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Literature Review

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Visualization, language, and problem-solving: A review of research on multilingual learners in geometry

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Abstract

Visualization is widely recognized as a powerful support for problem-solving across disciplines such as mathematics and science. However, research on visualization has often treated it as a language-neutral cognitive tool, paying limited attention to the linguistic and cultural resources of multilingual learners (MLs). This review synthesizes empirical research on the role of visualization in problem-solving for MLs, clarifying its conceptualization, its study, and the evidence regarding its effectiveness for this population. Drawing on cognitive, cognitive-linguistic, and sociocultural perspectives, the review examines how visual representations interact with language demands, disciplinary practices, and learners' meaning-making processes. The analysis reveals several tensions in the literature, including limited consideration of learners' language proficiency and a predominance of monolingual research designs. The review also identifies methodological and theoretical gaps, particularly the need for integrated frameworks that account for both visual and linguistic dimensions of problem-solving. Implications for future research and instructional practice are discussed, highlighting directions for more inclusive, theoretically grounded investigations of visualization in multilingual learning contexts.

Keywords: visualization, language, problem-solving, multilingual learners, geometry

INTRODUCTION

In our current fast-paced world, characterized by unprecedented innovation–particularly with the advent of artificial intelligence–mathematics plays a vital role in shaping learners' thinking. Mathematics–particularly, geometry–enables us to understand our changing world and align with developing technological knowledge (Moru et. al., 2021). Geometry and spatial thinking are valuable in their own right but also provide a critical conceptual foundation for learning other topics in mathematics. Spatial thinking plays an essential role in mathematical problem-solving; however, it requires a strong understanding of mathematical concepts. Various teaching strategies, such as visualization, have been identified as necessary for improving learners' mathematical thinking during problem-solving. Encouraging the use of visualization during problem-solving can empower learners to experience success in mathematics. Visualization is widely recognized as an essential tool for promoting problem-solving skills and enhancing learners' understanding of mathematical concepts (Hlongwana et al., 2025; Tiwari et al., 2024).

For multilingual learners (MLs) who may struggle with language-based explanations, visualization offers an alternative pathway to understanding complex geometric concepts, thereby enhancing their ability to solve geometry problems (Samosa, 2021). Visualization is also a significant aspect of spatial reasoning and enhances problem-solving skills (Harris et al., 2023; Lowrie et al., 2019). Visualization plays a substantial role in problem-solving in Euclidean geometry. Learners develop the ability to visualize, problem-solve, think critically, reason deductively, clarify concepts, and prove theorems as they progress (A. Jones, 2007). Visualization is related to transferring both verbal and non-verbal knowledge to learners and to creating a theoretical understanding of geometry (Mnguni, 2014; Schoenherr et al., 2024). Learners need to visualize and understand when solving geometry problems to complete geometry tasks.

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Contribution to the literature

- This review contributes to the literature by synthesizing research on the role of visualization in mathematics problem-solving among MLs and by challenging the common assumption that visualization is a language-neutral cognitive tool. It highlights how visual representations interact with language demands, disciplinary practices, and learners' meaning-making processes.
- The review also identifies key gaps in existing research, including inconsistent definitions of visualization, limited attention to learners' language proficiency, and the dominance of monolingual research designs.
- By stressing the need for integrated theoretical frameworks that account for both visual and linguistic dimensions of problem-solving, the paper provides directions for more inclusive research and instructional practices that better support MLs.

Having discussed the significant role of visualization in geometry problem-solving, one of the considerable challenges in mathematics education, especially in geometry, is that many students display conceptual misunderstandings even when they give correct answers on traditional tests (Orhani, 2025). These misunderstandings stem from how concepts are interpreted, taught, or applied, creating an uncertain foundation for further learning. Similarly, there are cases in which learners provide incorrect answers with high confidence, indicating that they hold strong, deeply rooted misconceptions, which make it more difficult to correct through traditional teaching (Fadhilatullathifi et al., 2020).

In the context of this paper, multilingualism has also been identified as one of the main reasons learners experience challenges in understanding mathematics (Mudhefi et al., 2024). According to Barwell (2009) and Sharma and Sharma (2023), learning mathematics in multilingual classrooms is complex because it requires teachers to support learners with diverse educational needs, each of which, when considered individually, could require distinct interventions. Expanding on Barwell's (2009) notion, Sharma and Sharma (2023) add that learning mathematics in a language that is neither the learners' first language nor the teachers' can create challenges such as code-switching, mediation, and a lack of transparency in communication.

Mudhefi et al. (2024) conducted a study in South Africa that applied van Hiele's (1957) model to analyze learners' difficulties in geometry. The findings of the study identified challenges faced by learners in geometry, such as poor understanding of the properties of geometric figures and their relationships (conceptual understanding), difficulty interpreting and mentally manipulating geometry shapes (visualization challenges), inability to use correct mathematical terms when describing geometric properties (terminology confusion), and difficulty constructing logical arguments or proofs (lack of logical reasoning). A study conducted by Mukuka and Alex (2024) identified additional challenges, including a poor understanding of fundamental geometry concepts, misinterpretation of questions due to incorrect decoding of the problem statement, and a lack of prior mathematical knowledge.

Another challenge is that learners' logical reasoning and problem-solving skills in geometry remain weak, further impacting performance (WAEC, 2018, 2023, 2024). In rural schools, limited access to learning materials and inadequate teacher training further hinder learners' ability to grasp and apply geometric concepts (Adler & Ronda, 2015).

Given these complexities, Essien (2013) emphasizes that teachers must equip learners with the necessary understanding and skills to navigate the challenges posed by learning mathematics in multilingual classrooms.

In this paper, we argue that learners in multilingual contexts must be adequately prepared to manage the complexities of learning mathematics in a language that is not their native tongue. The challenges identified above caught my attention as a researcher and motivated me to explore this issue further to develop potential solutions. However, before doing so, it was essential to examine the research conducted to build on existing knowledge in this area. This paper bridges two gaps in the literature: first, it shows how existing studies view visualization as the key to problem-solving for MLs; second, it provides an overview of how visualization and language are connected to help MLs understand geometry problem-solving questions.

Literature Strategy

The literature presented in this paper is extracted from peer-reviewed articles. The studies focused on problem-solving in mathematics education, on geometry or spatial reasoning, and on the role and importance of visualization. Furthermore, research on multilingual or second-language learners. Followed by studies focusing on the effects of language on MLs' ability to solve geometry questions. Lastly, strategies for supporting MLs and the role of visualization in enhancing problem-solving in geometry.

LITERATURE REVIEW

Problem-Solving in Mathematics Education

This section presents a broader understanding of problem-solving and its relevance to this paper and to research on mathematics problem-solving, providing an overview of the research on problem-solving in mathematics education.

Problem-solving plays an important role in mathematics education, enabling learners to practice and integrate the concepts, theorems, and skills that have been learned (Hudojo, 2005). In the context of this study, MLs are expected to integrate geometry concepts, theorems, routines, and skills when solving questions using visualization in the form of diagrams (physical) and word problems (mentally). In this paper, the researcher views problem-solving as a crucial aspect of mathematics education, demonstrating learners' understanding of various aspects of mathematics, including mathematical concepts and the visuals used to communicate them in areas such as geometry. In this paper, the analysis of problem-solving studies is used to explore possible factors that contribute to poor performance on geometry problem-solving questions.

Polya (1945) views problem-solving as an art in which the learner engages by replicating what the teacher does in the classroom and applying what they see the teacher doing. Polya (1945) states:

> Mathematics problem-solving is a practical art like swimming or playing the piano: you can learn it only by imitation and practice … If you wish to learn to swim, you have to go to the water, and if you wish to become a problem-solver, you must solve problems (p. 49).

When MLs interact with geometry questions, their problem-solving strategies are the practice of the 'art' they have learnt. Furthermore, Polya (1945) describes problem-solving as "the ability to solve non-routine problems that are not easy, that require a degree of independence, judgment, originality, and creativity" (p. 56). Donaldson (2015) views problem-solving as a process that comprises solving non-routine exercises that combine creativity and intuition. This study focuses on the literature to analyze existing debates regarding the effects of visualizing both non-routine and routine geometry problems in multilingual contexts. In addition, Govender (2015) defines problem-solving as one distinct attribute that brands mathematics as essential.

In view of these positions on visualization and problem-solving, in the context of this paper, the researcher believes there is a strong correlation between visualization and Polya's (1945) problem-solving strategies, described next. Polya's (1945) problem-solving steps help the researcher navigate the effects of visualization on learners' performance across diverse linguistic settings. This aligns with Polya's (1945) problem-solving steps.

The next section presents problem-solving strategies and approaches used in mathematics education.

Polya's (1945) Problem-Solving Steps

In everyday life, individuals constantly encounter problems, whether personal or work-related. Romberg (1994) and Enrique-Romero et al. (2016) define a problem, or problem situation, as one that presents difficulties and prompts an individual to seek a solution. Enrique-Romero et al. (2016) further explain that a situation qualifies as a problem when the individual is aware of it, recognizes the need for action, but is unable to resolve it immediately. In such cases, problem-solving involves gathering relevant information and connecting it to the problem at hand. The more comprehensive the information available, the more effectively an individual can address the problem. However, not everyone is equally adept at retrieving and applying relevant information to solve problems efficiently.

This phenomenon is also evident in mathematics education, where learners frequently struggle to understand and solve mathematical problems. Problem-solving has long been a central focus of mathematics education research. According to Halmos (1980), the essence of mathematics lies in its problems and the pursuit of solutions. Research has underscored the importance of both problem posing and problem-solving in fostering mathematical thinking and teaching expertise. Kilpatrick (1987) questioned the origins of well-structured mathematical problems, leading to a surge of interest in problems posing as a means of enhancing mathematics instruction (Ernest, 1989; Silver, 1994). Teachers have increasingly adopted problems posing as a strategy to facilitate learners' mathematical thinking (Cain, 2020). However, the ability to pose effective mathematical issues depends on a teacher's level of mathematical training and teaching experience (Klein & Leikin, 2020). The literature suggests that teachers and the context of mathematics instruction play crucial roles in shaping learners' mathematical thinking during problem-solving.

According to Hanegem (2017), mathematics education is about encouraging mathematical thinking and understanding and developing problem-solving abilities and research skills. Many mathematical problems can be solved using several different procedures (Gcasamba, 2014). Polya's (1945) four steps of problem-solving are not designed to restrict learners from using a particular strategy to solve a task, but rather to provide a guide for understanding the problem and choosing a strategy based on the learner's understanding. Polya's (1945) 4-step problem-solving process is shown in **Figure 1**.

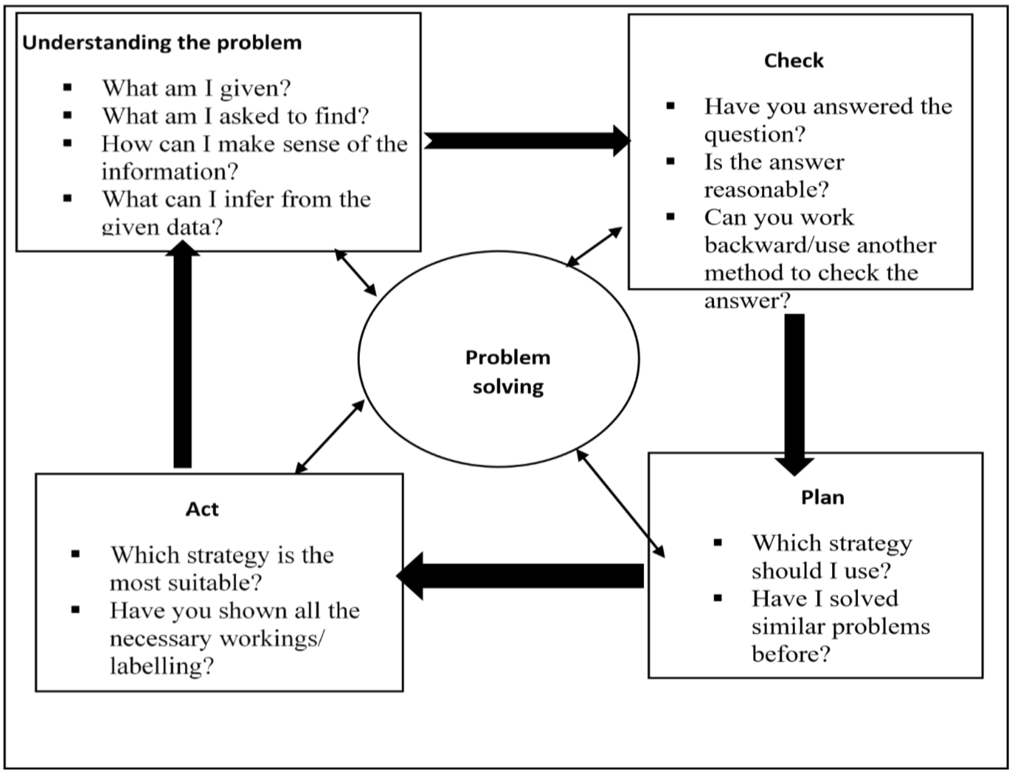

Figure 1. Polya's (1945) 4-step problem-solving processes (adapted from Caglayan, 2014, p. 15)

The first step of Polya's (1945) four-step problem-solving model is to understand the problem, which involves carefully reading the problem statement, listening attentively, and identifying key terms embedded in the problem. This step is crucial because it determines whether a learner has fully grasped the situation. In mathematical problem-solving, learners must understand the concepts involved, establish connections through visual perception, and enhance their critical thinking skills. At this stage, learners should visualize the problem with a strong conceptual understanding to develop an appropriate strategy. Furthermore, this step aligns with two key elements of Sfard's (2008) commognition theory: word use and visual mediators. For word-based questions, word use plays a critical role in learners' comprehension. In contrast, for questions accompanied by diagrams, learners integrate their understanding through visual mediators (such as diagrams) to develop a clearer conceptual grasp. F. Bautista et al. (2015) argue that using visual tools (visual mediators) in a structured and complementary manner can significantly enhance learners' understanding of mathematical concepts (word use) and relationships during problem-solving.

The second step, devising a plan, involves learners formulating a strategy or setting up an equation to solve the problem. At this stage, learners must determine an appropriate approach before attempting a solution. They may also draw on prior problem-solving experiences and integrate their existing knowledge into their approach to the new problem. For MLs, this step may involve selecting a strategy on their comprehension of the problem statement, the accompanying diagram, or their prior knowledge related to the question.

The third step, carrying out the plan, requires learners to implement their chosen strategy. If they have formulated an equation or identified a relevant approach, this is the stage where they execute their solution process. Here, in the context of this paper, MLs engage in problem-solving using the strategy they deem most effective.

Finally, the fourth step, looking back, involves reviewing the solution to determine whether all the given data was used appropriately and whether the answer is reasonable. At this stage, learners verify their solutions using mathematical principles. In the context of geometry, they assess their solutions using geometry shapes, properties, and theorems to determine whether their answers make sense or if alternative methods should be considered. Notably, this step integrates the remaining two elements of commognition: narratives and routines. Learners apply geometry theorems and axioms (narratives) to solve problems and develop systematic approaches (routines) during problem-solving.

Polya's (1945) four-step problem-solving process is particularly relevant to studying how language of instruction and visualization influence MLs approach geometry problems. By observing learners' responses from step one to step four, the researcher can assess their comprehension of the problem before solving it. This, in turn, enables an analysis of how visualization influences problem-solving for MLs in geometry.

The next section presents the existing research on mathematics problem-solving.

Research on Mathematics Problem-Solving

This section reviews existing literature on problem-solving strategies in mathematics, highlighting their significance in fostering conceptual understanding. By analyzing existing studies, the section identifies gaps in research, particularly regarding the integration of visualization techniques into problem-solving in multilingual settings. This demonstrates the value of the current review analysis and positions it within the broader academic discourse.

Polya (1945) asserts that learners develop problem-solving skills by modelling the behaviors of experienced problem-solvers. However, contemporary research indicates that solving numerous problems over time is more effective than simply being taught problem-solving heuristics (Lester, 1994). Schoenfeld (1992) critiques problem-solving heuristics such as those proposed by Polya (1945), arguing that their success depends on learners' familiarity with problem-solving strategies. Similarly, Verschaffel et al. (2020) conclude that following a step-by-step heuristic does not necessarily enhance learners' problem-solving competence.

Additionally, research has explored the role of metacognition in mathematical problem-solving. Early studies (Garofalo & Lester, 1985; Lester et al., 1989; Schoenfeld, 1983, 1987, 1992) identified metacognition as a critical factor influencing problem-solving performance, suggesting that metacognitive knowledge, experience, and skills shape problem-solving behaviors. Recent studies continue to affirm the significance of metacognition in mathematical problem-solving (Azevedo, 2020; Desoete & De Craene, 2019; Zhao et al., 2019), with some linking it to the ability to predict problem-solving success (Ebomoyi, 2020; Kuzle, 2018). However, Shilo and Kramarski (2019) argue that failures in problem-solving do not always stem from poor metacognitive knowledge but rather from an inability to activate relevant knowledge during the problem-solving process. This suggests that multiple factors may influence mathematical problem-solving, including those explored in this study–particularly, the role of visualization in MLs' problem-solving processes.

Erbaş and Okur (2012) argue that learners' success in solving mathematics problems is not determined solely by metacognition but also by their mathematical knowledge and repertoire of problem-solving strategies. Efklides (2001) highlights the importance of metacognition in understanding one's own thinking. Furthermore, research emphasizes the need for teachers to recognize and support learners' problem-solving efforts to enhance their mathematical reasoning (Cai & Leikin, 2020). These perspectives underscore the need to explore factors that influence MLs' problem-solving abilities to develop appropriate interventions. This paper, therefore, reviews how the literature the effects of visualization on MLs' problem-solving processes in geometry.

Aljura et al. (2025) conducted a study on the role of mathematical connections in problem-solving, concluding that these connections serve as essential tools for mathematical reasoning. The study also identified factors that can weaken learners' mathematical connections. Pambudi et al. (2018) found that learners who are passively engaged in mathematics classes often lack sufficient learning experience, leading to weak mathematical connections, whereas those who participate actively acquire stronger problem-solving abilities. These findings are relevant to the present study, as they highlight how various factors can either enhance or hinder problem-solving. This study analyzes the effect that visualization has on MLs' ability to solve geometry problems.

The total mathematics usage factor (Department of Education, 2019), which is based on the South African National Research Council's (2001) mathematics proficiency framework, emphasizes problem-solving as a key component of learner-centered mathematics instruction. However, the impact of this approach remains largely unexplored, particularly in the South African context. Research on mathematical problem-solving in South Africa has been relatively scarce in recent years. Nevertheless, studies investigating mathematical connections in problem-solving (Pambudi et al., 2020) and the implementation of problem-solving approaches (Naroth & Luneta, 2015) have provided valuable insights. Naroth and Luneta (2015) found that both learners and teachers faced significant challenges adopting a curriculum that prioritized problem-solving. Similarly, Biccard (2020) reported that South African teachers often lack a deep understanding of problem-solving strategies and the classroom norms that support them.

Chirinda and Barmby (2017) investigated the impact of professional development on problem-solving instruction, finding that it improved learners' performance and encouraged teachers to adopt a facilitator role. However, they later concluded that contextual factors such as overcrowding, language barriers, and learners' lack of problem-solving strategies often compelled teachers to revert to traditional teacher-centered instruction (Chirinda & Barmby, 2018). These findings align with the present study's focus on language-related challenges in problem-solving, particularly among MLs solving geometry problems.

To date, research on mathematical problem-solving has demonstrated that problem posing is a higher-order mathematical skill dependent on both experience and content knowledge (Leavy & Hourigan, 2020).

Additionally, studies have provided theoretical frameworks for analyzing problem-solving behaviors, metacognitive processes, and instructional approaches. While visualization, Polya's (1945) four-step model, and dialogic discourse have been widely studied, limited research has examined the specific effects of visualization on MLs' process of solving geometry problems. This paper seeks to contribute to the existing body of knowledge by offering deeper insights into the role of language in problem-solving and its impact on learners' ability to visualize and solve geometry problems in multilingual settings. The following section discusses the literature on the use of visualization in learning mathematics.

Visualization in Mathematics

Zimmermann and Cunningham (1991) describe visualization as "a process of constructing or using geometrical or graphical representations of mathematical concepts, principles, or problems" (p. 1). Mudaly and Rampersad (2010) point out that "visualization can be a physical or mental process" (p. 38). These definitions emphasize that visuals can be either mental or physical and that visualization of mathematics problems can take place in the mind or in a physical or digital form. In this paper, both mental and physical visualization are required for solving geometry problems posed to MLs. Yin (2010) states that while the ability to solve problems "is at the heart of mathematics", visualization "is the heart of mathematical problem-solving" (p. 2).

Visualization is recognized as an essential tool in mathematics, with a rich history of enhancing understanding and problem-solving. The use of visual reasoning in mathematics can be traced back to ancient Mesopotamia and Greece (Stylianou, 2002). Euclid, a mathematician in ancient Greek who is considered the 'father' of geometry, wrote "The elements," a work that presents the definitions, axioms, and theorems of geometry. Euclid's arguments rely on the use of geometric figures or shapes. Thus, visualization has been utilized in mathematics for a very long time.

The importance of visualization has been explored by several researchers (see Arcavi, 2003; Hanna, 2007; Pedersen et al., 2005; Presmeg, 2006; Zimmerman & Cunningham, 1991). According to Presmeg (2006), research on visualization began in the late 1970s. From its beginnings and through the 1980s, research on the use of visualization in the teaching and learning of mathematics focused on theoretical psychology. During the 1990s, visualization became a significant area of research in mathematics education in its own right, but it was only in the 21st century that research on visualization developed to include semiotic aspects, such as signs and symbols (Presmeg, 2006).

Some researchers have pointed out that visualization is an essential aspect of thinking (Bansilal & Naidoo, 2012; Marlewski & Zarzycki, 2004; Presmeg, 2006). In support of this view, Marlewski and Zarzycki (2004) conclude that mathematical concepts cannot be introduced without being illustrated through pictures, drawings, graphs, and manipulatives.

The Role of Visualization in Learning with a Focus on Geometry

In mathematics education, the introduction of mathematical concepts is structured through a concrete-illustrative-symbolic sequence of instruction. This begins with the concrete stage, progresses through the illustrative stage, and culminates in the symbolic (abstract) stage, with transitions from one stage to the next (Volk et al., 2017). During the first stage, learners solve problems through action, manipulating concrete materials or objects in various activities. Through this action, learners develop procedural knowledge by imitating and practicing procedures. The next stage, known as the iconic or illustrative stage, involves representing the world through sensory perception–particularly, visual perception–and organizing these perceptions into meaningful structures. This stage serves as a foundation for the final symbolic stage, in which abstract representations–such as words, numbers, and other agreed symbolic systems–are used to express ideas, objects, and relationships.

In the context of constructing conceptual representations, one of the growing areas of research in mathematics education is the visualization of mathematical concepts (Presmeg, 2014). While definitions of visualization vary slightly across didactic and pedagogical studies, the core emphasis remains on its role in understanding mathematical concepts. Kosslyn (1996) defines visualization as the creation of mental images to represent a concept. Lipovec and Podgoršek (2016) describe visualization as the natural identification of mathematical relationships through graphical representations. Similarly, Atanasova-Pachemska et al. (2016) define visualization in mathematics as the process of forming images–whether sketches or diagrams on paper, mental images, or virtual representations–and successfully applying them in mathematical reasoning and problem-solving.

Through visualization, learners are expected to create, interpret, and manipulate visual representations, applying them meaningfully to solve problems and reflect on their reasoning. Lipovec and Podgoršek (2016) distinguish between static geometric representations (e.g., images, diagrams, and schematics) and dynamic geometric representations (e.g., videos and applets). In the context of learning geometry, concepts can be visualized in various ways, including concrete models, static geometric representations, dynamic geometric representations, and computer-generated representations.

The next section reviews studies on visual-spatial abilities in developing geometric concepts.

Visual-Spatial Abilities in the Development of Geometric Concepts

A review of the literature reveals that the ability to imagine static objects and to manipulate them mentally is fundamental to the teaching and learning of geometry. This ability requires spatial and visualization skills. Spatial abilities have been found to facilitate the development of geometric reasoning during the primary school years (Cheng & Mix, 2014; Danişman & Erginer, 2017; Sinclair & Bruce, 2014). In the case of 2D and 3D shapes, research has shown that visual-spatial abilities positively contribute to the learning of geometric shapes and their properties. Furthermore, research suggests that geometric and spatial skills learned in primary school prepare learners for mathematical and abstract reasoning in later education (Sparks et al., 2019). This suggests that by grade 11, learners should have a full understanding of geometric vocabulary, shapes, and their properties. However, in South Africa, this is not the case, as learners continue to demonstrate difficulty with these aspects when solving geometry problems. This calls for further research to understand the specific challenges affecting learners' understanding in the South African context. Hence, this paper reviews the existing debates regarding the role and language of visualization for learners of mathematics in geometry who use a language that is not their native language.

Cohrssen and Church (2017) designed a project-based program that provided early childhood learners with six activities to develop spatial thinking. Their study involved 19 learners aged 4 to 6 years, five of whom attended all sessions. The activities required learners to:

(1) draw the school,

(2) draw school signs,

(3) draw 2D maps of 3D environments,

(4) build 3D constructions from 2D images,

(5) compare and discuss maps, and

(6) identify shapes on peers' maps.

Learners used atypical early childhood materials such as blocks. The emphasis was on modelling consistent geometric vocabulary for the names of 2D and 3D shapes (e.g., cube, square, and cone) and for spatial directions (e.g., in, on, up, and down). The findings revealed that placing young learners in real-world spatial environments enhanced their understanding of 2D and 3D shapes. While this study focused on young children, its findings are relevant to this study as they suggest that a foundational understanding of geometric concepts can be laid in the early years of education. Early exposure to 2D and 3D shapes and their properties enhances learners' reasoning and spatial thinking in later grades when solving geometric problems (Cohrssen & Church, 2017).

Various aspects of spatial abilities have also been found to support learners' understanding of geometric concepts and their representations. For instance, Bruce and Hawes (2015) conducted a lesson study with a team of seven teachers and 42 primary school learners (aged 4 to 8 years) in ON, Canada. Learners participated in pre- and post-test assessments, including a 2D mental rotation task and a 3D mental rotation block task. The findings revealed that learners' spatial abilities in mental rotations were flexible and improved with practice. The researchers found that the use of manipulative and well-structured teacher-led lessons on 2D and 3D shapes enhanced learners' understanding of geometric concepts.

This finding is supported by a study by Hawes et al. (2017), who implemented a 32-week teacher-led spatial reasoning intervention in a K-2 classroom involving 12 female teachers and 39 learners (aged 4 to 7 years). They explored the extent to which in-class spatial activities improve learners' spatial geometry performance. Learners engaged in three activities:

(1) a task assessing spatial language with a focus on shape recognition and positional language,

(2) a 2D mental rotation task, and

(3) a reasoning task involving transformations, symmetry, and composition/decomposition of 2D shapes.

Hawes et al. (2017) suggested that spatial training might also enhance learners' numerical skills.

The ability to perceive objects from different perspectives has also been found to foster geometric understanding in 2D and 3D. Wijaya et al. (2015) examined the relationship between imaginary perspective-taking and mathematical ability among 334 learners from the Netherlands and 304 learners from Cyprus, aged 4 to 5 years. They assessed learners' visibility competence (i.e., reasoning about which objects are visible from different perspectives) and 24 appearance competence (i.e., reasoning about how objects appear from different viewpoints). Their findings indicated that mathematical ability is meaningfully related to learners' capacity to visualize objects from multiple perspectives, which in turn supports the learning of 2D and 3D shapes and their properties.

In addition to mental rotation and spatial orientation, nets are also considered effective in developing 3D geometric concepts by fostering spatial abilities (Wright & Smith, 2017). Wright and Smith (2017) conducted task-based interviews with six learners, asking them to predict whether a given net would fold into a target solid (e.g., a cube or a square-based pyramid). They found that learners employed various strategies, such as selecting a base and mentally rotating faces to determine the folding sequence. They argued that directing learners' attention to specific target shapes through the use of nets supported their understanding of the diverse properties of 3D solids.

Sack and Vazquez (2016a) argued that spatial orientation (i.e., the position and direction of objects) plays a crucial role in determining learners' representations of 3D models. They conducted a seven-year longitudinal study in an elementary school in the USA, involving 11 grade 3 and 14 grade 4 learners (aged 9 to 10 years). The study examined the development of 2D and 3D geometry concepts using the Geocadabra construction box dynamic computer interface integrated with the spatial operation capacity model (Sack & van Niekerk, 2009). This model consists of three sub-models:

(1) full-scale models, which learners can manipulate physically,

(2) conventional graphic models, which provide 2D representations of 3D objects, and

(3) semiotic models, which are abstract and symbolic representations, such as floor plans.

Their findings suggested that for learners to develop geometric understanding, they must acquire competencies across these three representational models. Additionally, they found that learners struggled with the use of spatial terminology, such as 'horizontal' and 'vertical', when describing figures.

Fujita et al. (2020) conducted a study with 1,357 grade 9 learners (aged 9 to 14 years) to explore how learners use spatial reasoning to solve geometry problems involving 2D representations of 3D shapes. They found that successful problem-solving requires the integration of spatial visualization skills (e.g., recognizing that the diagonal of a cube's square face forms two triangles), analytical reasoning (e.g., understanding that three diagonals of a cube form an equilateral triangle), and domain-specific knowledge (e.g., knowing that equilateral triangles have angles of 25 60°). They argued that learners need more opportunities to consolidate their spatial reasoning skills alongside domain-specific skills. These findings are particularly relevant to this study, as they suggest that to solve geometric problems effectively, learners must be able to visualize geometric shapes and their properties.

These studies highlight the crucial role of spatial abilities in developing an understanding of geometric shapes and their properties. However, none of these studies explored how learners used language to explain and construct their mathematical thinking, even when data were collected through interviews. This raises the question of whether language plays a role in learners' understanding of geometry and why instruction is typically delivered in a single language. Sack and Vazquez (2016b) provided some insights into the role of language in explaining learners' thinking, but they did not explore how learners navigate multiple cultural and linguistic meanings when discussing shapes.

An exploration of how learners integrate their understanding of geometric concepts into problem-solving while interacting in a multilingual setting has not been conducted. Investigating these processes in a multilingual context may provide valuable insights into how learners use multiple languages to develop their understanding of geometric shapes and concepts during problem-solving. The lack of research on the effects of multilingualism on visualization and spatial reasoning in geometry is evident.

Types of Visualization Tools Used in Mathematics Problem-Solving

This section discusses the review of visualization tools commonly used in rural South African schools, for relevance to this paper.

Pictures, graphs, figures, and gestures

Graphs and pictures enable learners to visualize mathematical problems and may serve the same function as diagrams and images (A. Bautista et al., 2015). When using graphs, learners can incorporate verbal statements, number lines, data tables, or symbolic notation. According to Duval (1995), a figure provides a geometric representation of a mathematical situation that is more concise and easier to understand than a verbal explanation. However, gestures–defined as body movements produced in a silent, non-communicative problem-solving context–also play a role in visualization (Chu & Kita, 2011). Existing research indicates that individuals perform visualization tasks more effectively when encouraged to move their hands in a manner congruent with what they need to visualize (Pereira & Smith, 2009; Zahn-Waxler et al., 1998). This has been observed in teachers' use of gestures as visualization tools during problem-solving lessons.

Manipulatives

Manipulatives are physical objects designed to represent abstract mathematical concepts concretely (Furner & Worrell, 2017). However, Ball (1991) questions their effectiveness in fostering correct mathematical understanding, emphasizing instead the importance of the context in which manipulatives are used to create meaning. Baroody (1989) supports Ball's (1991) argument, noting that while manipulatives do not guarantee success in solving mathematical problems, they remain valuable instructional tools.

Some teachers do not consider manipulatives essential for problem-solving and encourage learners to rely on mathematical reasoning instead (Furner & Worrell, 2017). Golafshani (2013) examined how grade 9 mathematics teachers use manipulatives and found that, although teachers appreciated their benefits, they often lacked the time and knowledge needed to integrate them effectively into lessons. Nevertheless, Furner and Worrell (2017) emphasize the importance of manipulatives as valuable tools for mathematical problem-solving, advocating for their use in classrooms to support conceptual understanding and enhance problem-solving skills.

The Use of Videos, Computers, and Other Technologies

The National Council of Teachers of Mathematics (2000) in the USA considers technology one of its six core principles; it states that "technology is important in learning and teaching mathematics: it influences the mathematics that is taught and enhances the learners' learning" (p. 24). Similarly, Shoba et al. (2020) views technology as a tool that directly affects learners' interactions with problems and their conceptual understanding, thereby contributing to the development of mathematical competencies and to how these can be explained. This highlights the importance of integrating computers and digital tools into mathematics teaching and learning.

Moreover, computer-aided algebra resources, particularly those incorporating symbolic manipulators, can serve as effective visualization tools (Chiu & Churchill, 2015). Lois and Milevicich (2010) argue that technology can act as a mediation strategy, allowing teachers to monitor, motivate, and stimulate learning while encouraging learners to reason and seek knowledge. Sutch (2010) asserts that technology enhances the relationship between teachers and learners by fostering conceptual co-learning. Given these benefits, this study recognizes technology as a crucial visualization tool that can support learners' understanding of mathematical problems during problem-solving. Furthermore, digital tools facilitate the introduction and reflection of new mathematical concepts, aligning with learners' cognitive development and fostering critical thinking. Caglayan (2014) suggests that materials designed to visualize algebraic expressions and numerical relationships can help learners construct mathematical formulae more meaningfully. To maximize learning with text and images, Chiu and Churchill (2015) propose three key principles: removing irrelevant words and graphics, highlighting essential elements, and presenting words with corresponding graphics simultaneously. Whiteboards can also be used to implement these principles effectively. Lois and Milevicich (2010) argue that whiteboards and video equipment provide valuable platforms for problem-solving, facilitating the teaching and learning of mathematical concepts in a more interactive manner.

Calculator

The use of calculators in mathematics classrooms shifts the teacher's role to that of a facilitator (Karadeniz & Thompson, 2018). The use of calculators in mathematics helps promote learner achievement and positive attitudes toward mathematics (Tan, 2012). Lazakidou and Retalis (2010) state that calculators save time, increase learners' efficiency, and improve their outcomes in mathematics. Hunter (2011) notes that calculators promote learners' reasoning skills. For this reason, in this review, calculators were regarded as a critical visualization tool.

Diagrams

A diagram is a physical form of a mental image (Mudaly, 2012). Mudaly (2012) explains that "diagrams allow the person viewing the diagram to see a complete picture in their mind" (p. 22). Mesaros and Weinkopf (2012) calls a diagram an "illustrative tool of visualization" (p. 321), and Winn (1991) portrays a diagram as a visual representation that uses spatial arrangement in a profound way. In the context of this study, diagrams can play an essential role during problem-solving as allowing MLs to view the problem using a diagram enables them to see a complete picture in their minds and better understand the problem to be solved. Diezmann (1995) states that "diagrams may consist of words and/or abstract pictures" (p. 223). In multilingual context, for learners to devise a suitable problem-solving strategy, they have to master both language of teaching and learning for them to make a good connection between the problem statement and accompanied diagram.

Swanson et al. (2011) argue that the use of diagrams during problem-solving enhances learners' capability to successfully solve mathematics problems. Sfard (2008) argues that diagrams make objects accessible, allowing learners to create and verify mathematical narratives. A benefit of diagrams is that they display relationships between variables or concepts (K. Jones, 2013; Sfard, 2008). However, Maries and Singh (2018a) argue that it is not just the use of diagrams, but their effective use, that leads to successful problem-solving, as failure to draw the diagram correctly may result in an inability to solve the problem.

These sources emphasize that diagrams should provide learners with a clear picture to enhance their thinking during problem-solving and enable them to make sense of their solutions by verifying them with diagrams. Similarly, if MLs are given only word problems and draw inaccurate diagrams, they might experience difficulties solving them.

Additionally, diagrams act as an essential step in organizing and simplifying information (van Garderen et al., 2014). Once information has been simplified into a diagram, it becomes easier to solve a problem.

Confirming this, Maries and Singh (2018b) argue that they view diagrams as a practical problem-solving heuristic. Similarly, A. Bautista et al. (2015) emphasize that teachers with strong mathematical backgrounds typically use diagrams to explain concepts. If learners are taught geometry concepts through diagrams, it becomes easier for them to solve geometry problems using their strong mathematical background. However, this general claim has not been tested in a particular context of learning and teaching. Hence, this study focuses on the context in which learners learn mathematics in a language that is not their native language.

In this section, various visualization tools were discussed in terms of their relevance to mathematical problem-solving. There has been growing research into the use of visuals in mathematics education. The use of visuals in mathematics is regarded as a powerful tool for teaching and learning, particularly in geometry. In addition, visual tools play a significant role in connecting mathematical concepts with understanding during problem-solving.

This raises the question of whether visuals are used to the expected standards during the teaching and learning of geometry. If they are used to the expected standard, where are we now?, It was just important to consider this review concerning exploring the effects of visualizing during problem-solving for MLs in geometry. Hence, a gap in the effects of visualizing when solving geometry questions in multilingual contexts is identified. In the next section, I present the literature review regarding the cognitive processes involved in visualization and problem-solving.

Cognitive Processes Involved in Visualization and Problem-Solving

This study adopted the visualization process described by Mudaly and Rampersad (2010). This visualization process starts with MLs seeing a diagram or forming a mental image after reading the problem statement. This image holds meaning for the learner and influences their thinking based on prior knowledge. Drezmann and Lowries (2012) support this view, stating that the absence of prior knowledge and the inability to construct mental representations result in learners facing challenges in visualizing and assessing their thinking. The meaning of the image is created through reflection, interaction with new stimuli, and engagement with other given data. Through this process, knowledge is internalized and externalized. New knowledge is formed as learners reflect and interact with their prior knowledge. Mudaly and Rampersad (2010) explain that "internalized new knowledge is utilized to effect what is seen or added to new diagrams" (p. 39). This new knowledge subsequently influences the way the learner perceives the same image or diagram.

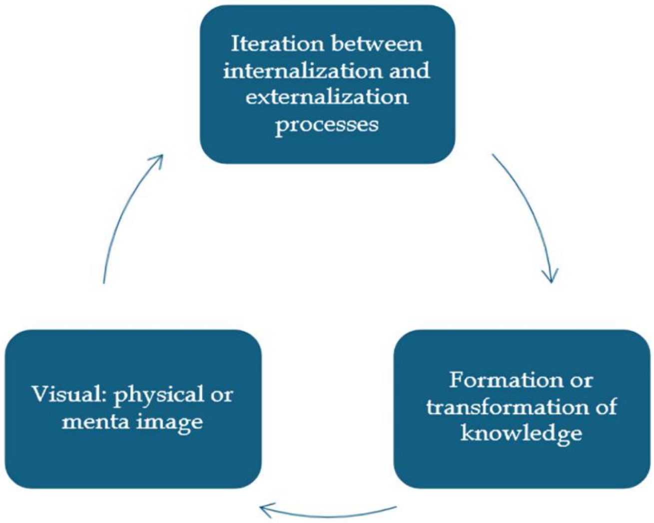

Figure 2. Visualization process (adapted from Mudaly & Rampersad, 2010, p. 39)

In this study, MLs' understanding during the problem-solving process in geometry may have been influenced by their prior knowledge. They were likely to comprehend a question more effectively if they had encountered similar problems before. However, if they had not been exposed to such problems, they may have struggled to establish connections due to a lack of relevant prior experiences with the geometry concepts and diagrams used in the study. According to Mudaly and Rampersad (2010), "diagrams allow the learner to change or cultivate new knowledge" (p. 39). In this study, diagrams are regarded as a visualization tools used to support the development of new knowledge and facilitate a deeper understanding of geometry problem-solving.

The image in **Figure 2** represents the process involved in visualization.

This article further adopted the five processes and seven roles of visualization identified by Yin (2010): understanding, connecting, constructing, using visual representation to solve the problem, and encoding to answer the problem. The process of understanding refers to the connection between the components of the problem. The connecting process involves creating connections to problems that were solved previously. The construction process entails creating a visual image. Visual representation is then used to solve the problem, and encoding is the process by which the problem is answered. These five processes identified by are relevant to this study: when MLs solve geometry questions, all these processes play a role in helping them make sense of the problem statement and use diagrams effectively.

The seven roles that Yin (2010) identifies are understanding the problem, shortening the problem, linking to a related problem, catering for individual learning styles, substituting for computation, verifying the solution, and transforming the problem into a mathematical form. Visualization comes into the first role–understanding the problem–as it helps learners grasp the connections between different components of a problem. In the role of simplifying a problem, visualization allows learners to recognize a less complicated version of the problem. Learners solve and identify the best procedure to successfully solve all similar types of questions. In geometry, two types of problems might be used: word problems and problems with diagrams. At this point, MLs are expected to understand the problem statement and find the approach that would best assist them in solving both types of questions. In the role of connecting to related problems, learners may link the given problem to their previous problem-solving experiences. Regarding accommodating individual learning styles, each learner has their own way of using visualization based on their understanding of the question. In this case, MLs may have employed different approaches when solving questions.

Visualization also serves as an alternative strategy for calculation during problem-solving. Learners can solve a problem visually without performing written calculations. Another function of visualization is as a tool to verify solutions. It can be used to determine whether an answer is reasonable. Diezmann (2001) clarifies that "writing information on a diagram is a conversion process that is made up of decoding linguistic information and encoding visual information" (p. 1). During this process, learners reorganize their knowledge, and new knowledge can be formed. This perspective suggests that, even when learners use languages that they are not fluent in, visualization serves as a universal tool that accommodates them regardless of their home language.

This section has critically reviewed the literature on the role of visualization processes in mathematical problem-solving. Visualization processes such as understanding, connecting, constructing, using visual representation to solve the problem, and encoding to answer the problem are recognized as crucial for mathematical problem-solving. Although researchers have identified key steps in visualization during problem-solving, few studies have explored the factors that may hinder learners from using these processes when solving geometry questions. Hence, the current paper explores the possible connection between visualization and language during geometry problem-solving in multilingual settings. The next section presents the historical development of mathematics education research regarding conceptions from a multilingual perspective and their influence on problem-solving in mathematics.

Multilingualism in Mathematics Education

Multilingualism has been defined in various ways in the literature. Kemp (2009) notes that monolinguals are speakers who use one language, while bilinguals are those who use two languages. Multilinguals generally use or know three or more languages, with varying degrees of proficiency in these languages (Barwell, 2018; Clarkson & Idris, 2006). However, scholarship in the field of multilingual research in the last decade has expanded the definition of multilingualism to include "various forms of social, institutional and individual usage as well as individual and group competences, plus various contexts of contact and involvement with more than one language" (Franceschini, 2009, p. 29). This understanding of multilingualism acknowledges language diversity, includes sensitivity towards socio-cultural diversity, and appreciates society's heterogeneity. It is now used as an umbrella term that encompasses research on bilingualism. For this paper, a contextual understanding of multilingualism is adopted. Language plays a critical role in teaching, learning, and mathematical problem-solving (Barwell, 2014; Morgan et al., 2015; Moschkovich, 2017; Planas et al., 2018).

Having knowledge of more than one language has been recognized as a 'resource' for learning mathematics (Adler & Ronda, 2015; Moschkovich, 2013, 2015a, 2015b), which was not the case in the earlier years of multilingual studies in the field of mathematics education research. It is evident that multilingualism research has significantly influenced the field of mathematics education.

Research on multilingual education has moved beyond deficit theories of multilingualism to address how the linguistic resources MLs bring to class can be effectively harnessed to provide high-quality mathematics education. In other words, more research now focuses on multilingualism as a resource in multilingual mathematics classrooms (Barwell, 2018; Planas, 2018; Prediger, 2019; Ryan & Parra, 2019). However, many researchers highlight the complexity of learning and teaching in multilingual classrooms, where learners are simultaneously learning the language of learning and teaching (LoLT) and learning mathematics as both a discipline and a language.

In multilingual settings, learners who share the LoLT with their home language are more familiar with the linguistic structures encountered in mathematics classrooms (Barwell, 2018). However, research indicates that this is not the case for learners whose home language differs from the LoLT (Gohel et al., 2007; Robertson & Graven, 2018). When learners' home language differs from the LoLT, they must contend with the additional challenge of not being fluent in the language of instruction. Mathematics education involves problem-solving, which requires a firm grasp of mathematics vocabulary. This suggests that learners must develop proficiency in the language of instruction to effectively comprehend mathematical terminology during problem-solving. The LoLT is thus expected to play a crucial role in learners' understanding of mathematical concepts. In multilingual contexts, research has demonstrated that while multiple language use in the mathematics classroom can support learning (Moschkovich, 2015a), it is not a straightforward solution, as diverse language infrastructures within and around schools place varying demands on learners and their teachers.

While a review of the literature on MLs and mathematics highlights research conducted in various countries, there is a limited number of studies on this topic that specifically focus on understanding mathematical concepts during problem-solving (e.g., Barwell, 2014, 2016a, 2016b, 2018, 2020; Moschkovich, 2002, 2007, 2012, 2018, 2019; Takeuchi et al., 2016). While many studies examine the socio-political dimension of language as a resource for gaining access to mathematical knowledge, they do not provide insights into how understanding of mathematical concepts is enhanced (Moschkovich, 2013; Planas & Setati-Phakeng, 2014; Setati & Barwell, 2006). The emphasis on language in these studies often overshadows learners' understanding of mathematical concepts during problem-solving, thus shifting the central aim of mathematics education research to the periphery. Therefore, the objective of this paper is to build on existing research by providing an in-depth analysis of the effects of visualization during problem-solving for learners who study mathematics in a language that is not their native tongue.

The Effects of Language on the Visualization of Multilingual Learners During Problem-Solving in Geometry

Problem-solving is a complex cognitive process that involves visualization, creativity, abstraction, and the connection of information (Rahman & Ahmar, 2016). Developing strong problem-solving abilities is a crucial objective of learning mathematics (Muflihatusubriyah et al., 2021). In other words, at the heart of mathematics lies the ability to solve mathematical problems. Utilizing problem-solving strategies in mathematics learning has a positive impact on students' abilities and skills for solving mathematical problems (Wiryawan et al., 2014). Effective problem-solving strategies facilitate learners' ability to tackle complex problems more efficiently (Ozturk & Guven, 2016). In some studies, the academic performance of learners taught using problem-solving methods surpassed that of learners taught through conventional teaching approaches (Behlol et al., 2018). Effective problem-solving methods foster active learning, where learners engage with the subject matter, work collaboratively, and learn through hands-on experiences.

Additionally, these methods promote critical thinking by prioritizing application over memorization. However, in geometry problem-solving, MLs must understand geometric language and visualize problems effectively to develop an appropriate problem-solving strategy. Visualization is, thus, another essential strategy that enhances mathematics learning, particularly during problem-solving.

According to Polya's (1945) four-stage problem-solving process–(1) understanding the problem, (2) devising a plan, (3) carrying out the plan, and (4) reviewing the solution–visualization is a highly effective tool applicable to all stages. Polya (1945) contends that his problem-solving approach aligns well with visualization and enhances learners' problem-solving abilities. In the context of this study, all four stages contribute to improving learners' ability to solve geometry problems. However, understanding the problem–the first stage–is particularly critical, as MLs must comprehend the geometric concepts embedded in the question to formulate an appropriate algorithm for solving the problem.

In mathematical problem-solving, visualization is a powerful cognitive tool that supports learners' reasoning. The efficacy of visualization stems from its role in information generation, given that the primary source of information processing is visual perception (Ozkaya et al., 2016). The way individuals perceive an object significantly influences the information they extract from it. In this study, learners' ability to visualize geometry questions is expected to aid in formulating potential problem-solving strategies. Furthermore, visualization enhances the understanding of problem situations (Surya et al., 2013) and is most effective when presented through diagrams, flowcharts, timelines, films, and demonstrations (Sundari & Prabawati, 2019). In current school situations, the geometry questions learners are always given include both diagram-based problems and word problems without diagrams. The combination of problem statements and diagrams required learners to integrate textual and visual information to determine an appropriate problem-solving strategy.

Visualization facilitates comprehension, memory retention, recall, and problem-solving. It plays a critical role in problem representation; however, many learners do not employ these essential processes in their learning (Warger, 2018). For instance, when solving geometry questions, learners were expected to visualize problems, recall procedural routines, and apply geometric proofs to solve geometry problems. Visualization techniques also help learners comprehend problems, explore possible solutions, and develop new perspectives (Samosa, 2021). This process requires learners to mentally construct familiar scenarios. In geometry, visualization is instrumental in recalling potential strategies for problem-solving. Thus, visualization is expected to enhance learners' ability to solve geometry problems by integrating diagrams and problem statements.

Despite this, Euclidean geometry remains one of the most challenging topics for both learners and teachers (Tachie, 2020). According to diagnostic reports on the South African national senior certificate exam, learners' performance on mathematics paper 2–particularly in the area of Euclidean geometry–has been declining (Tachie, 2020). This study is concerned with learners' poor performance in geometry and investigates the effects of visualization on the problem-solving abilities of MLs in geometry.

Debrenti (2015) asserts that visualization is an effective approach that enables learners to connect textual information in mathematics problems with their prior knowledge or experiences, thereby facilitating the formation of mental images that enhance comprehension. However, various researchers (Battista & Clements, 1995; Teahen, 2015) have identified several challenges that learners face in their learning of geometry. Teahen (2015) suggests that poor mathematical proficiency or computational skills are not the sole contributors to learners' difficulties in solving mathematical problems: the language used in word problems and learners' inability to comprehend these problems significantly impact mathematical achievement. Although multiple problem-solving strategies are available, visualization is increasingly recognized as the most effective (Debrenti, 2015). However, visualization may be influenced by additional factors during geometry problem-solving, and this study examines explicitly how visualization affects MLs' understanding of geometry.

A study by Mudaly and Narriadoo (2023) on solving word problems through visual thinking found that visualization was the most effective tool available for learners tackling word problems. Additionally, learners indicated that incorporating measurement and shapes into word problems necessitated well-drawn diagrams, as these diagrams enabled them to label measurements and identify known and unknown values, thus facilitating the selection of an appropriate problem-solving procedure. However, a common challenge encountered by learners was the complexity of reading lengthy problem statements and the confusion caused by the language used. Mudaly and Narriadoo's (2023) study is relevant to the current research, in which MLs engaged with geometry diagrams, integrating textual problem statements with visual representations.

Similarly, Ngirishi (2015) investigated the understanding of geometry of further education and training (FET) learners in KwaZulu-Natal and found that learners' difficulties in solving geometry problems were largely attributed to their lack of understanding of geometric language. In this regard, the present study examines the role of language in influencing learners' ability to visualize and solve geometry problems.

Luneta (2014) argues that insufficient teacher knowledge hinders learners' understanding of geometric concepts during geometry instruction, thereby impeding their ability to solve problems. Research has also indicated that learners exhibit limited proficiency in solving geometric problems (Cesaria & Herman, 2019). van Hiele (1959) posits that a lack of understanding in visualized geometry often arises from a disconnect between teachers and learners. The language and symbols used at each level of geometric reasoning vary, meaning that a relationship that is valid at one level may not hold at another (van Hiele, 1959). Teachers must strive to communicate in a manner aligned with their learners' cognitive level. van Hiele (1959) emphasizes that the language employed in geometry instruction should clarify rather than confuse learners.

In many cases, teachers simplify mathematical terminology to facilitate comprehension. However, this strategy often leads to misconceptions among learners (van Hiele, 1959). For instance, if a teacher uses the abbreviation 'parm' for 'parallelogram', learners may fail to associate it with the correct geometric term. In the context of this study, such terminology might lead to misunderstandings and misconceptions among MLs about visualized geometry. While simplifying terminology is a common instructional strategy, it often leads to learners' misconceptions, as students often struggle to grasp these modified approaches (Luneta, 2015).

Mudaly and Rampersad (2010) support the view that specialized terminology confuses learners and increases anxiety among those who struggle with mathematics. This suggests that language plays a significant role in learners' understanding of mathematics. The use of proper geometric terminology enhances learners' ability to communicate their knowledge of geometric concepts effectively (Atebe & Schäfer, 2010). The current paper reviews the impact of language on learners' ability to visualize during problem-solving in geometry.

Language Factors Affecting the Visualization of Euclidean Geometry Problems

Visualization and the language used in geometry can help learners understand problem-solving questions. However, language barriers can pose a challenge when visualizing geometry problems, particularly when teachers and learners have different first languages. Even when mathematical concepts are visually represented, learners may still struggle to comprehend them if they are presented in a language that is not their home language (Maluleke, 2019). Visualization can assist learners in making connections between mathematical concepts, identifying patterns and relationships, and deepening their overall understanding of mathematics (Rathour et al., 2022). However, language barriers between teachers and learners can hinder teachers' ability to engage learners in visualization and mathematical problem-solving (Bermejo Fernandez et al., 2021).

This perspective on language and visualization suggests that language itself plays a significant role in solving geometry problems, particularly for MLs. This was demonstrated in an empirical study conducted by Mudaly (2021), which found that learners were able to organize data and develop strategies for solving mathematical problems using internally visualized diagrams. However, the presence of drawn diagrams may be even more beneficial, as they allow learners to verify their solutions when solving geometry questions. In this regard, Borbosa and vale (2021) emphasize that visual methods are essential in mathematical problem-solving. Without them, learners are more likely to attempt problem-solving without adequate understanding. According to Moleka (2021), proficiency in English, reading skills, and a sense of mathematical language are vital for success in visualizing mathematical problems. In this paper, if MLs lacked reading skills and did not understand mathematical language, their ability to visualize and solve geometry problems could be compromised.

A relevant study by Zulu and Mudaly (2023) explored the problem-solving strategies used by pre-service mathematics teachers through a visual and discursive lens. The study revealed that visualization and discourse play a crucial role in mathematics teaching and learning. It also found that language is key to self-discursive activity, which contributes to successful visualization. However, given these findings, the effect of language must be carefully considered, as it can influence learners' ability to visualize and comprehend geometry problems.

A study conducted by Chimire et al. (2022) in Asia examined the difficulties faced by secondary level students in learning geometry. The study identified three major contributing factors to learners' struggles: inadequate visualization abilities, ineffective instruction, and inadequate understanding of the language of geometry. These findings highlight that language and visualization are among the key challenges in learning geometry. In this context, language and visualization serve as fundamental pillars for understanding Euclidean geometry. When solving geometry problems, learners must not only understand the language of geometry but also link it with diagrams to arrive at appropriate solutions. This paper, therefore, focuses on a review of how language affects MLs' comprehension of visualized problem-solving questions in geometry.

Similarly, a study by Chen et al. (2020) in the USA examined students with specific difficulties in geometry. The study found that students who spoke English at home and connected mathematics with science were significantly less likely to struggle with geometry. In contrast, non-English-speaking students and those with a negative attitude toward mathematics and science were more likely to be in the lowest-achieving group and experience difficulties in geometry. These findings suggest that MLs' performance in geometry may depend on their language proficiency and the frequency with which they use mathematical language at home. In this paper, the effects of language are analyzed via the existing studies in the context of MLs in South Africa, whose home language is not English, which is the LoLT, when visualizing geometry problem-solving questions. According to Robertson and Graven (2019), language can either support or hinder certain groups of learners in making sense of mathematical concepts in geometry. If learners are not taught the terminology of Euclidean geometry adequately, they may struggle to grasp key concepts. van Hiele (1959) emphasizes the importance of developing the language of Euclidean geometry and argues that teachers must continuously support learners in constructing a deductive relational system in geometry.

A study by Jackson (2023) on problem-solving found that learners benefited from instruction and activities that emphasized the importance of definitions and understanding mathematical language. The study also noted that cooperative learning played a key role in helping learners process mathematical postulates. Given that language plays a key role in learning, MLs' ability to visualize and solve geometry problems may depend on their comprehension of the language used in problem statements. This study, therefore, aims to investigate the effects of language, specifically in the visualization of geometry problem-solving questions, among MLs.

The significance of language is even more pronounced in multilingual classrooms, which are common in South Africa. In these classrooms, mathematics is often taught in a language other than the learners' home language. This paper explores existing studies on the role of language in the visualization of geometry problem-solving questions by MLs. Language can either facilitate understanding or impede comprehension (Collett & Steyn, 2017). In the case of geometry, specifically, many learners struggle to grasp the relevant mathematical language (Sanders, 1994).

A study by Moleka (2021) on teachers' perspectives regarding language factors affecting the visualization of mathematical word problems found that insufficient reading skills, a lack of understanding of mathematical language and structure, and ambiguous language were key barriers to visualization. In the context of this study, the language used when visualizing geometry problems may also influence MLs' ability to engage with the material effectively. Research suggests that engaging in mathematical visualization enhances learners' problem-solving skills and ability to reason mathematically (Yilmaz & Argun, 2018). While visualization is crucial for developing problem-solving abilities, language also plays a fundamental role in ensuring successful visualization (Zulu & Mudaly, 2023).

The literature reviewed indicates that mathematical language significantly affects learner performance. However, this paper focuses specifically on how language affects the visualization of geometry problems by MLs. The review acknowledges that simplifying mathematical language can help reduce the challenges learners face in making sense of visualized mathematical concepts, including geometry. Visualization is a powerful tool for teaching and learning Euclidean geometry, as it helps learners generate meaningful mental representations necessary for solving geometry problems (Moleko, 2021). It is also important to note that there is a scarcity of literature on the effects of language visualization during problem-solving, as limited research has been conducted in this area.

Given the strong connection between mathematical language and visualization, this study sought to determine whether mathematical language is a contributing factor to learners' poor performance in visualized problem-solving in geometry or if other factors in the South African curriculum also play a role.

The next section presents a review of the literature on strategies for supporting MLs in mathematics education.

Strategies for Supporting Multilingual Learners in Mathematics

Language plays a significant role in developing a deep conceptual understanding of mathematical ideas. Learners studying mathematics in a language other than their home language require explicit and deliberate language support. Various strategies employed by teachers to assist learners in such contexts have been highlighted by scholars such as Clarkson (2009). Clarkson's (2009) research identified three key approaches:

1. Teachers should encourage different types of language use, including informal discussions in the language of instruction.
2. The language pathways of learners in complex multilingual settings should be traced to understand their linguistic transitions.
3. Teachers should actively utilize academic mathematical language and foster an expectation that learners will progressively adopt it.

Encouraging learners to discuss mathematical ideas in their own language has been shown to enhance their conceptual understanding (Moschkovich, 2012). However, oversimplifying the language of instruction can hinder the acquisition of mathematical concepts by causing confusion. Moschkovich (2012) proposes more inclusive approaches to addressing language-related challenges in mathematics classrooms. These include focusing on learners' mathematical reasoning rather than merely simplifying language, recognizing and supporting engagement with linguistic complexities in mathematics, treating everyday language as a resource rather than an obstacle, and uncovering the mathematical reasoning embedded in learners' speech and actions.

Furthering this discussion, Moschkovich (2019) introduces a framework of "academic literacy in mathematics" (p. 89) to examine how a mixture of linguistic practices enables learners to engage in mathematical processes. She explores how different meanings of the same word can influence mathematical thinking. Within this framework, she argues that solving mathematical problems requires not only mathematical proficiency but also competence in mathematical discourse and practices (Moschkovich, 2019).

Moschkovich (2019) analyzed a lesson on two-dimensional geometric shapes in a grade 4 bilingual classroom where instruction was delivered in both Spanish and English. She argues that bilingual learners' pronunciations often reflect hybrid language practices rather than simple code-switching. Additionally, she observed that the teacher rephrased learners' responses to encourage participation rather than enforcing rigid mathematical terminology. This perspective aligns with Planas and Chronaki (2021), who noted that discussions about code-switching often overlook the dynamic nature of language use, including variations in vocalizations, expressions, intonations, styles, and bodily gestures that contribute to meaning-making. This insight is particularly relevant to mathematics education, as strategies such as language simplification may inadvertently cause confusion when learners encounter mathematical problems that do not align with the simplified terminology.

Multilingualism in mathematics classrooms is not a challenge unique to South Africa (Kahiya & Brijlall, 2021a). Countries such as Sweden, the USA, Catalonia, and Cyprus face similar issues due to demographic shifts and international migration. For example, studies in Sweden have revealed that learners from foreign backgrounds perform less successfully in mathematics compared to their Swedish peers (Achuonye, 2015). Second-language learners require scaffolding to progress in both linguistic and mathematical skills (Moschkovich, 2012). Earlier research emphasized the importance of teachers taking responsibility for fostering both language and mathematical development. However, in high school settings in South Africa, particularly in the FET phase (grade 10-grade 12), where classrooms are highly multilingual, teachers may face time constraints in addressing these needs effectively.

In mathematics, discipline-specific language plays a crucial role in learners' academic success; this means learners must master both the language of instruction and the specialized vocabulary of the discipline. The ability to shift between these linguistic registers is considered key to developing mathematical understanding. Various researchers advocate code-switching as a strategy that enables MLs to integrate their linguistic competencies to grasp mathematical concepts effectively (Moschkovich, 2013, 2015a; Setati & Moschkovich, 2013). However, tensions arise between the language used for thinking and the language used for demonstrating knowledge. Making mathematical language more explicit can sometimes shift learners' attention away from the underlying mathematical concepts. Barwell et al. (2016) examined these tensions by comparing mathematics classrooms in Canada, Malaysia, and South Africa, concluding that teachers play a crucial role in managing these linguistic challenges. Their study identified multiple strategies that teachers employ, including mixing languages to support meaning-making, regulating language use in the classroom, using multiple discourses, and interpreting learners' verbal expressions. These findings highlight the importance of teachers' involvement in facilitating the comprehension of mathematical concepts in multilingual classrooms to enhance learners' critical thinking skills.

Another widely recognized strategy for supporting MLs in mathematics is the use of gestures. Research suggests that gestures serve as an essential tool for reasoning and problem-solving (Alibali & Nathan, 2012; Flood, 2021; Flood & Banks, 2021). Flood and Banks (2021) found that learners often express their confidence in mathematical claims through gestures. Additionally, Flood (2021) observed that instructors use representational gestures to provide learners with cues about problem-solving approaches, methods, or answer formats. This highlights the significance of gestures in helping MLs select appropriate problem-solving strategies. Similarly, Ng et al. (2016) found that MLs use gestures as visual mediators when words are unavailable to engage in mathematical discourse.

Several specific instructional strategies have been identified as effective in multilingual mathematics classrooms (Achuonye, 2015; Hughes et al., 2017; Kahiya & Brijlall, 2021b; Moschkovich, 2018). These include:

- **Explicit or direct instruction:** Providing clear and structured teaching methods that break down mathematical concepts for MLs.
- **Cooperative learning:** Encouraging group-based learning activities that allow students to discuss and solve mathematical problems collaboratively.
- **Problem-solving approaches:** Engaging learners in problem-solving tasks that foster critical thinking and the application of mathematical concepts to real-world contexts.

These strategies highlight the importance of integrating language support with mathematics instruction to ensure that MLs can engage meaningfully with mathematical concepts. Each is described in detail below.

Explicit or direct Instruction

Explicit instruction is characterized by a clear description of the skill followed by supported practice and feedback given on time (Hughes et al., 2017). In this strategy, the teacher leads by solving problems on the board, guiding learners through the questions, and gaining the learners' attention by asking questions (Bender, 2009). Learners may discuss ideas with each other in any language. The teacher verifies learners' solutions and reteaches any concepts they struggle with. This method had relevance to this study.

Cooperative learning

Working in groups helps develop learners' self-esteem and motivates them to engage in their learning. Research indicates that this method benefits both non-English speaking learners and those with learning disabilities. According to Alexander and Van Wyk (2014), cooperative learning supports learners from diverse backgrounds in developing intellectual autonomy. Hossain and Rezal (2018) implemented the "learning together" model of cooperative learning and found a significant positive impact on mathematics achievement. In this approach, learners are given individual tasks, discuss their solutions in pairs, and collectively arrive at the best answer to share with the class.

Cooperative learning models have been shown to improve learner performance (Brijlall, 2008). Tindale et al. (2012) argue that learners working in groups solve problems more effectively than those working individually. Similarly, Karali and Aydemir (2018) suggest that cooperative learning fosters cognitive development, enhances social relationships, boosts learners' confidence, and improves their attitudes toward school. Moreover, cooperative learning provides opportunities for positive social interaction and interpersonal communication, both of which contribute to better learning outcomes (Johnson et al., 1994; Slavin, 1996).

Problem-solving approaches

In mathematics instruction, these approaches emphasize learners' active engagement in exploring and solving mathematical problems rather than passively receiving information. In multilingual classrooms, these approaches help bridge language barriers by promoting conceptual understanding through visual representations, collaborative learning, and contextualized problem-solving tasks (Setati & Barwell, 2008). Visual aids, manipulatives, and real-world

Table 1. Pragmatic perspectives during visualization

Pragmatic perspectives	Meaning
Transformational pictures	A component of mnemonic to enhance learners recall memory of information, especially text
Organizational pictures	Structural framework that is handy for text content
Decorative pictures	More for designation of page than relating to text content
Representational pictures	Illustration of part or the whole text content
Interpretational pictures	Comprehends the understanding of difficult question text

problem contexts help MLs grasp abstract concepts (Moschkovich, 2015b).

This section has discussed key strategies recommended by scholars in the field of multilingualism in mathematics and their relevance to this review. The effective implementation of these strategies in teaching and learning can enhance MLs' mathematical thinking, particularly in problem-solving contexts.

Role of Visualization in Enhancing Problem-Solving Skills

According to Bruner's (1996) three models of representation, the learning process is structured into three levels: enactive, iconic, and symbolic. The enactive level is particularly significant for visualization, as it bridges practical experiences and formal understanding, effectively serving as a mediator in communication (Deliyianni et al., 2009; Sintonen, 2024). Diagrams or images that learners use or construct to enhance their understanding help form mental representations, which in turn facilitate problem-solving (Deliyianni et al., 2009; Rösken & Rolka, 2006). Visualization not only aids in establishing relationships among mathematical concepts but also provides an effective means for systematically, semantically, and pragmatically solving problems (Sintonen, 2024).

Supporting this perspective, a study by Mudaly (2016) on the role of visualization in Euclidean geometry proofs found that learners who drew diagrams during the proving process had a clearer understanding of the problems. This finding aligns with the study's focus, highlighting the effectiveness of visualization in mathematical problem-solving. Similarly, Naidoo (2011) describes visualization as the ability to create and transfer mental images, which are essential for mathematical problem-solving. These mental images, represented through diagrams, pictures, or other visual tools, simplify complex mathematical problems and enhance problem-solving capabilities.

The structured use of pictorial signs contributes to different perspectives in mathematical understanding. Well-formed pictorial signs support a syntactic perspective, meaningfully used signs convey a semantic perspective, and pictorial signs employed for thinking, communication, and learning provide a pragmatic perspective (Schnotz, 2002). According to Carney and Levin (2002), pictures in text processing serve five key perspectives within a pragmatic framework: transformational, organizational, decorative, interpretational, and representational. These perspectives are outlined in **Table 1**.

Referring to the table above, Schnotz (2002) found that learners recalled information better when text was illustrated with a picture. According to Schnotz (2002), advanced mathematics also benefits from visual representation when solving problems in areas such as geometry, algebra, and probability. In this study, the focus was on geometry problem-solving. Compared to the novice mathematician, the expert often uses images to visualize the problem and guide planning of a solution (Stylianou & Silver, 2004), which is consistent with Polya's (1945) problem-solving steps adopted in this study.

In this section, the importance of visualization in enhancing mathematical problem-solving skills was discussed. Visualization helps learners better understand mathematical concepts, thereby enhancing their mathematical reasoning skills.

Gaps in the Literature

This critical review of the research literature on visualization in geometry education reveals a scarcity of research examining the effects of visualization during problem-solving in multilingual geometry classrooms. While international studies framed by commognition exist (Barquero et al., 2022; Wang, 2016), they primarily focus on elementary geometry or specific activities such as defining geometric concepts, rather than on problem-solving in geometry. In South Africa, Mahlaba (2021) investigated learners' participation in discourse during geometry problem-solving at the high school level; however, his study focused on pre-service mathematics teachers. Notably, his research did not address the complexity of problem-solving in multilingual classrooms. In the reviewed literature, visualization is treated as language-neutral, which is problematic for MLs.

Furthermore, studies on multilingualism revealed an overreliance on monolingual assessment tools and limited attention to learners' own meaning-making strategies. This paper aims to fill this gap by proposing further research to elucidate the process by which MLs engage with geometry concepts during problem-solving. This paper seeks to contribute to the growing body of knowledge on how language and discourse shape mathematical understanding in multilingual settings.

Implications and Future Research Direction

Research over the years has focused on the teaching strategies used in geometry instruction; however, this paper sought to analyze existing studies on the correlation among visualization, language, and problem-solving in geometry for MLs. Based on the findings of the literature, the following suggestions are made for further research using the commognitive framework:

1. The effects of visualizing during problem-solving for MLs in geometry,
2. Examine the long-term effects of using visual aids on the geometry performance of MLs, and
3. Compare the effectiveness of different types of visual aids (e.g., static and interactive diagrams) in improving the problem-solving skills of MLs in geometry.

CONCLUSION

A critical review of existing research on problem-solving methodologies was conducted to establish a conceptual foundation for this paper. Next, the review explored the historical evolution of visualization in mathematics education, highlighting its pivotal role in facilitating the comprehension of geometry concepts during problem-solving. This discussion encompassed an analysis of visualization processes and the pedagogical tools employed to support mathematical reasoning and understanding. Next, the paper engaged with the scholarship on multilingualism in mathematics education, critically evaluating studies that investigate the intersection of language and mathematical cognition. Attention was given to the impact of linguistic diversity on visualization in problem-solving, as well as to pedagogical strategies designed to enhance mathematical understanding in multilingual geometric classrooms. The available literature on multilingual knowledge in South Africa indicates that geometry skills are among the least researched areas in multilingual contexts.

From the analysis, it can be concluded that although visualization is beneficial in multilingual settings, learners need to be taught to use visuals effectively to reason mathematically with accuracy, paying closer attention to the linguistic factors that might influence their ability to make meaning on their own, thereby enhancing their visual abilities. This finding resonates with Moschkovich (2010), who argues that MLs must find mathematical meanings and unfamiliar linguistic translations. Similarly, Prediger (2019) asserts that the syntactic structure of mathematical problems can increase the cognitive load for learners who are not fluent in the language of teaching and learning.

Given the challenges learners face in multilingual contexts and the strong correlation between the language of teaching and learning, this discussion underscored the need to investigate how language shapes learners' problem-solving processes in geometry within the multilingual secondary school setting in South Africa and to provide necessary support, where applicable, for MLs. This must be done to achieve goal 4 of educational sustainability (quality education).

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