MODESTUM
EURASIA Journal of Mathematics, Science and Technology Education, 2024, 20(5), em2441
ISSN:1305-8223 (online)
OPEN ACCESS
Research Paper
https://doi.org/10.29333/ejmste/14466

Using video to develop pre-service teachers' noticing within a mathematical modelling context

Siyu Zuo [1], Lizhe Liu [1], Chunxia Qi [1*]

[1] Faculty of Education, Beijing Normal University, Beijing, CHINA

Received 10 September 2023 ▪ Accepted 10 April 2024

Abstract

Teacher education should foster teachers' competencies in teaching mathematical modelling since it is a demanding task. Mathematical modelling requires spontaneous reactions, which are related to the core of teachers' competencies–teacher noticing. The study designed a video-based course to develop these competencies in the context of mathematical modelling. The findings revealed a noticeable improvement in their noticing competencies after participating in the semester-long course. In particular, pre-service teachers' topical focus shifted towards modelling thinking and pedagogy. They made significantly more interpretive comments and used significantly more task-dimensional knowledge for teaching mathematical modelling to reason about events. These results appear to support the viability of using video for this purpose.

Keywords: mathematical modelling, pre-service teachers, teacher education, teacher noticing, video-based course

INTRODUCTION

Mathematical modelling has been a core goal of mathematics education worldwide (Kaiser, 2017). In many national curricula, modelling competencies play a central role (Faull, 2010; NGACBP & CoCSSO, 2010). In China, these have been identified as key competencies in mathematics curriculum standards (Ministry of Education of the People's Republic of China [MoE], 2017, 2022). However, the development of students' mathematical modelling competencies is a challenge to teachers who are comfortable with traditional instruction models (Doerr, 2007). Preparing pre-service teachers (PSTs) for how to teach mathematical modelling is a pressing task in teacher preparation.

Unlike conventional teaching, students' trajectories in doing mathematical modelling activities are complex and varied. Previous studies have also found that novice teachers experience difficulties concerning teaching mathematical modelling (Niss & Blum, 2020). This means that for teaching in a mathematical modelling context, teachers need an in-the-moment decision-making skill that is closely related to teacher noticing. Concept of noticing offers a framework (including skills of selective attention and knowledge-based reasoning) for understanding and supporting teachers' in-the-moment decisions and offers a mechanism to analyze situation-specific competencies (van Es & Sherin, 2002).

In the context of mathematical modelling, researchers have confirmed the validity of video instruments to measure teachers' noticing competencies (Alwast & Vorhölter, 2022). But very few studies have examined whether using video can develop noticing competencies in this context. Thus, a course dedicated to the development of PSTs' noticing in the context of mathematical modelling is necessary to familiarize PSTs with the pedagogy of mathematical modelling and to understand the student behaviors that occur in this process. For this reason, we designed a video-based course for developing PSTs' noticing in the context of mathematical modelling. To validate the effectiveness of this video-based course intervention, the central questions for this study were, as follows:

1. What changes occurred in PSTs' skills of selective attention within the context of mathematical modelling during the video-based course?
2. What changes occurred in PSTs' skills of knowledge-based reasoning within context of mathematical modelling during video-based course?

✉ zuosyu@126.com ✉ bnuliulizhe@163.com ✉ qichxia@126.com **(*Correspondence)**

Contribution to the literature

- Rather than most research focusing on training teachers in mathematical modelling knowledge, this study developed an instrument to assess PSTs' noticing within a mathematical modelling context.
- This study designed a video-based curriculum to promote this skill among PSTs to examine the changes of PSTs.
- This study validated the feasibility and effectiveness of using videos in a PST training course on transforming teacher noticing under mathematical modelling.

LITERATURE REVIEW

Teacher Noticing

Goodwin (1994), analyzing the professional activities of archaeologists and police officers, first introduced the concept of professional vision to explain why professionals and amateurs understand the same events and phenomena very differently. Sherin (2001) expanded the idea of professional vision to encompass teaching for teachers. Teachers' professional vision focuses on understanding the events happening in the classroom. When teachers observe the classroom, they constantly consider the events that they are concerned about and interpret the events based on their professional knowledge. Teachers' professional knowledge also influences what they attend. Sherin and van Es (2009) simplified this phenomenon into two processes:

(1) selective attention, where the classroom is viewed as a complex field featuring many coinciding events, such that teachers cannot consider everything and thus tend to specific events in the classroom and

(2) knowledge-based reasoning, where teachers reason about an event based on their professional knowledge and understanding as they focus on the event in question.

In addition, Jacobs et al. (2011) identified "deciding how to respond based on students' understanding" as the third component of teachers' attention and referred to it as "decision making." Currently, there is a growing body of research on teacher noticing (Weyers et al., 2023a) that focuses mostly on conceptualizations (e.g., Choy & Dindyal, 2021; Kaiser et al., 2017), intervention methods (e.g., Weyers et al., 2023b), and relationships with other constructs (e.g., Depaepe et al., 2020; Kaiser et al., 2015; Krauss et al., 2020), including teacher knowledge and beliefs.

Since the subjects of this study were PSTs who had relatively little teaching knowledge and experience, they were less able to respond to students and make instructional decisions. Therefore, based on the definition provided by Sherin and van Es (2009), this study considered selective attention and knowledge-based reasoning as the two essential components of teacher noticing. Meanwhile, it has been usual for researchers to ask questions that are situated within one or both of these categories (Stahnke et al., 2016).

Pre-Service Teachers' Noticing in a Video-Based Environment

Teaching is an interaction between teachers and students based on specific content (Potari & Jaworski, 2002). Therefore, many complex interactions occur in the actual classroom teaching process. However, classroom videos can illustrate the interactions that occur during the course, such as classroom discourses. Thus, such videos have been widely used in teacher noticing studies (Kleinknecht & Gröschner, 2016), especially in PSTs (Amador et al., 2021). Videos are more beneficial for PSTs about enhancing their professional knowledge and skills in teaching and learning and helping them understand the developmental process associated with students' mathematical thinking than textual materials, such as those focused on instructional design (Wang & Hartley, 2003). Specifically, this approach does not require PSTs to react instantly to events that occur in the classroom, thus offering them more time to reflect on students' discussions and enhancing teachers' explanatory and reflective skills (Sherin, 2000, 2003).

Video clips were the primary tool used by most programs to involve PSTs in noticing tasks (Santagata et al., 2021). Typically, these videos depicted the classroom practices of other teachers, although a few programs utilized videos created by the teachers themselves (e.g., Teuscher et al., 2017). Additionally, some video clips were solely dedicated to showcasing students as they worked through a mathematical problem. Structured frameworks or viewing guides were the predominant methods used by the majority of programs to aid teachers in developing the competence of noticing. Open-ended prompts were also utilized by some programs. Each program's framework was unique and provided prompts to assist teachers in analyzing the videos. Some frameworks were based on research regarding students' learning of specific mathematics concepts (e.g., Shin, 2021), while others directed participants to pay attention to the nuances of classroom interactions (e.g., Walkoe & Levin, 2018). The researchers used the videos to address two main types of teacher noticing (Santagata et al., 2021): A number of studies have explored the impact of programs on teacher noticing, with some examining the extent to which these interventions influenced the development of teachers' noticing competencies (e.g., Suh et al., 2021), while others have specifically focused on how these interventions supported the growth of such competences (e.g., Walkoe et al., 2020).

Relationship Between Teacher Noticing & Professional Knowledge

Researchers have agreed that teacher noticing should be linked to their professional knowledge, as this notion bridges teachers' beliefs, knowledge, and performance in teaching practice (König et al., 2022; Mason, 2002). According to the seminal framework developed by Shulman (1987), mathematics teacher expertise is divided into three main components that have been developed over decades: mathematics content knowledge (MCK), mathematics pedagogical content knowledge (MPCK), and general pedagogical knowledge (GPK) (Ball et al., 2008).

The relationship between these three types of knowledge and teacher noticing has been studied in different cultural contexts, although the results of these studies have not been entirely consistent. König et al. (2014) examined the relationship between teacher noticing and GPK and found a correlation between PSTs' GPK and their ability to interpret classroom situations. However, there was no relationship between their ability to perceive meaningful events in the classroom and GPK. Dreher and Kuntze (2015), who examined the relationship between teacher noticing and MCK, found a weak correlation between PSTs' MCK and their ability to pay attention, while in-service teachers did not exhibit a significant correlation in this context. In addition, recent studies have found that MCK may affect different components of teacher noticing. For example, Sánchez-Matamoros et al. (2019) found that the connected, inferential part (e.g., knowledge-based reasoning) of PSTs' noticing is strongly related to their MCK, while the perceptual attention aspect (e.g., selective attention) is not related to mathematical content knowledge. In studies of the relationship between teacher noticing and MPCK, Dreyer and Kunze (2015) described a more substantial relationship among in-service mathematics teachers but not among PSTs. Some studies have suggested a strong relationship between PSTs' MPCK and teacher noticing (Dreher & Kuntze, 2015). Overall, teachers' professional knowledge is significantly related to the reasoning and explanation components of teacher noticing, a finding, which is consistent with the theoretical framework constructed by Sherin and van Es (2009). Yang et al. (2021) found an overall weaker link between Chinese PSTs' noticing and their professional knowledge than the link found in Western countries, but MPCK and GPK were significantly correlated with the explanation and decision-making aspects of teacher noticing. However, little is known regarding how Chinese PSTs' noticing relates to MPCK and how they use MPCK to explain and reason about the events that they notice.

Knowledge for Teaching Mathematical Modelling

Scholars have reached a consensus that mathematical modelling uses mathematical knowledge to solve real-world problems. This activity focuses on the connection between mathematics and the real world, translating real-world problems into mathematical problems, returning to real-world problems, and critically analyzing the corresponding steps and results. The modelling cycle is a crucial feature of mathematical modelling and can be based on different views of mathematical modelling. For example, Blum and Ferri (2009) created a four-stage mathematical modelling cycle based on Pollak's view, including idealization, mathematization, mathematical considerations, and validation of the interpretation (Maaß, 2006). From a cognitive perspective, Blum and Ferri (2009) proposed a seven-stage mathematical modelling cycle, idealizing the modelling activity in terms of constructing, simplifying, mathematizing, working mathematically, interpreting, validating, and exposing, which are conceptualized as seven processes. In this study, Blum and Ferri's (2009) seven-step cycle process was used to represent the basic steps of mathematical modelling for the subsequent analysis.

Previous studies have identified potential cognitive conflicts encountered by students when engaging in mathematical modelling activities at each step. For example, Galbraith and Stillman (2006) noted that students might fail to articulate the modelling problem situation, make assumptions that simplify the problem, identify modelling strategies, determine dependent and independent variables, express content using mathematical formulas, use appropriate procedures, and situation the mathematical solution results in a realistic context for examination.

Although researchers have not found decisive strategies for supporting students engaging in modelling activities independently, it is commonly agreed that teachers should balance between teachers' guidance and students' self-exploration when implementing modelling tasks. This balance requires a high level of competence in teaching mathematical modelling.

Ball (2000) deconstructed MCK and MPCK into their essential parts, building on the work of Shulman (1987). Similar to Shulman (1987), Ball (2000) maintained that effective teaching requires understanding both the subject at hand and the relevant pedagogy. These two components are also key to teaching mathematical modelling. Ferri (2018) identified four dimensions of teacher modelling instruction:

(1) the theoretical dimension, including modelling cycles or the aims and perspectives of modelling as background knowledge,
(2) the task dimension, including multiple solutions or cognitive analyses of modelling,
(3) the instruction dimension, including teachers' need to know how to make appropriate interventions, give students proper support, and provide timely feedback, and
(4) the diagnostic dimension, such as the knowledge necessary to recognize students' difficulties and mistakes.

This framework includes both theoretical and task-related dimensions concerning MCK and pedagogical and diagnostic dimensions related to MPCK, which in turn can help enhance teachers' content knowledge and students' ability to recognize potential learning obstacles and the corresponding prerequisite modelling skills and competencies.

Cetinkaya et al. (2016) further explicated MPCK for mathematical modelling from the perspective of "mathematical modelling pedagogy":

(1) understanding the cognitive demands of a given modelling task,
(2) knowing how to organize classroom discourse and how to manage the classroom during modelling activities,
(3) providing appropriate strategic interventions that allow students to complete the modelling activity independently,
(4) listening interpretively, identifying, and responding to student thinking,
(5) identifying the strengths and weaknesses of different paths during the modelling process, and
(6) identifying unexpected solutions and approaches and developing coping strategies.

These frameworks share certain commonalities: they all emphasize the need for teachers to deeply understand students' modelling thinking during the modelling process to intervene on their behalf using appropriate scaffolds. Therefore, this study constructs a theoretical framework for teacher noticing in the specific content area of mathematical modelling based on a combination of the definition provided by Sherin and van Es' (2009) with Ferri's (2018) framework for knowledge for teaching mathematical modelling and analyses PSTs' selective attention and knowledge-based reasoning by video analysis task.

CONCEPTUAL FRAMEWORK

Our current study focuses primarily on the use of video to support PSTs' noticing concerning a mathematical modelling context. Furthermore, we used Sherin's (2007) two main subprocesses, i.e.,

(a) selective attention and
(b) knowledge-based reasoning, to characterize the ability to notice.

Selective attention pertains to how the teacher decides where to focus his or her attention at a given moment (Sherin, 2007). Selective attention includes two components–actor and topic. The term actor refers to who or what was the object of PSTs' noticing. The initial framework developed by Sherin and van Es (2009) included three codes for the actor. However, some vital information may be lost using this encoding method. Therefore, Stockero et al. (2017) improved this perspective by refining it into six codes building on the original code. Therefore, their encoding method is used in this study. The topic includes four codes, and the fourth code–mathematical thinking–has been altered to modelling thinking.

Knowledge-based reasoning refers to how a teacher reasons about what is noticed based on his or her professional knowledge and understanding, which includes two components–stance and professional knowledge. The term stance refers to the level of PST's reasoning, which includes describing, evaluating, and interpreting. The component of Sherin and van Es's (2009) noticing framework did not consider teachers' professional knowledge in teaching mathematical modelling. The model developed by Ferri (2018) for knowledge of mathematical modelling teaching was utilized to examine PSTs' professional knowledge. The theoretical dimension refers to the teacher's knowledge regarding the modelling cycle, modelling goals and perspectives, and types of modelling tasks. The task dimension refers to the teacher's knowledge related to solving, analyzing, and creating modelling tasks. The instruction dimension refers to the teacher's knowledge that is necessary to intervene appropriately in the modelling process. The diagnostic dimension refers to the knowledge related to identifying the modelling process stage and diagnosing the difficulties that students face during this process.

Selective attention and knowledge-based reasoning interact dynamically. The kinds of interactions that teacher notices are likely to influence how the teacher reasons about those events. In addition, teachers' professional knowledge and disposition can be expected to drive the factors that stand out to the teacher in any given situation. The conceptual framework of this study is shown in **Table 1**.

METHOD

Participants

The participants in this study were 25 PSTs (five men, 20 women; individually coded from PST1-PST25) with a professional background in pedagogy or mathematics. The purpose of the study was explained to participants

Table 1. Analytical framework for coding PSTs' noticing within a mathematical modelling context

Coding
Selective attention
Actor
Teacher (T)
Teacher/student (T/S)
Student/teacher (S/T)
Student group (SG)
Individual student (IS)
Others (O)
Topic
Management (M)
Climate (C)
Pedagogy (P)
Modelling
Thinking (MT)
Knowledge-based reasoning
Stance
Describe (D)
Evaluate (E)
Interpret (I)
Knowledge for teaching mathematical modelling
Theoretical dimension (TH)
Task dimension (TA)
Instruction dimension (IN)
Diagnostic dimension (DI)

and all participants voluntarily agreed to join and signed a consent form. All educational research guidelines and ethical rules have been followed and approved by the University's research committee. The participants were required to have taken mathematical modelling courses or to have experience in mathematical modelling competitions. They must also have completed educational internship activities and have practical mathematics instruction experience. Prior to the implementation of the study, all PSTs had completed a lesson on teaching mathematical modelling. Moreover, as a teaching assistant for the course, one of the researchers knew each PST's participation in activities and classroom tasks.

Procedures

This study was conducted in the *middle school mathematics pedagogy* course, which is part of a teacher preparation program at a Chinese university. This course was in the first semester of the four-semester teacher preparation program to give PSTs the professional knowledge and skills they need to succeed in the classroom. Research on mathematical modelling, noticing, teaching, and learning served as the foundation for the *middle school mathematics pedagogy* course (e.g., Alwast & Vorhölter, 2022). Over the course of three months, the class met 12 times, once a week for three hours. There were three primary sections to this course.

The first phase (week 1-week 4) introduced students to instructional theory in mathematics modelling. The instructor explained the primary connotations, value, significance, specific manifestations, and mode of instruction of mathematical modelling to ensure that PSTs had a preliminary understanding of mathematical modelling instruction.

During the second phase of the program (week 5-week 8), particular emphasis was placed on paying close attention to the intricate details of classroom interactions and developing precise portrayals of these events. In addition, PSTs were encouraged to consider various interpretations of the observed phenomena, such as the insights that could be gained from a student's explanation regarding their understanding, or how different types of mathematical modelling thinking might be promoted by the questioning strategies employed by different teachers.

In the concluding phase of the course (week 10-week 12), the emphasis was placed on promoting PSTs' noticing of classrooms as dynamic and interactive spaces that establish connections among students, teaching, and content, by delving into the correlations between student thinking, the characteristics of modelling tasks, specific teaching strategies, and classroom discourse. Afterwards, they were required to complete an instructional design task in mathematical modelling as a final assignment for the course.

During the initial phase, PSTs were instructed to view the first video clip and complete the pre-video analysis task. Similarly, in the final phase, they were prompted to watch the second video clip and complete the post-task.

Description of Video Clip

The videos used in this study were selected from a basic education quality improvement project in China, which provided a rich source of authentic classroom interactions. Both video clips feature the full mathematical modelling cycle, offering insights into the complexities of teaching and learning mathematical modelling in real classroom settings.

The selection process for the video clips was meticulous and aimed at ensuring their appropriateness and representativeness for assessing PSTs' noticing within a mathematical modelling context. The criteria for selection included not only the presence of a large number of interactions between the teacher and students but also the alignment with the learning objectives and content of the course.

In the first video clip, the teacher engaged junior high school students in a mathematical modelling activity focused on designing a scoring scheme for selecting a mobile phone based on various data points. 30 students were divided into six groups containing five students each (**Figure 1**). High-definition cameras were installed in the front and back of the classroom. The video captured the teacher's facilitation of group discussions, student presentations, and the use of technology such as

Figure 1. First video clip (Source: Field study)

Figure 2. Second video clip (Source: Field study)

a projector to showcase students' work, providing a comprehensive view of classroom dynamics during mathematical modelling tasks.

Similarly, the second video clip showcased a mathematical modelling activity with high school students, where they applied knowledge of quadratic functions to calculate safe stopping distances for different vehicles. Thirty students are seated in groups of six and there is special camera equipment to capture the behavior of the students (**Figure 2**). The video highlighted group collaboration, problem-solving strategies, and student engagement, offering valuable insights into how PSTs can develop their noticing skills in a diverse mathematical modelling context.

The videos were selected based on the researchers' familiarity with the lesson and the detailed interactions captured during the activities. Their inclusion was guided by the project's overarching goals and the relevance of the content to the targeted learning outcomes.

Description of Video Analysis Task

The researchers developed two open-ended prompts in the video analysis task to investigate PSTs' noticing. The two open-ended prompts were designed in line with the three open-ended tasks that van Es (2011) asked PST to complete, which can be summarized as

(1) describing teacher behaviors that give different requirements to facilitate student learning,

(2) evaluating the effectiveness of the teacher's teaching and explaining these evaluations, and

(3) creating alternative options that teachers could use to improve their teaching and enhance student learning.

These three tasks are focused on "promoting student learning," which leads to an evaluation of teachers' teaching behaviors and consideration of ways of improving teaching. This process is consistent with Santagata et al.'s (2007) "three-step framework" for analyzing a lesson: lesson goals and structure, student learning, and instruction documentation. Thus, based on Santagata et al.'s (2007) framework, this study adapted and deconstructed the tasks proposed by van Es (2011) tasks and included documentation of "important parts of the classroom" to scaffold other tasks for PSTs. The two specific prompts were, as follows:

(1) Watch the video and record the critical teaching/learning moments (i.e., events, behaviors, or situations in the classroom) in as much detail as possible and

(2) In what ways do you think the intended moments contribute to students' learning of mathematical modelling? Why? Please write down your thoughts (**Table 2**).

Prompt (1) focuses on investigating PSTs' selective attention, while prompt (2) focuses on investigating PSTs' knowledge-based reasoning.

To ensure the similarity in difficulty between the pre-test and post-test, we conducted a pilot test of the video analysis task involving six experienced mathematics teachers. Their feedback was used to revise the tasks and video clips.

Data Collection

The main source of data for this study consists of a pre-and post-video analysis task in the second phase and the last phase of the *middle school mathematics pedagogy* course. The task was created with the specific goal of allowing participants in the course to gain an understanding of the nature and development of PSTs' noticing.

PSTs were asked to watch the whole-class video clip and then finish the video analysis task. In actual instruction, teachers notice the moment without a rewind button, so they were permitted to watch the video clip only once without interruption. Additionally, in the real world, it is not possible to inform teachers in advance of the specific classroom moments they should pay attention to. While watching the video clip, PSTs were permitted to highlight some significant events and make notes. This was permitted not only because the video clips were too long for PSTs to remember everything they saw, but also due to the challenging mathematics modelling activities in the video. They

Table 2. Video analysis task

Name:	Video:	Major:	Date:
Cautions -When you have completed one record, record next one in a new row (using automatic numbering in table) -Please clearly record actor of codes -Record as much as possible in (1)			
(1) Watch video & record critical teaching/learning moments (i.e., events, behaviors, or situations in classroom) as much detail as possible.			
1. 2. 3. 4. …			
(2) In what ways do you think identified moments contribute to students' learning of mathematical modeling? Why? Please write down your thoughts.			
1. 2. 3. 4. …			

watched the video online in the Zoom meeting together and were given 30 minutes to finish the video analysis tasks and subsequently returned the completed tasks to the researchers. All the mentioned activities were finished in class. The same format was followed for viewing the first and second clips.

Data Analysis

In the analysis process, the research team initially segmented the transcripts of each PST into one or more analysis units, following the approach outlined by Sherin and van Es (2009). An analysis unit refers to a specific idea pertaining to the classroom event(s) captured in the video clip. Subsequently, each idea unit was assigned a code based on the designated coding categories. A total of 707 units were collected from 25 PSTs.

The units in prompt (1) were read to categorize PSTs' selective attention using a coding scheme: actor (teacher, teacher/student, student/teacher, student group, individual student, other) and topic (management, climate, pedagogy, mathematics modelling). The first step is to determine the actor. For example, PST1 made the following note: "The teacher asked the students if they had ever used cell phones and whether they knew anything about them, highlighting the fact that they needed help choosing a cell phone in this class." Although this record mentions the teacher and the students, there is no interaction between the teacher and the students. Therefore, this text is encoded as T. Subsequently, the researchers need to identify the topic of PST's selective attention.

Specifically, modelling thinking is related to the students' thinking of the seven parts of the modelling cycle. For example, PST3 made the following note: "The whole lesson is oriented towards solving practical application problems, which helps students improve their problem-solving skills". She highlighted the simplifying phase of the modelling cycle. Furthermore, she noted the characteristics of solving practical problems in modelling activities and believed that this approach could enhance students' problem-solving skills.

To investigate PSTs' knowledge-based reasoning, researchers used PSTs' understanding of classroom-specific momentary concerns collected from open-ended prompt (2) for analysis. For example, as PST1 noted, "The teacher asked students to make their operational comparisons, giving students much room for independent measurement in terms of scoring high or low, which can promote the development of noncognitive skills." This was coded as I because PST described, explained, and provided an inference in the video clip. This study utilized Ferri's (2018) framework to scrutinize PSTs' reasoning about the events that occur in mathematical modelling lessons based on their knowledge for teaching mathematical modelling. For example, as PST14 noted, "the teacher throws out one task after another. Each task is interrelated, fully driving the students to integrate the knowledge of averages into the story context." This unit was coded as TA since PST14 evaluated the characteristics of the task and used professional knowledge to explain the facilitation of student modelling learning.

To ensure the reliability of the coding process, initial coding was conducted by a teaching assistant and a researcher with a PhD in mathematics education, both of whom possessed proficient coding skills and familiarity with the coding framework. The coding process involved independently coding the video data to establish an initial level of agreement. The data reliability was assessed using the Cronbach's alpha coefficient, which yielded a high value of 0.91 after the two researchers completed the independent coding. This indicates a strong level of agreement between their coding assessments.

Table 3. Pre-service teachers' selective attention by actor

Actor	Pre-task (n=25)		Post-task (n=25)		Z	p
	Number (n)	Percentage (%)	Number (n)	Percentage (%)		
Teacher	95	36.68	89	31.90	t=0.358	0.723
Teacher/student	84	32.43	102	36.56	-1.143	0.253
Student/teacher	15	5.79	12	4.30	-0.711	0.477
Student group	55	21.24	67	24.01	t=-0.993	0.331
Individual student	1	0.39	1	0.36	0.000	1.000
Others	9	3.47	7	2.51	-0.324	0.746

Note. t: t-value from a paired-sample t-test (normally distributed data)

Subsequently, a rigorous process of coder training and consensus-building was implemented to address areas of doubtful coding and enhance inter-rater reliability. This involved the two initial coders and a third experienced researcher collaborating to review and reconcile any discrepancies in coding decisions. Through discussions and iterative revisions, consensus was achieved on all coding categories and interpretations.

Furthermore, to enhance transparency and ensure the reliability of the coding process, detailed documentation of coding decisions, coding guidelines, and examples of coded segments were maintained throughout the coding process. This documentation facilitated consistency in coding practices and provided a reference for resolving any coding discrepancies.

For the quantitative analysis of the data, we employed both parametric and non-parametric tests to assess the difference between pre-test and post-test results. Specifically, we utilized either a paired-sample t-test or a non-parametric Wilcoxon signed-rank test based on the normality of the data. To determine the appropriate test, we initially conducted a Shapiro-Wilk test to assess the normality of the data distribution. The results indicated that the majority of the data did not follow a normal distribution ($p<0.05$), except for specific variables (T, SG, P, D, E, and I) in both the pre-task and post-task assessments (all $p>0.05$).

As a result, for non-normally distributed data, we opted to use the Wilcoxon signed-rank test, which focuses on medians and is robust against non-normality. Conversely, for normally distributed data, we employed the paired-sample t-test, which focuses on means and is suitable for normally distributed data.

Additionally, we took measures to control potential confounding variables and ensure the validity of our analyses. These included conducting sensitivity analyses, checking assumptions for each statistical test used, and addressing any outliers or influential data points that may have affected the results. By utilizing a combination of parametric and non-parametric tests and conducting thorough data checks and sensitivity analyses, we aimed to provide robust and reliable statistical findings regarding the differences observed between pre- and post-test results.

RESULTS

Changes in Pre-Service Teachers' Selective Attention

Actor

The most notable aspect of this section was a predominant focus on teacher behaviors among PSTs, with limited instances of attention directed towards individual student interactions or contributions.

Among the actors, two types of codes, i.e., teacher and teacher-led student-teacher interaction, belonged to the teacher subject category, going down from 69.11% to 68.46%. Three types of codes, i.e., individual student, student group and student-led student-teacher interaction, belonged to the student subject category, rising from 27.39% to 28.67%. Finally, there was a decrease in other types of attention from 3.47% to 2.51% (**Table 3**).

The results show that PSTs paid more attention to the teacher's behavior than to that of the students. Only two units were coded as individual students. For example, PST2 made the following note: "A student answered that it depends on how many people buy." This phenomenon is simply not widespread. According to the individual analyses of the records of each PST, all PSTs were more concerned with teacher behavior.

The results suggested that while PSTs continued to prioritize observing teacher behaviors, there was a noticeable increase in attention towards student interactions and contributions. This shift could be attributed to the intervention's emphasis on promoting student-centered learning and active engagement within the mathematical modelling context.

To further examine the changes in actor between the pre-test and post-test, we conducted either a Wilcoxon signed-rank test or a paired-sample t-test. However, the results indicated that there were no significant differences (all $p>0.05$).

Topic

The most significant aspect observed in this section was a of PSTs' focus towards modelling thinking, indicating a progressive immersion into the intricate processes of mathematical modelling.

Table 4. Pre-service teachers' selective attention by topic

Topic	Pre-task (n=25)		Post-task (n=25)		Z	p
	Number (n)	Percentage (%)	Number (n)	Percentage (%)		
Management	21	8.11	15	5.38	-1.221	0.222
Climate*	37	14.29	14	5.02	-2.549	0.011
Pedagogy	128	49.42	154	55.20	t=-1.267	0.218
Thinking*	73	28.19	96	34.41	-1.973	0.048

Note. t: t-value from a paired-sample t-test (normally distributed data) & *p<0.05

Table 5. Pre-service teachers' knowledge-based reasoning by stance

Stance	Pre-task (n=25)		Post-task (n=25)		Z	p
	Number (n)	Percentage (%)	Number (n)	Percentage (%)		
Describe	25	32.47	25	27.17	0.822	0.420
Evaluate*	22	28.57	29	31.52	-2.522	0.019
Interpret*	30	38.96	38	41.30	-2.097	0.047

Note. *p<0.05

Regarding the topic, PSTs demonstrated two primary areas of focus in the pre-task: pedagogy (49.92%) and modelling thinking (28.19%). However, in the post-task, there was a substantial increase in the proportion of attention dedicated to modelling thinking (from 28.19% to 34.41%), while the percentage related to climate experienced a notable decrease (from 14.29% to 5.02%). This shift indicates a heightened engagement with the intricacies of mathematical modelling processes.

Individual analysis of each PST revealed a consistent recognition of pedagogy and modelling thinking. Furthermore, they all took note of the teacher's facilitation of various group discussions. For instance, as expressed by PST1, "the teacher instructed the students to engage in group discussions regarding their recommendations for cell phones, considering the provided information."

Additionally, PSTs demonstrated attentiveness to the teacher's scaffolding techniques within the classroom environment. These observations highlight PSTs' acute awareness of instructional strategies employed within the classroom.

In terms of modelling thinking, PSTs demonstrated a focus on various aspects of the mathematical modelling cycle, such as mathematizing. Additionally, several PSTs made observations regarding the aspect of simplifying. For instance, PST1 recorded the following note: "The teacher asked, 'How do we choose a cell phone?' The students responded with survey, performance, and price." This heightened attention to modelling thinking signifies an evolving understanding and application of mathematical modelling principles among PSTs.

The Wilcoxon signed-rank test revealed significant differences between the two tasks in the "climate" (Mdpre=1, Mdpost=0, Z=-2.549, and p=0.011) and the "modelling thinking" dimension (Mdpre=3.0, Mdpost=4.0, Z=-1.973, and p=0.048). All the results are shown in **Table 4**.

Changes in Pre-Service Teachers' Knowledge-Based Reasoning

Stance

The most significant aspect observed in this section was the notable shift towards enhanced evaluation skills among PSTs during the post-task phase (**Table 5**).

Within the category of stance, pre-test comments primarily revolved around description (32.47%) and interpretation (38.96%). An illustrative example can be found in PST9's statement: "The teacher efficiently introduced the lesson, providing a concise focus", which was coded as a description. However, during the post-test, a notable shift was observed. More comments focused on evaluation (increasing from 28.57% to 31.52%) and interpretation (rising from 38.96% to 41.30%), while the proportion of comments related to description slightly decreased (from 32.47% to 27.17%) compared to the pre-test phase. Moreover, the total number of comments increased overall, from 77 to 92 (+19.00%), between the pre- and post-task.

The paired-sample t-test revealed significant differences between the two tasks in the "evaluate" (Mpre=2.08, Mpost=3.17, t=-2.522, and p=0.019) and the "interpret" (Mpre=2.13, Mpost=2.83, t=-2.097, and p=0.047). The results demonstrated that PSTs have developed significantly in the competence of knowledge-based reasoning.

The heightened emphasis on evaluation suggests that PSTs have developed a deeper capacity to critically assess and analyze instructional practices, pedagogical strategies, and student interactions. This shift indicates a progression from descriptive and interpretive commentary towards more evaluative and reflective engagement, showcasing a maturation in knowledge-based reasoning and analytical acumen among PSTs.

The heightened emphasis on stance suggested that PSTs have developed a deeper capacity to assess and analyze instructional practices, pedagogical strategies,

Table 6. Pre-service teachers' knowledge-based reasoning by knowledge for teaching mathematical modelling

KKTM	Pre-task (n=25)		Post-task (n=25)		Z	p
	Number (n)	Percentage (%)	Number (n)	Percentage (%)		
Theoretical dimension	4	5.19	9	9.78	-1.232	0.238
Task dimension*	2	2.60	7	7.61	-1.890	0.049
Instruction dimension	9	11.69	7	7.61	-0.577	0.564
Diagnostic dimension	0	0.00	2	2.17	-1.414	0.157

Note. KTMM: Knowledge for teaching mathematical modelling; Number of other knowledges that is not related to teaching mathematical modelling was 62 (80.52%) in pre- & 67 (72.83%) in post-task; t-test shows there were no significant differences (t=-0.534 & p=0.598)

and student interactions in the context of mathematical modelling. This shift indicated a progression from descriptive commentary towards more evaluative engagement, showcasing a maturation in knowledge-based reasoning skill among PSTs.

Knowledge for teaching mathematical modelling

The most significant aspect observed in the analysis of this section was the enhanced utilization of the task dimension by PSTs. This shift reflected a deeper understanding of pedagogical strategies specific to mathematical modelling, thereby demonstrating PSTs' growing proficiency in applying theoretical knowledge to practical teaching situations.

The four-dimensional framework utilized in this study, developed by Ferri (2018), demonstrates a robust thematic character concerning mathematical modelling and distinguishes itself from GPK framework.

In terms of the theoretical dimension, the percentage increased (from 5.19% to 9.78%) between the pre-task and post-task. PSTs exhibited knowledge for teaching mathematical modelling, with a focus on the modelling cycle. They were able to recognize specific events within the modelling process and apply the modelling cycle to analyze student progress within that process. A notable trend was observed with a steady increase in the percentage of the task dimension (from 2.60% to 7.61%) in the post-task. PSTs utilized this knowledge to make judgments regarding the difficulty of the modelling tasks and anticipate potential areas of student confusion. However, their noticing was not specifically directed towards exploring multiple approaches to completing the modelling tasks. There was a noticeable rise in the percentage of the diagnostic dimension (from 0.00% to 2.17%). Only two out of the 25 PSTs utilized this knowledge to analyze the events they observed. Conversely, the percentage of the instruction dimension slightly decreased (from 11.69% to 7.61%) between the pre-task and post-task. PSTs employed this knowledge to interpret the teacher's feedback, particularly interventions made during the modelling process. They recorded a series of questions posed by the teacher to analyze scaffolded instruction.

Furthermore, there was an overall increase in the total number of instances showcasing knowledge for teaching mathematical modelling, from 15 to 25 (+19.00%), between the pre-task and post-task. The Wilcoxon signed-rank test showed that there was a significant difference between these two tasks in task dimension (Mdpre=0, Mdpost=1, Z=-1.890, and p=0.049). All the results are shown in **Table 6**.

DISCUSSION

The findings from our video analysis task revealed that PSTs exhibited improvements in their noticing competencies, specifically in selective attention and knowledge-based reasoning, following their participation in the *middle school mathematics pedagogy* course spanning 12 weeks. This indicates that PSTs developed enhanced skills in recognizing critical events related to mathematical modelling. These results align with previous studies conducted by Qi et al. (2022) and Stockero et al. (2017) in a general context. Our previous study observed a significant increase in the percentages of noticing related to analyzing student math learning and math interaction. Similarly, the latter study reported that intervention facilitated development of PSTs' ability to notice various aspects of classroom interactions.

Development of Pre-Service Teachers' Selective Attention within a Mathematical Modelling Context

The comparison of PSTs' responses to our open-ended prompt (2) between the two tasks revealed a shift in selective attention. Specifically, PSTs' focus transitioned from the teacher to the students, indicating an improvement in their attending-to skills. However, it is noteworthy that there was no significant increase in attention directed towards students, suggesting a limitation in the effectiveness of the course intervention in the actor of selective attention. The results of previous studies have shown that PSTs tend to pay more attention to teacher behaviors than to students when viewing videos (e.g., Amador et al., 2021; Shin, 2021). Consistent with previous studies, PSTs were primarily focused on teacher behaviors. Simultaneously, PSTs also paid more attention to teacher teaching behaviors (i.e., teachers' teaching styles, teaching decisions, and use of teaching resources) rather than management and climate. This finding could have as its explanation that PSTs viewed the video more as mathematical modelling learners than as future teachers for mathematical modelling.

As the research by Star and Strickland (2008) made clear, looking at a lesson from the viewpoint of the learner may deter the observer from paying close attention to the teaching and learning.

The term "contextual teaching" appeared more frequently in PSTs' records of teachers' behaviors. This finding differs from those of previous studies, according to which PSTs tended to pay more attention to general teaching actions that promote students' participation in classroom discussions (e.g., Males, 2017; Sánchez-Matamoros et al., 2019) and less attention to teaching in specific contexts. This difference may be due to the content in question; the mathematical modelling lesson creates an authentic context that is closely related to students' lives, and PSTs pay much attention to this unusual mode of teaching. On the other hand, this approach is also required for modelling instruction. Modelling uses mathematical knowledge to solve real-world problems, so creating an authentic and life-like context is a crucial aspect of teaching that deserves PSTs' selective attention (e.g., Alwast & Vorhölter, 2022; Cerqueira, 2001).

The topical focus of PSTs shifted towards modelling thinking and pedagogy, which also indicated the development of their attending-to skills (Sherin & van Es, 2009). The findings revealed a significant decrease in their selective attention to classroom climate, accompanied by a significant increase in their emphasis on pedagogy and modelling thinking. Thus, on the actor of selective attention, the intervention of the course in this study was successful. This was similar to previous findings in other mathematical content areas (e.g., Choy & Dindyal, 2021; Suh et al., 2021; Walkoe, 2015) further validates the effectiveness of using video to promote PSTs' attentional skills in the context of mathematical modelling.

In addition, PSTs' selective attention to modelling thinking involved several components of the mathematical modelling cycle in which context PSTs paid the most attention to the constructing and simplifying processes. Constructing and simplifying represent the real-world part of the modelling cycle. PSTs were more concerned with the practicality of the modelling task, perhaps because they perceived the real-world element to be more challenging to teach. Specifically, they focused on how the teacher created the situation and how the students transformed the problems in the real-world situation into mathematical problems while watching the video.

This suggested that when using videos to develop PSTs' noticing span in the context of mathematical modelling, great care should be taken to ensure that the mathematical modelling activities in the selected videos are centered around problems in real contexts.

Development of Pre-Service Teachers' Knowledge-Based Reasoning within a Mathematical Modelling Context

PSTs' answers to our open-ended prompt (2) between two tasks showed changes in the extent to which they were able to engage in the skills of knowledge-based reasoning. After finishing the post-task, PSTs exhibited middle to high level of reasoning, according to the definition proposed by Sherin and van Es (2009). More specifically, they make fewer descriptive comments and significantly more evaluative and interpretive comments. This result implied their interpreting skills had improved over the course of the intervention, which was similar to previous research findings (e.g., Keppens et al., 2021; Suh et al., 2021). Previous studies have shown that mathematics teachers' ability to explain students' thinking contributes to the improvement of teaching and development of teachers (Jacobs et al., 2010). In contrast, in this study, PSTs mainly reasoned about the teachers' behaviors. A possible reason for this difference is the fact that PSTs, who are on the verge of entering a natural teaching environment, are more inclined to focus on the teachers and analyze their teaching behaviors.

Effective noticing is more likely when teachers draw on professional knowledge in specific content areas for their noticing (Sherin & van Es, 2009). Because we used Ferri's (2018) model of teaching mathematical modelling to scaffold noticing, we were also able to document changes in the noticing of particular instances of professional knowledge. Despite their more affluent knowledge base, PSTs did not often use their knowledge of teaching mathematical modelling to evaluate or interpret in both pre-and post- task, which is most relevant to teaching mathematical modelling (Ferri, 2018; Yang et al., 2022). In contrast, they used other knowledge (mostly GPK) to interpret the events they noticed. Just as Schoenfeld (2011, p. 232) noted, "what teachers notice and how they act on it is a function of teachers' knowledge and resources, goals and orientations." The comparison between the two tasks revealed that there was an increase in the frequency of using knowledge for mathematical modelling teaching by PSTs through the intervention implemented in the course. Particularly, there was a significant difference in the utilization of knowledge in the task dimension. However, they did not significantly decrease the use of other types of knowledge. This suggests that the intervention had some effect in this aspect, but its impact remains limited.

This suggests that when using video to develop PSTs' noticing skills in the context of mathematical modelling, facilitators' pedagogical intervention strategies should be more focused on the domain of mathematical modelling itself, enhancing PSTs' ability to use domain-specific pedagogical knowledge to explain key events.

CONCLUSIONS

The study designed a video-based course to develop PSTs' noticing in the context of mathematical modelling. The results showed that their noticing competencies increased somewhat after they participated in a semester-long course. PSTs' topical focus shifted towards modelling thinking and pedagogy. Also, they exhibited a significant increase in interpretive comments and utilized task-dimensional knowledge for teaching mathematical modelling to reason about classroom events. These results provide support for effectiveness of using video-based approaches in enhancing PSTs' competencies in this domain.

Limitations

This study was conducted solely by reference to a small group of PSTs. Future research can expand the scope of the study group by selecting different groups of teachers, such as both novice and experienced teachers. Simultaneously, the features of PSTs' noticing about mathematical modelling revealed in this study can be used to develop teachers' competencies in mathematical modelling in a more targeted manner in future teacher training programs. Finally, it is important to exercise caution when utilizing or interpreting findings due to the limitations of a small and non-random sample.

Author contributions: SZ, LL, & CQ: data collection & analysis; **SZ:** literature review, research design, & writing; & **CQ:** conceptual framework & writing. All authors have agreed with the results and conclusions.

Funding: No funding source is reported for this study.

Ethical statement: The authors stated that the study was approved by the Research Ethics Committee of Faculty of Education of Beijing Normal University on 15 March 2023 with approval code BNU202303100004. Written informed consents were obtained from the participants.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Alwast, A., & Vorhölter, K. (2022). Measuring pre-service teachers' noticing competencies within a mathematical modeling context–An analysis of an instrument. *Educational Studies in Mathematics, 109*(2), 263-285. https://doi.org/10.1007/s10649-021-10102-8

Amador, J. M., Bragelman, J., & Superfine, A. C. (2021). Prospective teachers' noticing: A literature review of methodological approaches to support and analyze noticing. *Teaching and Teacher Education, 99*, 103256. https://doi.org/10.1016/j.tate.2020.103256

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*(3), 241-247. https://doi.org/10.1177/0022487100051003013

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education, 59*(5), 389-407. https://doi.org/10.1177/0022487108324554

Blum, W., & Ferri, R. B. (2009). Mathematical modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Application, 1*(1), 45-58.

Cerqueira, B. J. (2001). Mathematical modelling in pre-service teacher education. In *Modelling and mathematics education* (pp. 185-194). Elsevier. https://doi.org/10.1533/9780857099655.4.184

Cetinkaya, B., Kertil, M., Erbas, A. K., Korkmaz, H., Alacaci, C., & Cakiroglu, E. (2016). Pre-service teachers' developing conceptions about the nature and pedagogy of mathematical modeling in the context of a mathematical modeling course. *Mathematical Thinking and Learning, 18*(4), 287-314. https://doi.org/10.1080/10986065.2016.1219932

Choy, B. H., & Dindyal, J. (2021). Productive teacher noticing and affordances of typical problems. *ZDM-Mathematics Education, 53*(1), 195-213. https://doi.org/10.1007/s11858-020-01203-4

Depaepe, F., Verschaffel, L., & Star, J. (2020). Expertise in developing students' expertise in mathematics: Bridging teachers' professional knowledge and instructional quality. *ZDM-Mathematics Education, 52*(2), 179-192. https://doi.org/10.1007/s11858-020-01148-8

Doerr, H. M. (2007). What knowledge do teachers need for teaching mathematics through applications and modelling? In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study* (pp. 69-78). Springer. https://doi.org/10.1007/978-0-387-29822-1_5

Dreher, A., & Kuntze, S. (2015). Teachers' professional knowledge and noticing: The case of multiple representations in the mathematics classroom. *Educational Studies in Mathematics, 88*(1), 89-114. https://doi.org/10.1007/s10649-014-9577-8

Faull, A. (2010). The Australian curriculum: A look through the lens of Christian education. *TEACH Journal of Christian Education, 4*(2), 8-13. https://doi.org/10.55254/1835-1492.1067

Ferri, R. B. (2018). *Learning how to teach mathematical modeling in school and teacher education*. Springer.

Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modelling process. *ZDM-Mathematics Education, 38*(2), 143-162. https://doi.org/10.1007/BF02655886

Goodwin, C. (1994). Professional vision. *American Anthropologist, 96*(3), 606-633. https://doi.org/10.1525/aa.1994.96.3.02a00100

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202. https://doi.org/10.5951/jresematheduc.41.2.0169

Jacobs, V. R., Lamb, L. L. C., Philipp, R. A., & Schappelle, B. P. (2011). Deciding how to respond on the basis of children's understandings. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 97-116). Routledge.

Kaiser, G. (2017). The teaching and learning of mathematical modeling. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 267-291). National Council of Teachers of Mathematics.

Kaiser, G., Busse, A., Hoth, J., König, J., & Blömeke, S. (2015). About the complexities of video-based assessments: Theoretical and methodological approaches to overcoming shortcomings of research on teachers' competence. *International Journal of Science and Mathematics Education, 13*(2), 369-387. https://doi.org/10.1007/s10763-015-9616-7

Kaiser, G., Blömeke, S., König, J., Busse, A., Döhrmann, M., & Hoth, J. (2017). Professional competencies of (prospective) mathematics teachers—Cognitive versus situated approaches. *Educational Studies in Mathematics, 94*, 161-182.

Keppens, K., Consuegra, E., & Vanderlinde, R. (2021). Exploring student teachers' professional vision of inclusive classrooms in primary education. *International Journal of Inclusive Education, 25*(9), 1091-1107. https://doi.org/10.1080/13603116.2019.1597186

Kleinknecht, M., & Gröschner, A. (2016). Fostering preservice teachers' noticing with structured video feedback: Results of an online-and video-based intervention study. *Teaching and Teacher Education,* 59, 45-56. https://doi.org/10.1016/j.tate.2016.05.020

König, J., Blömeke, S., Klein, P., Suhl, U., Busse, A., & Kaiser, G. (2014). Is teachers' general pedagogical knowledge a premise for noticing and interpreting classroom situations? A video-based assessment approach. *Teaching and Teacher Education, 38*, 76-88. https://doi.org/10.1016/j.tate.2013.11.004

König, J., Santagata, R., Scheiner, T., Adleff, A.-K., Yang, X., & Kaiser, G. (2022). Teacher noticing: A systematic literature review of conceptualizations, research designs, and findings on learning to notice. *Educational Research Review, 36*, 100453. https://doi.org/10.1016/j.edurev.2022.100453

Krauss, S., Bruckmaier, G., Lindl, A., Hilbert, S., Binder, K., Steib, N., & Blum, W. (2020). Competence as a continuum in the COACTIV study: The "cascade model". *ZDM-Mathematics Education, 52*(2), 311-327. https://doi.org/10.1007/s11858-020-01151-z

Maaß, K. (2006). What are modelling competencies?. *ZDM- Mathematics Education*, *38*, 113-142.

Males, L. M. (2017). Using video of peer teaching to examine grades 6-12 preservice teachers' noticing. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening Perspectives, contexts, and frameworks* (pp. 91-109). Springer. https://doi.org/10.1007/978-3-319-46753-5_6

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Routledge. https://doi.org/10.4324/9780203471876

MoE. (2020). *Mathematics curriculum standards for senior secondary schools.* People's Education Press.

MoE. (2022). *Mathematics curriculum standards for compulsory education*. Beijing Normal University Press.

NGACBP & CoCSSO. (2010). *Common core state standards for mathematics*. http://www.corestandards.org/math

Niss, M., & Blum, W. (2020). *The learning and teaching of mathematical modelling*. Routledge. https://doi.org/10.4324/9781315189314

Potari, D., & Jaworski, B. (2002). Tackling complexity in mathematics teaching development: Using the teaching triad as a tool for reflection and analysis. *Journal of Mathematics Teacher Education, 5*, 351-380. https://doi.org/10.1023/A:1021214604230

Qi, C., Liu, L., & Zuo, S. (2022). Fostering noticing in prospective teachers through a video-based course: Results of an intervention study from China. *Asian Journal for Mathematics Education, 1*(2), 204-220. https://doi.org/10.1177/27527263221107718

Sánchez-Matamoros, G., Fernández, C., & Llinares, S. (2019). Relationships among prospective secondary mathematics teachers' skills of attending, interpreting and responding to students' understanding. *Educational Studies in Mathematics, 100*(1), 83-99. https://doi.org/10.1007/s10649-018-9855-y

Santagata, R., König, J., Scheiner, T., Nguyen, H., Adleff, A. K., Yang, X., & Kaiser, G. (2021). Mathematics teacher learning to notice: A systematic review of studies of video-based programs. *ZDM-Mathematics Education, 53*(1), 119-134. https://doi.org/10.1007/s11858-020-01216-z

Santagata, R., Zannoni, C., & Stigler, J. W. (2007). The role of lesson analysis in pre-service teacher education: An empirical investigation of teacher learning from a virtual video-based field experience. *Journal of Mathematics Teacher Education, 10*(2), 123-140. https://doi.org/10.1007/s10857-007-9029-9

Schoenfeld, A. (2011). Noticing matters. A lot. Now what? In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 223-238). Routledge.

Sherin, M. G. (2000). Viewing teaching on videotape. *Educational Leadership: Journal of the Department of Supervision and Curriculum Development, 57*, 36-38.

Sherin, M. G. (2001). Developing a professional vision of classroom events: Teaching elementary school mathematics. In *Beyond classical pedagogy* (pp. 75-93). Erlbaum.

Sherin, M. G. (2003). New perspectives on the role of video in teacher education. In *Advances in research on teaching* (pp. 1-27). Emerald. https://doi.org/10.1016/S1479-3687(03)10001-6

Sherin, M. G. (2007). The development of teachers' professional vision in video clubs. In R. Goldman, R. Pea, B. Barron, & S. J. Derry (Eds.), *Video research in the learning sciences* (pp. 383-395). Erlbaum.

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education, 60*(1), 20-37. https://doi.org/10.1177/0022487108328155

Shin, D. (2021). Preservice mathematics teachers' selective attention and professional knowledge-based reasoning about students' statistical thinking. *International Journal of Science and Mathematics Education, 19*(5), 1037-1055. https://doi.org/10.1007/s10763-020-10101-w

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-23. https://doi.org/10.17763/haer.57.1.j463w79r56455411

Stahnke, R., Schueler, S., & Roesken-Winter, B. (2016). Teachers' perception, interpretation, and decision-making: A systematic review of empirical mathematics education research. *ZDM-Mathematics Education, 48*(1), 1-27. https://doi.org/10.1007/s11858-016-0775-y

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education, 11*, 107-125. https://doi.org/10.1007/s10857-007-9063-7

Stockero, S. L., Rupnow, R. L., & Pascoe, A. E. (2017). Learning to notice important student mathematical thinking in complex classroom interactions. *Teaching and Teacher Education, 63*, 384-395. https://doi.org/10.1016/j.tate.2017.01.006

Suh, J., Gallagher, M. A., Capen, L., & Birkhead, S. (2021). Enhancing teachers' noticing around mathematics teaching practices through video-based lesson study with peer coaching. *International Journal for Lesson & Learning Studies, 10*(2), 150-167. https://doi.org/10.1108/IJLLS-09-2020-0073

Teuscher, D., Leatham, K. R., & Peterson, B. E. (2017). From a framework to a lens: Learning to notice student mathematical thinking. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 31-48). Springer. https://doi.org/10.1007/978-3-319-46753-5_3

van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 164-181). Routledge.

van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education, 10*(4), 571-596.

Walkoe, J. (2015). Exploring teacher noticing of student algebraic thinking in a video club. *Journal of Mathematics Teacher Education, 18*(6), 523-550. https://doi.org/10.1007/s10857-014-9289-0

Walkoe, J., & Levin, D. M. (2018). Using technology in representing practice to support preservice teachers' quality questioning: The roles of noticing in improving practice. *Journal of Technology and Teacher Education, 26*(1), 127-147.

Walkoe, J., Sherin, M. G., & Elby, A. (2020). Video tagging as a window into teacher noticing. *Journal of Mathematics Teacher Education, 23*(4), 385-405. https://doi.org/10.1007/s10857-019-09429-0

Wang, J., & Hartley, K. (2003). Video technology as a support for teacher education reform. *Journal of Technology and Teacher Education, 11*(1), 105-138.

Weyers, J., König, J., Santagata, R., Scheiner, T., & Kaiser, G. (2023a). Measuring teacher noticing: A scoping review of standardized instruments. *Teaching and Teacher Education, 122*, 103970. https://doi.org/10.1016/j.tate.2022.103970

Weyers, J., König, J., Scheiner, T., Santagata, R., & Kaiser, G. (2023b). Teacher noticing in mathematics education: A review of recent developments. *ZDM-Mathematics Education*. https://doi.org/10.1007/s11858-023-01527-x

Yang, X., Kaiser, G., König, J., & Blömeke, S. (2021). Relationship between Chinese mathematics teachers' knowledge and their professional noticing. *International Journal of Science and Mathematics Education, 19*(4), 815-837. https://doi.org/10.1007/s10763-020-10089-3

Yang, X., Schwarz, B., & Leung, I. K. C. (2022). Pre-service mathematics teachers' professional modeling competencies: A comparative study between Germany, Mainland China, and Hong Kong. *Educational Studies in Mathematics, 109*(2), 409-429. https://doi.org/10.1007/s10649-021-10064-x