episode 1

Episode 1 were excerpted from the seventh session. In that session, the solver made efforts to solve the following problem:

> If A and B are fixed points on a given circle and XY is a variable diameter of the same circle, determine the locus of the point of intersection of lines AX and BY. You may assume that AB is not a diameter. (Klamkin, 1988, p. 5)

The problem was presented to the solver without drawings. First, the solver attempted to find out a mathematical expression of the locus of the intersection point for about 30 minutes. After he failed that attempt, he decided to explore the problem situation to have a 'vision' about the solution. He drew the drawing shown in Figure 1, and then drew Figure 2 after getting an insight through the use of Figure 1. The solver drew those drawings in the following manner.

(a) The solver drew the circle and the coordinate axes in Figure 1. Set the points A and B. Drew the diameter XY. Then, he drew AX and BY to make an intersection point, Q1. (Fig. 3(i))

(b) The solver drew AY and BX to make another intersection point, Q2. (Fig. 3(ii))

(c) After tracing X'Y', he drew BY' and the tangent at A to make an intersection point, P. Moreover, he drew BX' and AY' to make another intersection point, which is the same as the point A. (Fig. 3(iii))

(d) He uttered, "It can vary rather extensively" and "this may be a circle." Then, he drew another circle so that it could pass through the four intersection points he had constructed. (Fig. 3(iv))

(e) Saying, "It would be enough to prove that this angle and this angle are equal," he added dots to the angles at P and Q1,, as shown in Fig. 3(iv).

(f) The solver began to construct another drawing (Figure 2). He drew a circle, the fixed points A and B, and diameter XY. Then, he drew AX and BY to make an intersection point, Q. (Fig. 4(i))

(g) He drew a diameter X'Y'. Then, he drew BY' and the tangent at A to make another intersection, P. He added the angle marks to $\angle$AQY and $\angle$APY' and the right-angle mark to $\angle$PAY'. (Fig. 4(ii))

(h) He drew AY and BX and marked ∠BYA, ∠BY’A, and ∠BXA with small circles. After a while, he traced ΔAYQ and ΔAY’P with his pen. (Fig. 4(iii))

(i) He marked ∠YAY’ and ∠QAP with small x’s. After tracing ΔAY’P, he added the right-angle mark to ∠XAY. (Fig. 4(iv))

(j) He drew AB and added the right-angle marks to ∠ABY’ and ∠ABP. Finally, he added the right-angle mark to ∠QAY. (Fig. 4(v))

(k) After seeing this drawing, the solver began to write a proof.

Figure 1

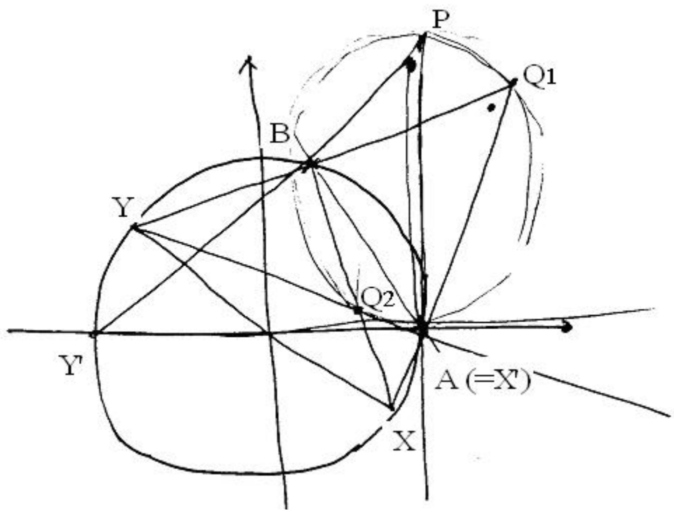


Figure 2

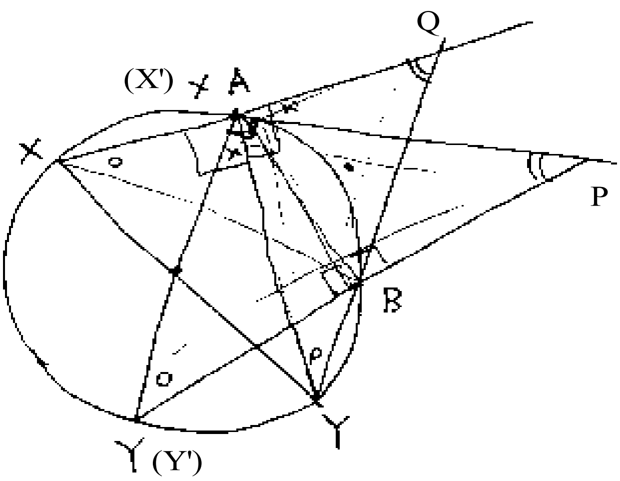


Figure 3

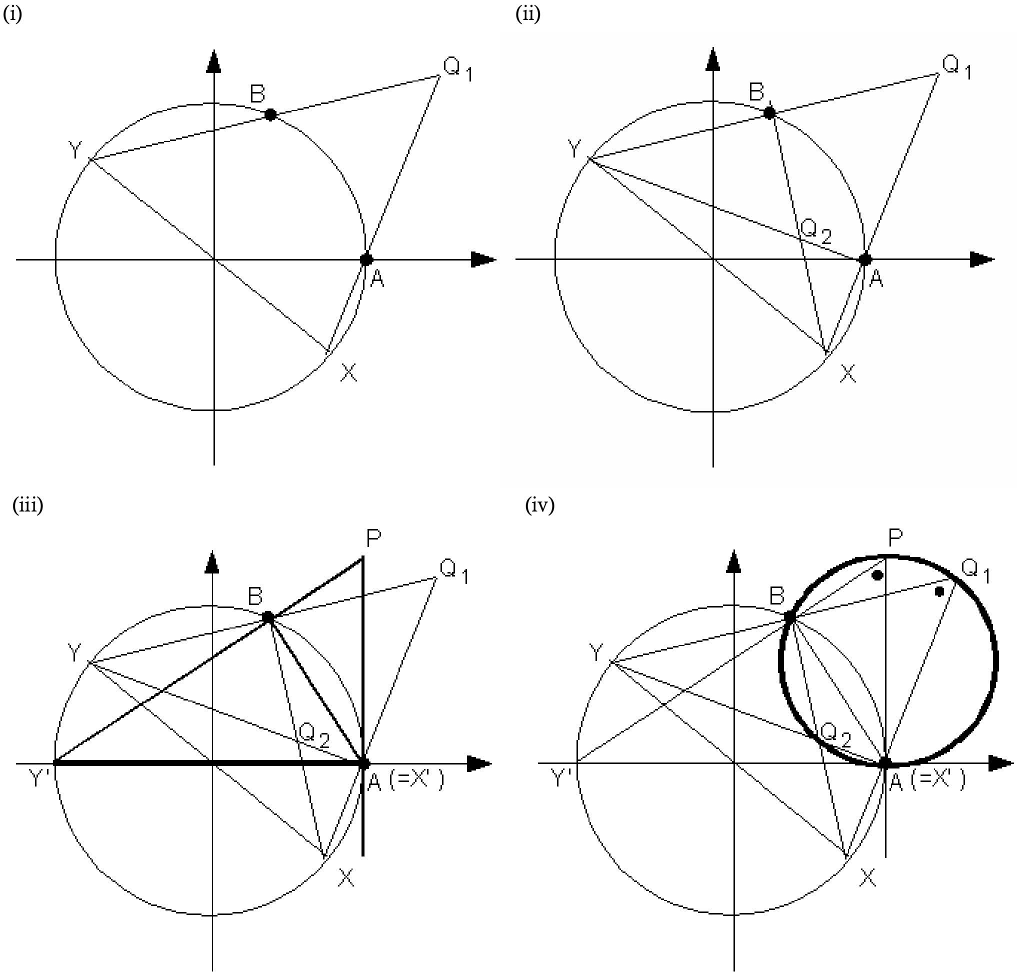
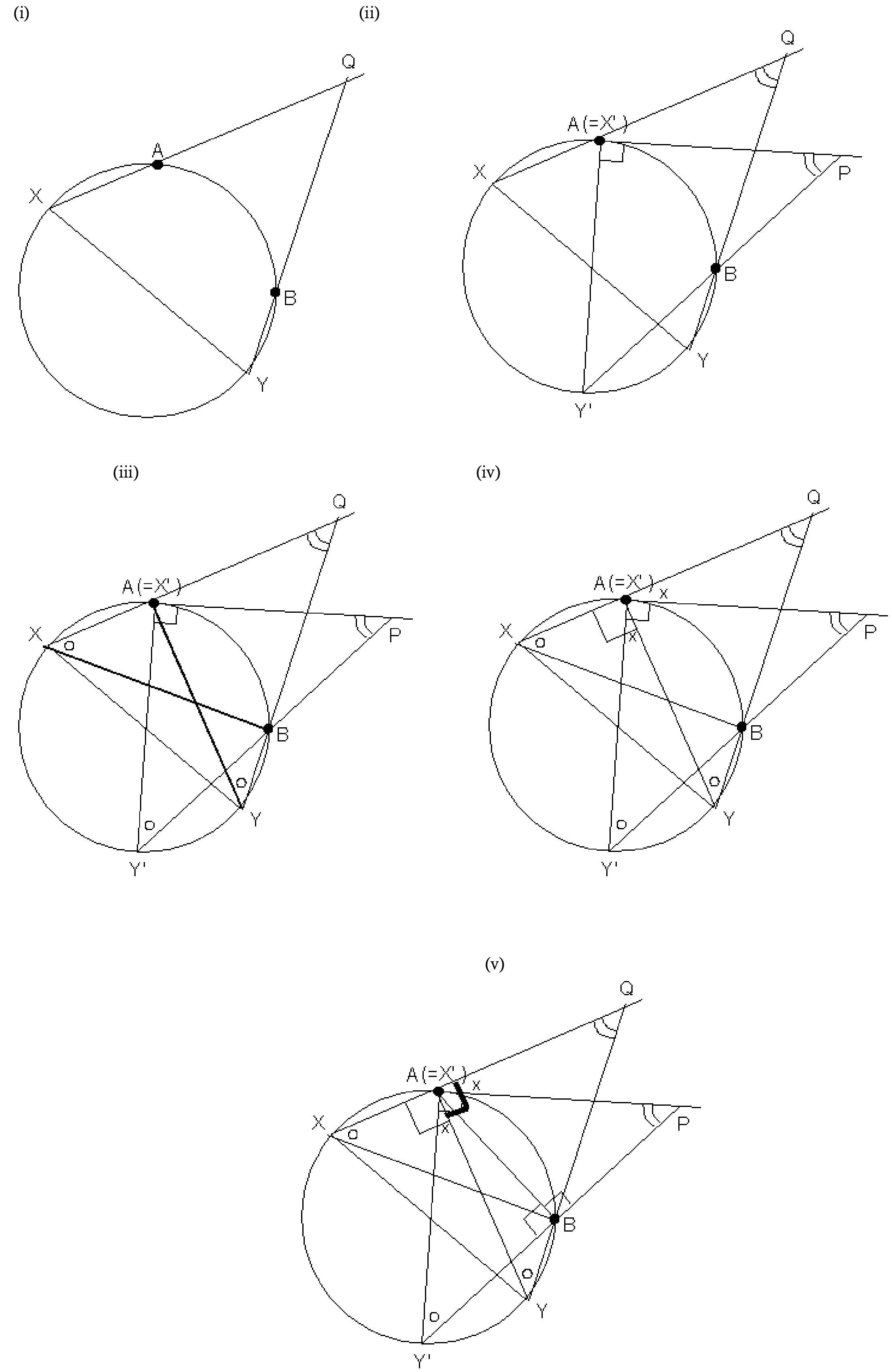


Figure 4



As the Figure 3(i), 3(ii), and 3(iii) show, the solver constructed some interaction points following the problem statement. Although he seemed to draw only one diameter in this drawing, he succeeded in representing the varying of the diameter by using the diameter and x-axis cleverly and constructing four intersection points (P, Q1, Q2, and A) in one drawing (Nunokawa, 2004). His action of representing what is given in the problem statement led to a kind of combination of four intersection points. This combination made him utter "it can vary rather extensively" and "this may be a circle" at stage (d). Some researchers of creativity pointed out that properties which emerge through combinations of existing elements are important for discovery (Roskos-Ewoldson, 1993; Sawyer, 2003). In this case, a certain extensity emerged through the combination of four intersection points. The solver recognized such an emergent pattern in the drawing and made sense of it as a circle.

It should be noted here that the solver saw this combination of elements with the information he had obtained during the previous activity. In the interview concerning episode 1, the solver told that he had noticed that the locus in question could be expressed only by sine and cosine functions through his calculations implemented before drawing Figure 1. Thus, it can be said that the above-mentioned emergent pattern occurred at the contact point between the combination appearing in the drawing and the solver with the related information.

The solver drew another circle at stage (d), as shown in Figure 3 (iv), to represent the information he had just obtained. This action in turn led to another combination of elements, a combination of the circle and two pairs of lines constituting the intersection points. The solver recognized a new emergent pattern here, two inscribed angles in a circle. AP, BP, AQ1, and BQ1 were the lines given in the problem statement and originally were drawn to make intersection points. The combination of the circle and these lines was not intended in advance and it changed the pairs of lines into inscribed angles of this circle. This emergent pattern suggested to the solver new information that the measures of angles at intersection points would be constant.

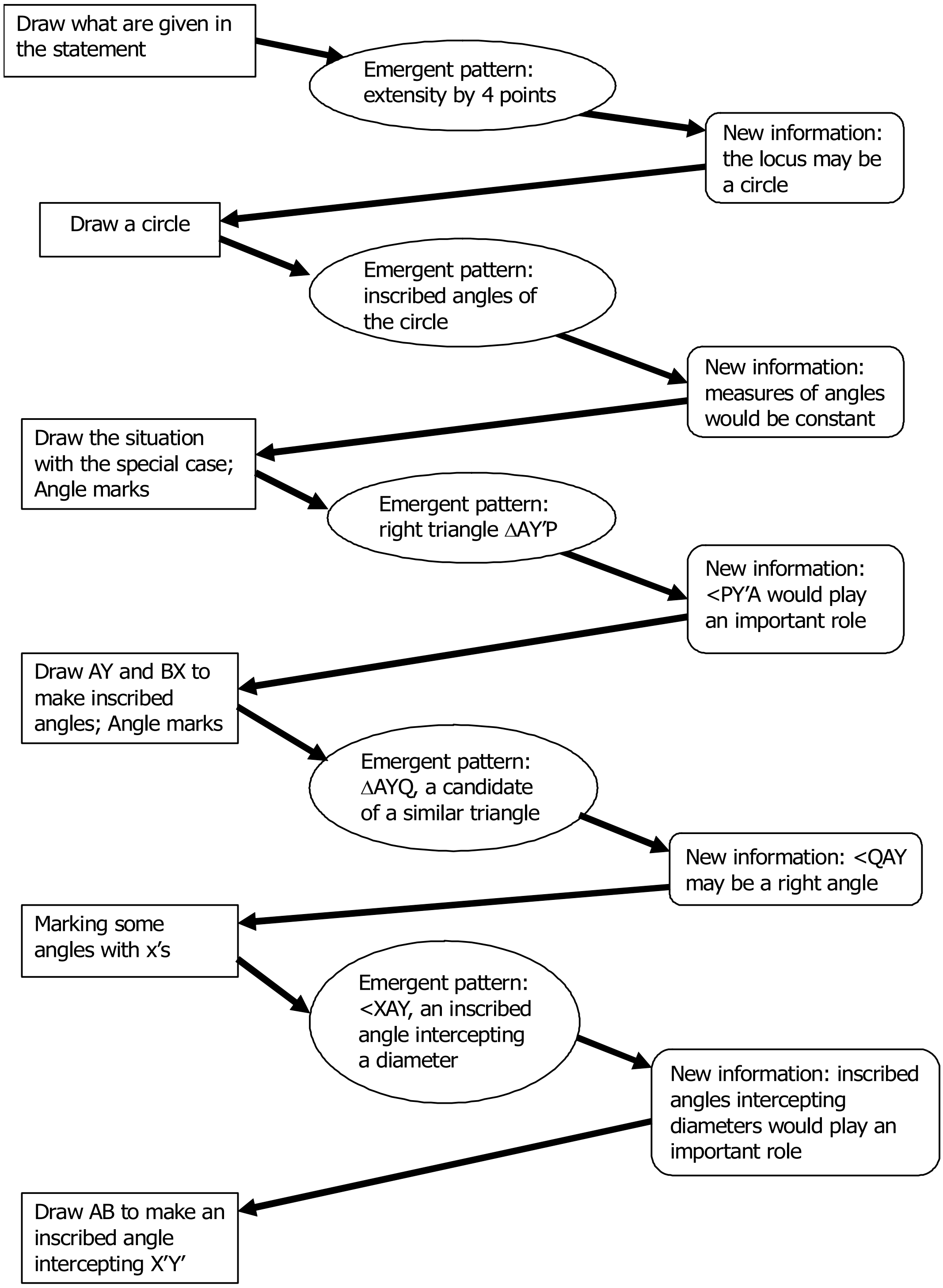
At stage (f), the solver constructed new drawing based on this information, saying "I will try a special case." As Figure 4 (ii) shows, this drawing (Figure 2) included a special case where one of the end points of the diameter coincided with the fixed point A. Here, it should be noted that this special case had been used long before he drew Figure 2. The special case was originally adopted by the solver when he calculated an expression of the locus of the intersection point before he drew Figure 1. He introduced it as the case where the denominator of one expression was equal to zero. This case was 'quoted' in Figure 1 in exploring the movement of the intersection point. Thus, it can be said that the solver drew Figure 4 (ii) by associating what he knew at that stage. Since he came to attend to angles at stage (e), however, he added an angle mark to $\angle$PAY' as well as $\angle$AQY and $\angle$APY'. This marking produced a combination of three lines, all of which were drawn to make $\angle$APY', and the angle marks. This combination led to a new emergent pattern, ΔAY'P. Because one of the angles of this triangle was an angle at an intersection point and another was a right angle, the remaining angle of this triangle, ∠PY'A, came to attract the attention of the solver.

This made the solver draw AY and BX to make inscribed angles intercepting the same arc as ∠PY'A. In fact, he marked the newly constructed inscribed angles with the same marks as the ∠PY'A. This marking made a combination of the other three lines and the angle marks. Two of these lines were drawn to make ∠AQY and one of them was drawn to make an inscribed angle. However, this combination led the solver to another emergent pattern, ΔAYQ. The fact that the solver also drew an extra line BX suggests that the above combination was not expected in advance. He recognized this emergent pattern and traced it with his pen.

This recognition directed the attention of the solver to angles around the point A, because he could prove the conclusion if ∠QAY was a right angle. The fact that the solver marked ∠YAY' and ∠QAP with small x's seems a representation of such a subgoal. His attention to ∠QAY highlighted a combination of XQ and AY. Moreover, a combination of these lines and the x marks at ∠YAY' and ∠QAP might highlight a kind of correspondence of the right- and left-side of AY, and this left-side composed an emergent pattern, an inscribed angle intercepting a diameter. Although XY, XA and AY were introduced independently, this combination led to an emergent pattern which could complete his proof. The solver tried to check inscribed angles intercepting another diameter, X'Y', and drew AB.

The flow of the solver's use of drawings can be summarized as Figure 5.

Figure 5


Draw what are given in the statement
Emergent pattern: extensity by 4 points
New information: the locus may be a circle
Draw a circle
Emergent pattern: inscribed angles of the circle
New information: measures of angles would be constant
Draw the situation with the special case; Angle marks
Emergent pattern: right triangle ∆AY'P
New information: <PY'A would play an important role
Draw AY and BX to make inscribed angles; Angle marks
Emergent pattern: ∆AYQ, a candidate of a similar triangle
New information: <QAY may be a right angle
Marking some angles with x's
Emergent pattern: <XAY, an inscribed angle intercepting a diameter
New information: inscribed angles intercepting diameters would play an important role
Draw AB to make an inscribed angle intercepting X'Y'

Episode 2

Episode 2 was excerpted from the ninth session, in which the solver tried to solve the following problem:

> The inscribed sphere of a given tetrahedron touches all four faces of the tetrahedron at their respective centroids [i.e. centers of gravity]. Prove that the tetrahedron is regular. (Klamkin, 1988, p.9)

First, the solver sketched the problem situation and explored the situation using that sketch. Through this exploration, he came to need to check the following: whether medians of two adjacent faces and radii orthogonal to those faces were on the same plane. After that, he said, "I will re-draw it clearly," and drew the drawing in Figure 6. The solver obtained new information about the problem situation using that drawing. Then he drew the other drawings like Figure 7 and Figure 8. The solver drew those drawings in the following manner.

(a) The solver drew two triangles in Figure 6 as adjacent faces of the given tetrahedron. Then, he drew the median lines AM and DM of these faces. He also added some dots on the median lines as trisecting points to indicate the centre of gravity of each face. (Fig. 9(i))

(b) The solver drew Oa and Od, each of which connected the centre of gravity of each face and the centre of the inscribed sphere. He added the right-angle marks to $\angle$DaO and $\angle$AdO. Writing "What I want to prove is that this angle is $\angle$R," he added the right-angle mark to $\angle$AMC. (Fig. 9(ii))

(c) The solver added the short double lines to Oa and Od for expressing Oa=Od. (Fig. 9(iii))

(d) After mentioning the equality of some right angles, the solver drew OM and traced Oa, OM, and $\angle$DaO with his pen. (Fig. 9(iv))

(e) After checking ΔOaM and ΔOdM, the solver marked dM and aM with small circles for expressing dM=aM. This relationship led him to concluding AM=DM. (Fig. 9(v))

(f) The solver attempted to extend this equality of medians to other pairs of medians and began to construct a new drawing. He drew the triangle ABC in Figure 7. He also drew its median, AM. (Fig. 10(i))

(g) He drew BD and CD to make another triangle. He drew the median of this triangle. (Fig. 10(ii))

(h) He drew A'B, A'D, A"C, and A"D to make two more triangles. (Fig. 10(iii))

(i) The solver drew the line A'C mentioning the equality of medians. He also drew A"B. (Fig. 10(iv))

(j) He marked AM, DM, BM, and CM with the pairs of double lines mentioning that AD and BC were bisected. (Fig. 10(v))

(k) The solver marked AC and BD with short lines saying that the opposite sides were equal. He also marked AB and CD. (Fig. 10(vi))

(l) The solver began to draw another drawing shown in Figure 8. He drew a larger triangle AA'A" first, then drew BC, CD, and DB by connecting points on the sides of this triangle. (Fig. 11(ii))

(m) The solver drew three long lines, AD, A'C, and A"B. (Fig. 11(iii))

(n) He marked A'B and DC with small circles. He also marked A'D and BC with small triangles and DB and A"C with small crosses, as shown in Figure 11(iv).

(o) The solver added another small cross to AC. He also added a small triangle to A"D and a small circle to AB. Saying that he could show that all four triangles were congruent, he drew a smaller triangle within each face. (Fig. 11(v))

(p) After he failed to show that each face was equilateral, the solver redrew a similar drawing as shown in Figure 11(vi). After he drew two lines A'C and AD, he marked the intersection of these lines with a dot and said that that point was a center of gravity and all the three lines intersected at one point. Then, he mentioned that showing that the lengths of the longer lines were equal (i.e. AD=A'C=A"B) would be enough for proving the conclusion.

Figure 6

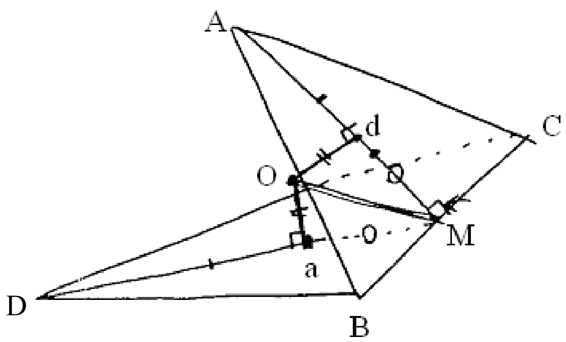


Figure 7

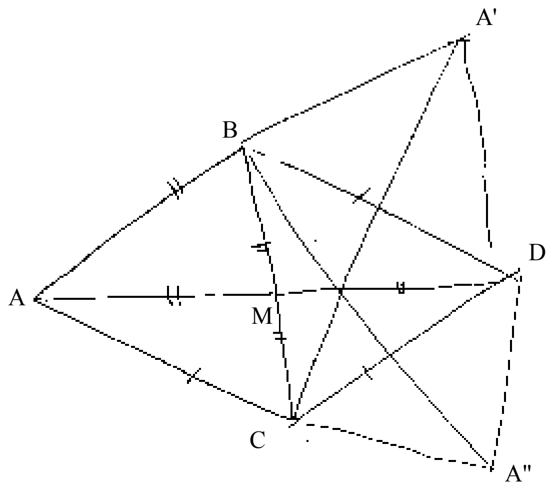


Figure 8

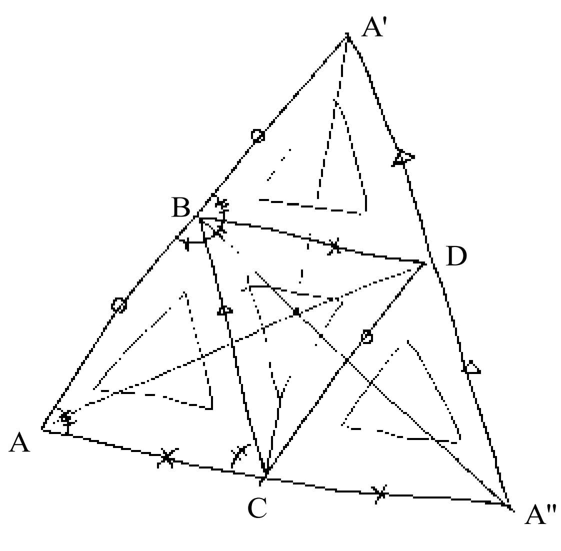


Figure 9

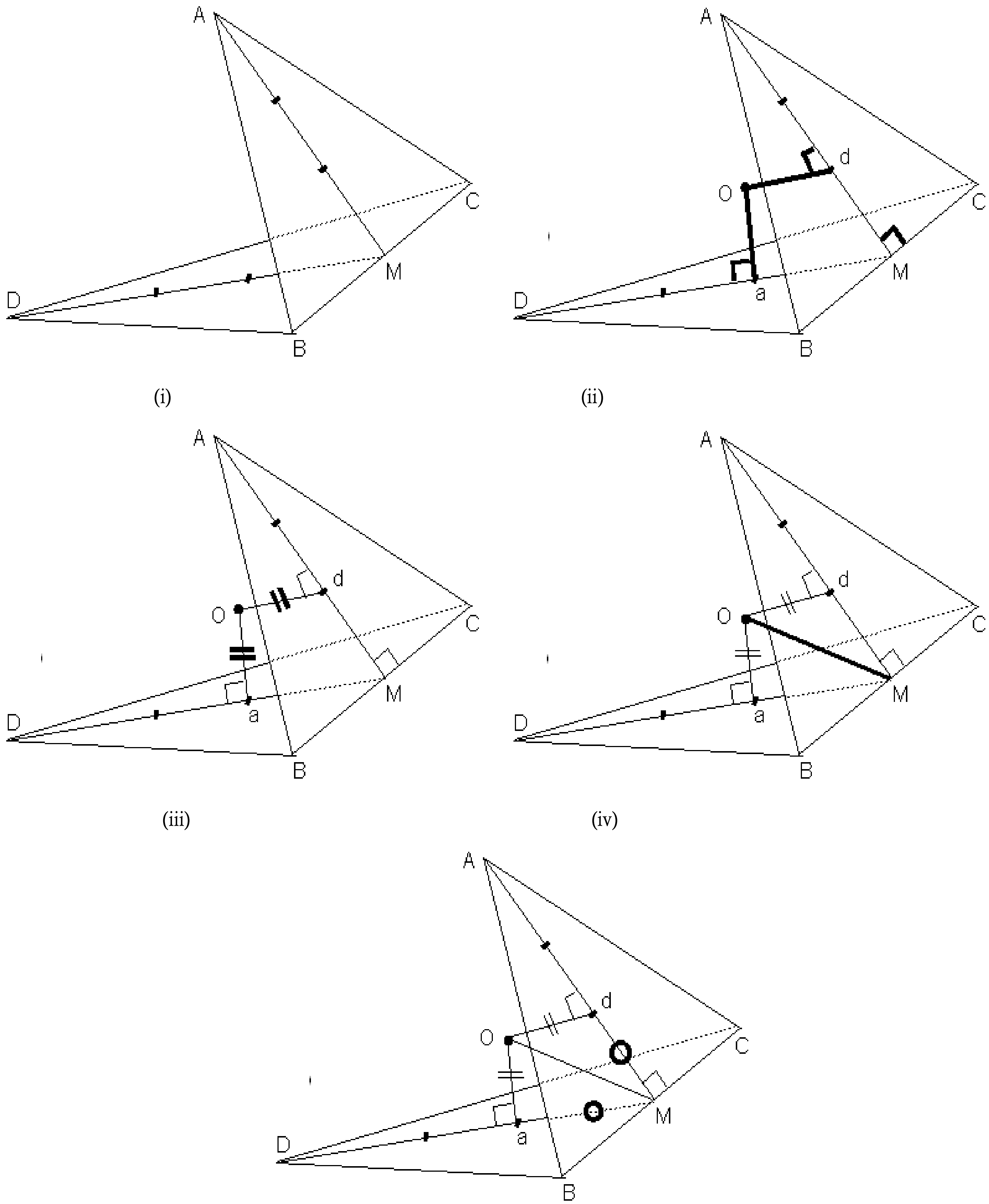


Figure 10

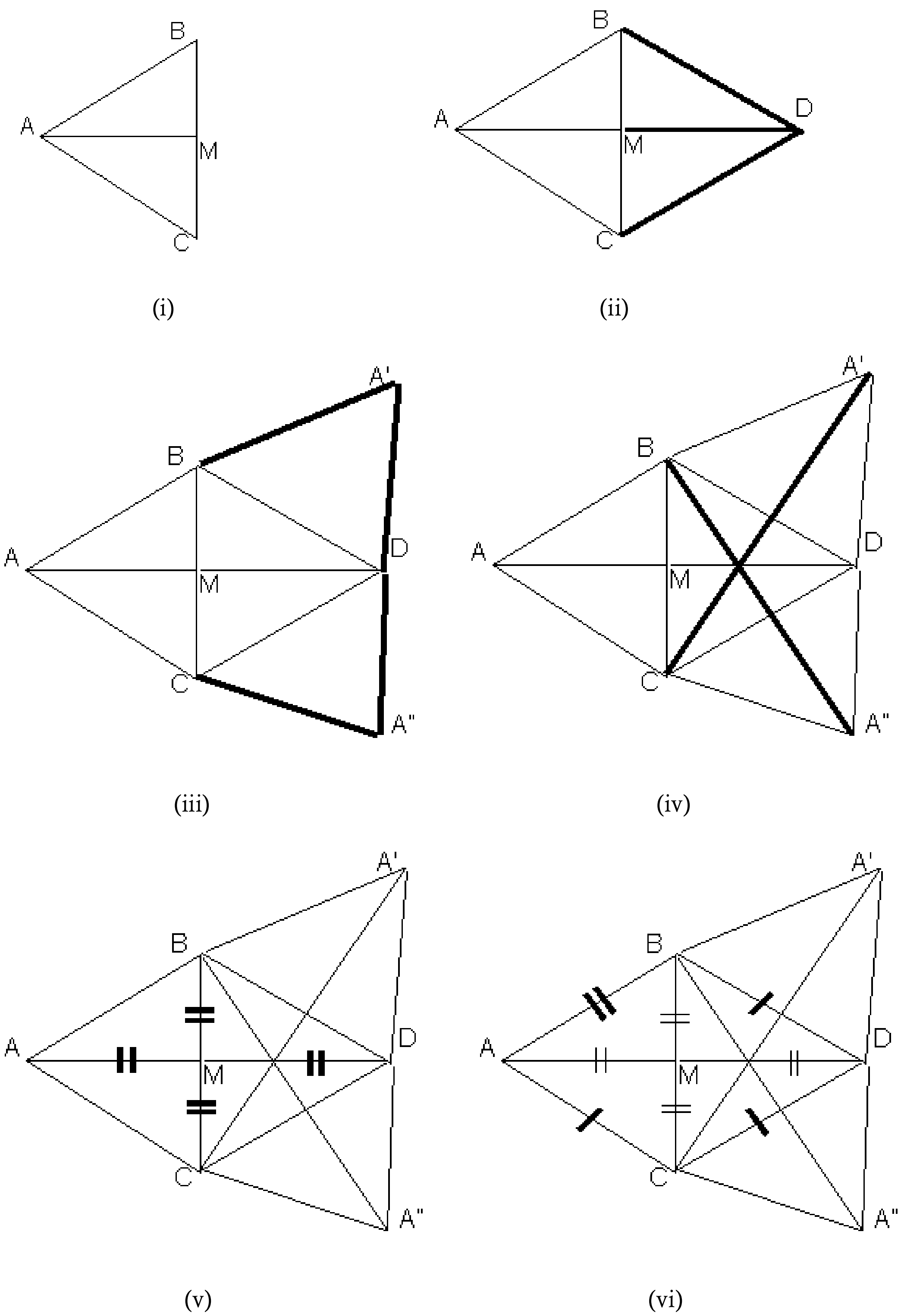
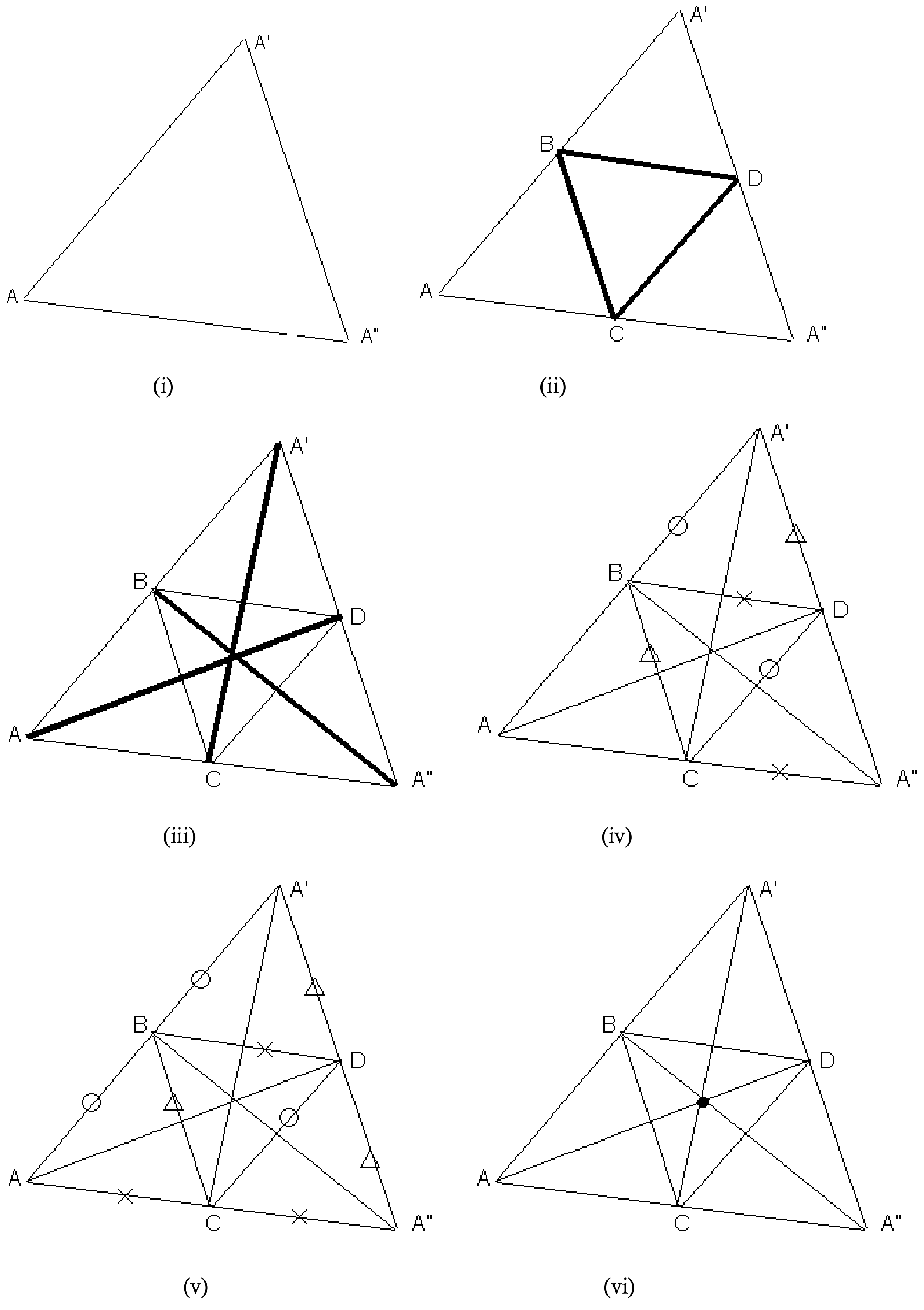


Figure 11



Two of the trisecting points added at stage (a) represented the centers of gravity of the drawn faces. Two line segments marked with the short double lines at stage (c) represented the sphere. The right-angle marks added at stage (b) indicated that this sphere was tangent to the faces of the given tetrahedron. Eliciting only two adjacent faces was influenced by the information he had obtained during his previous activity (see Nunokawa, 2004). But, the solver seemed to represent the problem situation given in the problem statement at stage (a), (b), and (c).

In representing the problem situation, the drawing produced a combination of the elements, AM, DM, Oa, Od, and some marks. And this combination led to an emergent pattern, a quadrilateral like a kite, which could become a cue of one of the diagram configuration schemata, "Congruent-Triangle-Shared-Side schema" (Koedinger & Anderson, 1990). His recognition of this emergent pattern informed him that there might be two congruent triangles in this pattern. Following this information, he drew line OM at stage (d). The combination of the above-mentioned pattern and line OM led to another emergent pattern, two right triangles with the common hypotenuse. Checking that these triangles were congruent to each other, the solver obtained new information that the median lines of two adjacent faces had the same lengths and he marked those lines with circles. He might also obtain the information that median lines of other pairs of adjacent faces would have the same lengths, because he had noticed the symmetric nature of the problem situation at the earlier part of his solving process.

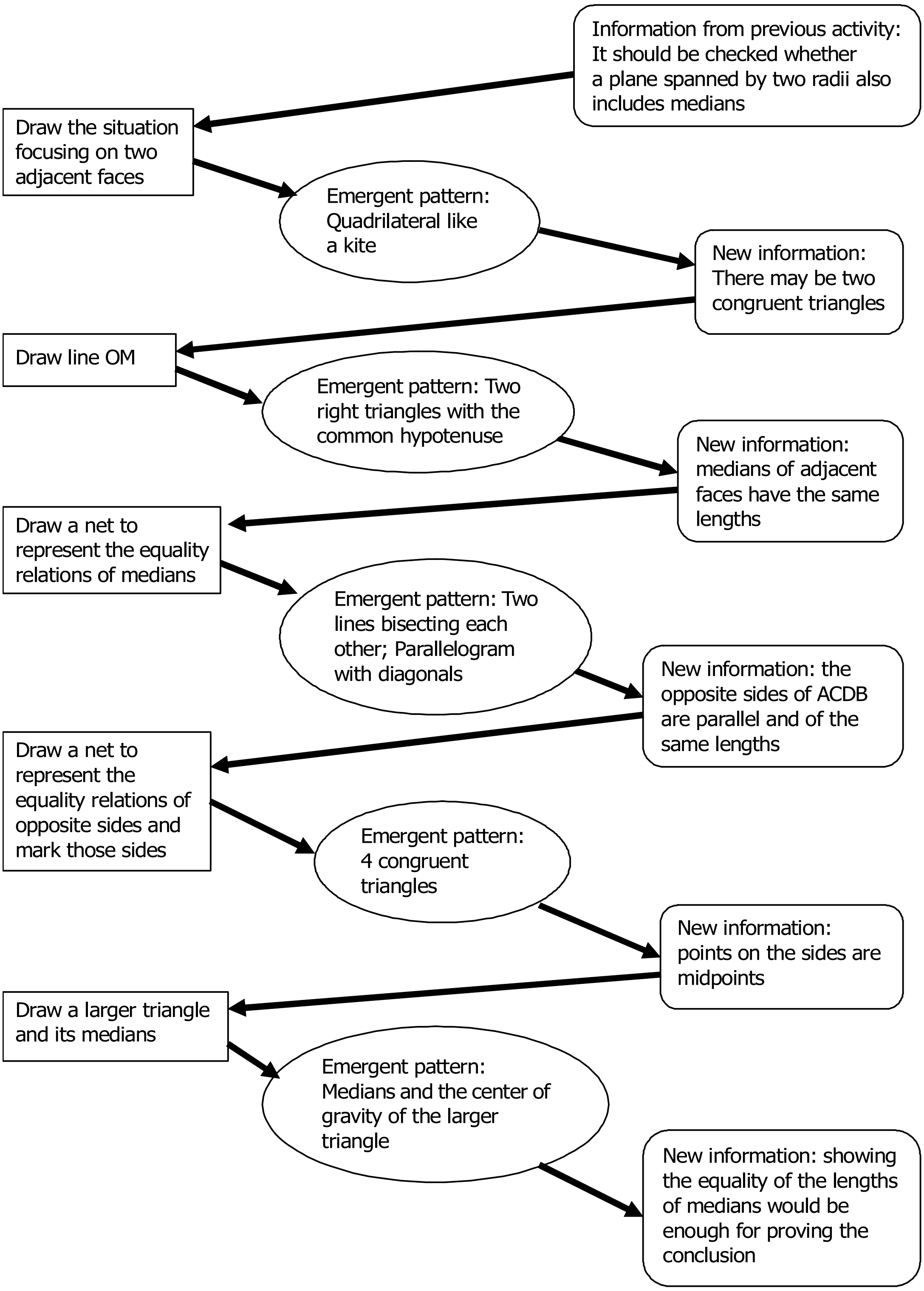
To represent such information, the solver drew a net including all the four faces of the given tetrahedron. After representing the equality of the medians he had found during the previous activity (Figure 10(i) and (ii)), he constructed two more triangles and drew median lines of those triangles (Figure 10(iii) and (iv)). When he marked some line segments to represent the equality relationships concerning those lines, a combination of two lines, AD and BC, and same-length marks occurred in the drawing. This combination led to an emergent pattern, two lines bisecting each other. Since the marks on BM and CM were added with the different intention from the marks on AM and DM, this combination might be unintended. This led to another emergent pattern, parallelogram ACDB and its diagonals. This pattern recognition was based on his implicit and incorrect assumption: Two medians of adjacent faces became one long line (see Figure 10(iv)). The recognized pattern informed him that the opposite sides of the quadrilateral were parallel to each other and of the same length.

To represent this information, the solver marked some sides at stage (k). Furthermore, he redrew the net at stage (l) to represent similar relationships among the sides of four faces of the tetrahedron. In drawing this net, the solver used the information that the opposite sides of each quadrilateral were parallel. This implied that, for example, AB was parallel to A'B and AB+BA' became one line. Thus, the use of the larger triangle AA'A" can be seen a representation of the information he had obtained before (Figure 11(i)). At this stage, the equality relations of the adjacent median lines were not expressed in the drawing. This means that Figure 8 was drawn to represent the information that the opposite sides of each pair of faces were of the same lengths. The solver marked all the sides of the four faces at stage (n) and (o). Through this marking, a combination of the sides and the three kinds of marks on them occurred in the drawing. This combination led to an emergent pattern, four congruent triangles in the net. It also led to another emergent pattern, the longer side bisected by a point on it (e.g. side AA' bisected by point B). Figure 11(iv) suggests that this bisection was not assumed in drawing Figure 8. The pattern implied that each point on the side of ΔAA'A" was a midpoint of that side.

To represent what he had found at the previous stage, he drew another net at stage (p). Although he did not mark any sides, the above information might let him consider that the lengths of the adjacent sides (e.g. AC and A"C) were equal. When he drew A'C and AD in Figure 11(vi), a combination of these lines and their intersection occurred in it and this combination led to an emergent pattern, the medians of the larger triangle and its center of gravity. When he found that four faces were congruent at stage (o), he mentioned that all the faces should be equilateral. It implied that a larger triangle, ΔAA'A", should be equilateral because it was similar to each face. This requirement and the above-mentioned emergent pattern, medians and a center of gravity of the larger triangle, might remind him of the mathematical fact that a triangle whose medians have the same lengths is equilateral. This is the mathematical knowledge that can be used for a solution of this proof problem (cf. Nunokawa, 2001). Although the solver could not solve this problem completely, he could arrive at some of the essential pieces of its solution through his use of drawings.

The flow of the solver's use of drawings can be summarized as Figure 12.

Figure 12


Information from previous activity: It should be checked whether a plane spanned by two radii also includes medians
Draw the situation focusing on two adjacent faces
Emergent pattern: Quadrilateral like a kite
New information: There may be two congruent triangles
Draw line OM
Emergent pattern: Two right triangles with the common hypotenuse
New information: medians of adjacent faces have the same lengths
Draw a net to represent the equality relations of medians
Emergent pattern: Two lines bisecting each other; Parallelogram with diagonals
New information: the opposite sides of ACDB are parallel and of the same lengths
Draw a net to represent the equality relations of opposite sides and mark those sides
Emergent pattern: 4 congruent triangles
New information: points on the sides are midpoints
Draw a larger triangle and its medians
Emergent pattern: Medians and the center of gravity of the larger triangle
New information: showing the equality of the lengths of medians would be enough for proving the conclusion

USE OF DRAWINGS AND INFORMATION GENERATION

The analyses of the above two episodes show some insights into how drawings can contribute to generating new information and promote problem solving processes.

When the solver represented what he understood (e.g. the given conditions or the information obtained in the previous activities) in a drawing, a kind of combination or configuration of elements occurred in this drawing. While there was a case (i.e. drawing OM at stage (c) of Episode 2) in which the solver added an element in the anticipation that a certain emergent pattern could be recognized, in the other cases, the solver did not seem to know what kind of emergent pattern would be recognized in advance. The combination of elements occurred beyond his intention.

For example, in episode 1, the solver drew the line AY to make an inscribed angle congruent to a certain angle. However, representing it in the drawing brought about the combination of this line and other lines given in the problem statement, which would be recognized as an emergent pattern, ΔAYQ. The short time lag between marking angles and tracing ΔAYQ implies that this triangle was not intended in advance. In episode 3, the solver drew the situation focusing on two adjacent faces. He drew the medians of these faces and the radii orthogonal to these faces in order to represent the centers of gravity and the given sphere respectively, both of which were given in the problem statement. However, representing them in the drawing brought about the combination of these lines, which would be recognized as an emergent pattern corresponding to a diagram configuration scheme. When he recognized this emergent pattern, the solver said, "I've come to an unexpected place." His utterance implies that this emergent pattern was unexpected in advance.

Although such combinations or configurations of elements were supervenient, they seemed to trigger the solver's recognition of certain emergent patterns. Larkin & Simon (1987) pointed out that drawings can group together all information and enable a solver to avoid large amount of search. However, at least according to the analyses of the above two episodes, drawings could not only group together the elements, but also produce certain combinations or configurations of them, and drawings enabled the solver not only to avoid large amount of search for necessary elements, but also to notice unexpected emergent patterns. This role of producing supervenient combinations or configurations can be seen an important contribution of drawings, because it could support the creative aspects of the solver's thinking through bridging what he knew and what did not know. Since the combinations or configurations can be beyond solvers' intentions, they can make them reach novel information about problem situations.

While drawings could contribute to the solver's thinking through producing supervenient combinations or configurations, the above episodes also showed the limitation of this contribution of drawings. As discussed in the analysis of episode 1, when the solver saw the combination of the four intersection points which occurred in Figure 3 (iii), he had the information that the locus in question could be expressed only by sine and cosine functions, which he had obtained through his calculations implemented before drawing it. It might be said that this information about the problem situation helped the solver to recognize the emergent pattern. In Episode 3, it might be critical for attending to the combination of elements like a kite that the solver had the Congruent-Triangle-Shared-Side schema. What the solver knew at that time seemed to play significant roles in recognizing emergent patterns in drawings, in a similar way in which arithmetic facts play an essential role in identifying patterns in numerical data (Haverty et al., 2000). Whether an emergent pattern can be recognized depends upon whether a solver has knowledge or information which can support his recognition of that pattern, as well as upon how easily a produced combination or configuration of elements makes the solver recognize that pattern (e.g. typical configuration for a certain pattern or configuration with much noise).

This limitation of the contribution of drawings also implies that, as a solver obtains more information about a problem situation in his problem solving process, how the solver interacts with drawings about that situation can change and he may get able to recognize emergent patterns he could not identify before. Yerushalmy (2000) reported that, as their learning on functions proceeded, the students could elicit appropriate information from rough sketches of graphs because they connected the sketches to the structure of the symbolic rules, mathematical knowledge they had learned (p. 137). Although it may be rare that the mathematical knowledge base of a solver is changing in one problem solving process, especially in a problem solving at elementary and secondary level, it must be plausible that the solver's knowledge base about the problem situation grows in that problem solving process and that the solver can recognize emergent patterns he might not be able to notice before.

For example, although the triangle ΔAY'P existed in drawings when the solver drew Figure 1 in episode 1, it was recognized as an emergent pattern after he had noticed that the measure of $\angle$APY' would be constant at stage (e). The inscribed angle $\angle$XAY in Figure 2 appeared when the solver drew the line AY. However, it was recognized as an emergent pattern after he had noticed that $\angle$QAY would be a right angle. In episode 2, it was the information that the point C, for example, bisected the side of the larger triangle that enabled him to recognize the emergent pattern of the medians and the center of gravity of the larger triangle, while the longer lines (e.g. A'C) and their intersection already existed even in Figure 7 and Figure 8.

Diagrammatic summaries of two episodes (Figure 5 and 12) clearly show a cyclic nature of use of drawings in mathematical problem solving. This cyclic nature was partially supported by the fact that while drawings produced combinations or configurations of elements, new information obtained from emergent patterns suggested the ways of modifying drawings. However, it was also supported by the fact that what kinds of emergent pattern the solver could recognize might change as his knowledge base about the problem situations grew in his solving processes. Some researchers have highlighted the relationship between two modes of thinking, visualizing and analyzing, in the use of drawings. The analyses of the above episodes suggest that we should also attend to the relationship between changes in solvers' use of drawings and changes in solvers' knowledge base about problem situations.

CONCLUDING REMARKS

The analyses of this paper highlighted one contribution of drawings in generating new information: to make an unintended combination or configuration of known elements which leads to an emergent pattern. Even when the solver can find an appropriate emergent pattern, the information the solver will obtain is not necessarily an idea that will directly imply solutions, but additional information that can deepen the solver's understanding only a little further. Therefore, the use of drawings should be considered in the cycle discussed above, where a series of information generation deepens the solver's understanding of the problem situation gradually and may make it more likely for him to reach solutions (cf. Nunokawa, 2005).

Taking account of the influence of what solvers know, developing students' repertoire of diagram configuration schemata may be one way to develop students' competence to effectively use drawings in mathematical problem solving (see also Stylianou (2002)). The same discussion, however, highlighted the importance of the information solvers have about problem situations. Also in this sense, use of drawings should be treated in relation to its cyclic nature or gradual deepening of solvers' understanding through those cycles.

Following the above analyses, use of drawings cannot be automatically helpful. Therefore, when encouraging their students to use drawings in solving mathematical problems, it may be necessary for teachers to pay attention to the following issues: (a) what kinds of emergent pattern can be expected to appear when students represent what they know in drawings; (b) what kinds of information about problem situations or what kinds of mathematical knowledge will be necessary for identifying those emergent patterns. Above all, it seems necessary for teachers not to expect that the initial drawing will show students the ideas critical to solutions, but to support them so that they can go through the cycle of their use of drawings.

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